On New Subclass of Multivalent Functions Involving a Generalized Al-Oboudi Differential Operator

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Abstract: The main aim of this paper is to introduce and investigate a new subclass of multivalent functions in the open unit disc \( U \) which is defined by a generalized Al-Oboudi differential operator. Coefficient inequalities, extreme points, Hadamard product and integral mean inequalities for fractional derivative of functions in this class are given.

Key words: Multivalent functions, extreme points, Hadamard product, fractional derivative, normalize.

1. Introduction

Let \( A \) denote the class of functions, \( f(z) \) is normalized by:

\[
f(z) = z + \sum_{j=2}^{\infty} a_j z^j
\]

(1.1)

which are analytic in the open unit disc, \( U = \{ z : |z| < 1 \} \).

For \( f(z) \in A \), Al-Oboudi [1] introduced and investigated the following operator:

\[
D^0_{\delta,p} f(z) = f(z)
\]

\[
D^1_{\delta,p} f(z) = (1 - \delta)f(z) + \delta zf'(z)
\]

\[
D^n_{\delta,p} f(z) = D_{\delta} \left( D^{n-1}_{\delta,p} f(z) \right) (n \in \mathbb{N})
\]

(1.2)

If \( f \) is given by Eq. (1.1), then from Eq. (1.2),

\[
D^n_{\delta,p} f(z) = z + \sum_{j=2}^{\infty} \left[ 1 + (j - 1) \delta \right]^n a_j z^j (n \in \mathbb{N}_0) \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})
\]

(1.3)

Note:

when \( \delta = 1 \), we get Salagean Operator [7].

Let \( A_p \) denote the class of functions of the form:

\[
f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j (p = 1, 2, 3, ...)
\]

(1.4)

which are analytic and \( p \)-valent in the open unit disc, \( U = \{ z : |z| < 1 \} \).

We can write the following equalities for the functions \( f \in A_p \):

\[
D^0_{\delta,p} f(z) = f(z)
\]

\[
D^1_{\delta,p} f(z) = \frac{\delta}{p} z f'(z) + \frac{1}{p} (1 - \delta) f(z) + \frac{(p - 1)}{p} (1 - \delta) z^p
\]

\[
D^n_{\delta,p} f(z) = D_{\delta} \left( D^{n-1}_{\delta,p} f(z) \right) (n \in \mathbb{N})
\]

(1.5)

If \( f \) is given by Eq. (1.4), then from Eq. (1.5), we see that:

\[
D^n_{\delta,p} f(z) = z^p + \sum_{j=p+1}^{\infty} \left[ 1 + (j - 1) \delta \right]^n a_j z^j
\]

(1.6)

(\( n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, p \in \mathbb{N} \))

Remarks:

(1) If \( p = 1 \), \( D^n_{\delta,p} f(z) = D_{\delta} f(z) \) is introduced by Al-Oboudi [1].

(2) If \( p = 1 \), \( \delta = 1 \), \( D^n_{\delta,p} f(z) = D^n f(z) \) is defined by Salagean [7].

Definition 1.1 Let \( C^{m,n}_{\delta,p} (\lambda) \) denote the subclass of \( A_p \) consisting of functions \( f \) satisfying the inequalities:

\[
\Re \left( \frac{D^n_{\delta,p} f(z)}{D^0_{\delta,p} f(z)} \right) > \lambda
\]

(1.7)

For some \( 0 \leq \lambda < 1 \), \( m \in \mathbb{N}, n \in \mathbb{N}_0, D^n_{\delta,p} f(z) \) as defined in Eq (1.6) and for all \( z \in U \).

Remarks:

If \( p = 1 \), the class \( C^{m,n}_{\delta} (\lambda) \) reduces to the class
2. Results and Discussions

Theorem 2.1 Let \( f(z) \in A_p \) satisfy:

\[
\sum_{j=p+1}^{\infty} \Psi_p(m, n, j, \delta, \lambda) |a_j| \leq 2(1 - \lambda) \quad (2.1)
\]

Where:

\[
\Psi_p(m, n, j, \delta, \lambda) = \left[ \frac{1 + (j - 1)\delta}{p} \right]^{m} + \lambda \left[ \frac{1 + (j - 1)\delta}{p} \right]^{n}
+ \left[ \frac{1 + (j - 1)\delta}{p} \right]^{m} + (1 - \lambda) \left[ \frac{1 + (j - 1)\delta}{p} \right]^{n}
\]

For some \( 0 \leq \lambda < 1, m \in \mathbb{N}, n \in \mathbb{N}_0, \delta \geq 0 \), then \( f \in c_{\delta, p}^{m,n}(\lambda) \).

Proof: Suppose that Eq. (2.1) holds true for some \( 0 \leq \lambda < 1, m \in \mathbb{N}, n \in \mathbb{N}_0, \delta \geq 0 \). Let us define:

\[
\frac{D_{\delta, p}^{m} f(z)}{D_{\delta, p}^{n} f(z)} - \lambda = F(z)
\]

By the definition of the class, \( F(z) < \frac{1 + z}{1 - z} \).

There exists a Schwarz function \( w(z) \), with \( w(0) = 0 \) and \( |w(z)| < 1 \), such that:

\[
F(z) = \frac{1 + w(z)}{1 - w(z)}
\]

This implies that:

\[
w(z) = \frac{F(z) - 1}{F(z) + 1}
\]

\[|w(z)| = \frac{|F(z) - 1|}{|F(z) + 1|} < 1\]

We note that:

\[
\frac{F(z) - 1}{F(z) + 1} = \frac{D_{\delta, p}^{m} f(z) - (1 + \lambda) D_{\delta, p}^{n} f(z)}{D_{\delta, p}^{m} f(z) + (1 - \lambda) D_{\delta, p}^{n} f(z)}
\]

\[
= \frac{z^p + \sum_{j=p+1}^{\infty} \left[ \frac{1 + (j-1)\delta}{p} \right]^{m} a_j z^j}{z^p + \sum_{j=p+1}^{\infty} \left[ \frac{1 + (j-1)\delta}{p} \right]^{m} a_j z^j + (1 - \lambda) \left[ \frac{1 + (j-1)\delta}{p} \right]^{n} a_j z^j}
\]

\[
= \frac{-\lambda z^p + \sum_{j=p+1}^{\infty} \left[ \frac{1 + (j-1)\delta}{p} \right]^{m} a_j z^j - (1 + \lambda) \left[ \frac{1 + (j-1)\delta}{p} \right]^{n} a_j z^j}{(2 - \lambda) z^p + \sum_{j=p+1}^{\infty} \left[ \frac{1 + (j-1)\delta}{p} \right]^{m} a_j z^j + (1 - \lambda) \left[ \frac{1 + (j-1)\delta}{p} \right]^{n} a_j z^j}
\]

\[
= \frac{\lambda - \sum_{j=p+1}^{\infty} \left[ \frac{1 + (j-1)\delta}{p} \right]^{m} a_j z^j - (1 + \lambda) \left[ \frac{1 + (j-1)\delta}{p} \right]^{n} a_j z^j}{(2 - \lambda) - \sum_{j=p+1}^{\infty} \left[ \frac{1 + (j-1)\delta}{p} \right]^{m} a_j z^j + (1 - \lambda) \left[ \frac{1 + (j-1)\delta}{p} \right]^{n} a_j z^j}
\]

\[
\leq \frac{\lambda + \sum_{j=p+1}^{\infty} \left[ \frac{1 + (j-1)\delta}{p} \right]^{m} a_j |z|^{j-p}}{(2 - \lambda) - \sum_{j=p+1}^{\infty} \left[ \frac{1 + (j-1)\delta}{p} \right]^{m} a_j |z|^{j-p}}
\]

The last expression is bounded above by \( 1 \) if:
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\[ \frac{z^p + \sum_{j=p+1}^{\infty} \left[ \frac{1+(j-1)\delta}{p} \right]^m a_j z^j}{z^p + \sum_{j=p+1}^{\infty} \left[ \frac{1+(j-1)\delta}{p} \right]^m a_j z^j + (1-\lambda) \left[ \frac{1+(j-1)\delta}{p} \right]^n a_j z^j} \geq \sum_{j=p+1}^{\infty} \left\{ \left[ \frac{1+(j-1)\delta}{p} \right]^m + (1-\lambda) \left[ \frac{1+(j-1)\delta}{p} \right]^n \right\} |a_j| \]

which is equivalent to the condition in Eq. (2.1). This completes the proof of Theorem 2.1.

3. Extreme Points

In view of Theorem 2.1, we now introduce the subclass:

\[ \mathcal{C}_{\delta,p}^{m,n}(\lambda) \subset \mathcal{C}_{\delta,p}^{m,n}(\lambda) \]

which consists of functions:

\[ f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j \ (a_j \geq 0) \]

where Taylor-Maclaurin coefficients satisfy inequality Eq. (2.1). Now, let us determine the extreme points of the class \( \mathcal{C}_{\delta,p}^{m,n}(\lambda) \).

**Theorem 3.1**

\[ f_p(z) = z^p, f_j(z) = z^p + \frac{2(1-\lambda)}{\Psi_p(m,n,j,\delta,\lambda)} z^j \]

\((j \geq p + 1)\) where \( \Psi_p(m, n, j, \delta, \lambda) \) is as given Eq. (2.2), then \( f \in \mathcal{C}_{\delta,p}^{m,n}(\lambda) \) if and only if

It can be expressed in the form:

\[ f(z) = \eta_p f_p(z) + \sum_{j=p+1}^{\infty} \eta_j f_j(z) \]  

(3.2)

where \( \eta_j \geq 0 \) and \( \eta_p = 1 - \sum_{j=p+1}^{\infty} \eta_j \).

**Proof:** Assume that:

\[ f(z) = \eta_p f_p(z) + \sum_{j=p+1}^{\infty} \eta_j f_j(z) \]

Then,

\[ f(z) = (1 - \sum_{j=p+1}^{\infty} \eta_j) z^p \]

\[ + \sum_{j=p+1}^{\infty} \eta_j \left( z^p + \frac{2(1-\lambda)}{\Psi_p(m,n,j,\delta,\lambda)} z^j \right) \]

(3.3)

Thus,

\[ \sum_{j=p+1}^{\infty} \Psi_p(m,n,j,\delta,\lambda) \eta_j \frac{2(1-\lambda)}{\Psi_p(m,n,j,\delta,\lambda)} \]

\[ = 2(1-\lambda) \sum_{j=p+1}^{\infty} \eta_j \]

\[ = 2(1-\lambda)(1-\eta_p) \leq 2(1-\lambda) \]

which shows that \( f \) satisfies the condition in Eq. (2.1) and therefore, \( f \in \mathcal{C}_{\delta,p}^{m,n}(\lambda) \).

Conversely, suppose that \( f \in \mathcal{C}_{\delta,p}^{m,n}(\lambda) \), since:

\[ a_j \leq \frac{2(1-\lambda)}{\Psi_p(m,n,j,\delta,\lambda)}, \ j \geq p + 1 \]

We may set:

\[ \eta_j = \frac{\Psi_p(m,n,j,\delta,\lambda)}{2(1-\lambda)} a_j \]

\[ \eta_p = 1 - \sum_{j=p+1}^{\infty} \eta_j \]

Then we obtain from:
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\[ f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j \]

\[ f(z) = (\eta_p + \sum_{j=p+1}^{\infty} \eta_j z^j = z^p + \sum_{j=p+1}^{\infty} \eta_j \frac{2(1-\lambda)}{\Psi_p(m,n,j,\delta,\lambda)} z^j \]

\[ \Rightarrow f(z) = \eta_p z^p + \sum_{j=p+1}^{\infty} \eta_j f_j(z) \]

This completes the proof of Theorem 3.1.

Corollary 3.2: The extreme points of the class \( \hat{C}^{m,n}_{\delta,p}(\lambda) \) are given by:

Let \( f_p(z) = z^p, f_j(z) = z^p + \frac{2(1-\lambda)}{\Psi_p(m,n,j,\delta,\lambda)} z^j (j \geq p+1) \)

where \( \Psi_p(m,n,j,\delta,\lambda) \) is as given Eq. (2.2).

Theorem 3.3: Let the functions,

\[ f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j \quad (a_j \geq 0) \]

\[ g(z) = z^p + \sum_{j=p+1}^{\infty} b_j z^j \quad (b_j \geq 0) \]

be in the class \( \hat{C}^{m,n}_{\delta,p}(\lambda) \), for \( 0 \leq t \leq 1 \), then the function \( h(z) \) defined by:

\[ h(z) = (1-t)f(z) + tg(z) \]

\[ = z^p + \sum_{j=p+1}^{\infty} c_j z^j \]

\[ (1-t)a_j + tb_j \geq 0 \]

is also in the class \( \hat{C}^{m,n}_{\delta,p}(\lambda) \).

Proof: suppose that each of the functions \( f, g \) is in the class \( \hat{C}^{m,n}_{\delta,p}(\lambda) \), then making use of Eq.(2.1)

We see that:

\[ \sum_{j=p+1}^{\infty} \Psi_p(m,n,j,\delta,\lambda)c_j \]

\[ = (1) \]

\[ - t \sum_{j=p+1}^{\infty} \Psi_p(m,n,j,\delta,\lambda)a_j \]

\[ + t \sum_{j=p+1}^{\infty} \Psi_p(m,n,j,\delta,\lambda)b_j \]

\[ \leq (1-t)2(1-\lambda) + t2(1-\lambda) \]

\[ = 2(1-\lambda) \quad (3.5) \]

Which completes the proof of Theorem 3.3.

4. Integral Mean Inequalities for Fractional Derivatives

We will make use of the definitions of fractional derivatives by Owa [6], Srivastava and Owa [8], see also Eker and Owa [4].

Definition 4.1: The fractional derivative of order \( l \) is defined, for functions \( f \) by:

\[ D_z^l f(z) = \frac{1}{\Gamma(1-l)} \int_0^z \frac{f(t)}{(z-t)^l} d(t), (0 \leq l < 1) \]

where \( f \) is an analytic function in a simply connected region of the \( z \)-plane containing the origin, and the multiplicity of \( (z-t)^{-l} \) is removed by requiring \( \log(z-t) \) to be real when \( (z-t) > 0 \).

Definition 4.2: Under the hypothesis of definition 4.1, the fractional derivative of order \( q + l \) is defined, for function \( f \), by:

\[ D_z^{q+l} f(z) = \frac{d^q}{dz^q} D_z^l f(z), (0 \leq l < 1, q \in \mathbb{N}_0) \]

It readily follows from 4.2 that

\[ D_z^l z^k = \frac{\Gamma(k+1)}{\Gamma(k-l+1)} z^{k-l}, (0 \leq l < 1, k \in \mathbb{N}) \]

Further, we need the concept of subordination between analytic functions and a subordination theorem of Littlewoods in our investigation.

Definition 4.3: Two functions \( f \) and \( g \) analytic in \( U \), and write:
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If there exist a Schwarz function \( w(z) \) analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \), such that:
\[
f(z) = g(w(z)), (z \in U)
\]
In particular, if the function \( g(z) \) is univalent in \( U \), the above subordination is equivalent to:
\[
f(0) = g(0), f(U) = g(U) \quad [2]
\]
In 1925, Littlewood [5], proved the following subordination theorem.

**Lemma 4.4:** If \( f(z) \) and \( g(z) \) are analytic in \( U \) with \( f(z) < g(z) \), then for \( \sigma > 0 \) and \( z = re^{i\theta} (0 < r < 1) \):
\[
\int_0^{2\pi} |f(z)|^\sigma d\theta \leq \int_0^{2\pi} |g(z)|^\sigma d\theta \quad (4.4)
\]

**Theorem 4.5:** Let \( f \in \mathcal{S}^{m,n}_{\delta,p} (\lambda) \) and suppose that
\[
f(0) = g(0), f(U) = g(U) \quad [2]
\]

\[
\sum_{j=p+1}^{\infty} (j-q)_{q+1}a_j = \frac{2(1-\lambda)\Gamma(k+1)\Gamma(2+p-l-q)}{\Psi_p(m,n,j,\delta,\lambda)\Gamma(k+1-l-q)\Gamma(p+1-q)} \quad (4.5)
\]
For some \( j \geq q, 0 \leq l < 1, (j-q)_{q+1} \) denotes the Pochhammer symbol defined by:
\[
(j-q)_{q+1} = (j-q)(j-q+1)\ldots j.
\]

Also let the function:
\[
f_k(z) = z^p + \frac{2(1-\lambda)}{\Psi_p(m,n,j,\delta,\lambda)}z^k \quad (k \geq p+1)
\]
If there exist an analytic function \( w(z) \) given by:
\[
(w(z))^{k-p} = \frac{\Psi_p(m,n,j,\delta,\lambda)\Gamma(k+1-l-q)}{2(1-\lambda)} \sum_{j=p+1}^{\infty} (j-q)_{q+1}\Phi(j)a_jz^{j-p} \quad (4.7)
\]
\[
k \geq q
\]
\[
\Phi(j) = \frac{\Gamma(j-q)}{\Gamma(k+1-l-q)} \quad (0 \leq l < 1, j \geq p+1)
\]
then for \( \sigma > 0 \) and \( z = re^{i\theta} (0 < r < 1) \):
\[
\int_0^{2\pi} |D_z^{q+1}f(z)|^\sigma d\theta \leq \int_0^{2\pi} |D_z^{q+1}f_k(z)|^\sigma d\theta \quad (4.9)
\]
**Proof:** Let
\[
f(z) = z^p + \sum_{j=p+1}^{\infty} a_jz^j
\]
By means of Eq. (4.3) and definition 4.1, we have:
\[
D_z^{q+1}f(z) = \frac{\Gamma(p+1)}{\Gamma(p+1-l-q)} \left[ 1 + \sum_{j=p+1}^{\infty} \frac{\Gamma(j+1)\Gamma(p+1-l-q)}{\Gamma(p+1)}a_jz^{j-p} \right] \quad (4.10)
\]
\[
= \frac{\Gamma(p+1)}{\Gamma(p+1-l-q)} \left[ 1 + \sum_{j=p+1}^{\infty} \frac{\Gamma(p+1-l-q)}{\Gamma(p+1)}(j-q)_{q+1}\Phi(j)a_jz^{j-p} \right]
\]
where:
\[
\Phi(j) = \frac{\Gamma(j-q)}{\Gamma(j+1-l-q)}, 0 \leq l < 1, j \geq p+1
\]
Since \( \Phi \) is deceasing function of \( j \), we get:
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Similarly, from Eqs. (2.1) and (4.3) and definition 4.1:

\[ D_{z}^{\alpha} f_{k}(z) = \frac{\Gamma(p+1)z^{k-l-q}}{\Gamma(p+1)\Gamma(j+1-l-q)} \left[ 1 + \frac{2(1-\lambda)\Gamma(k+1)\Gamma(p+1-l-q)}{\Psi_{p}(m,n,j,\delta,\lambda)\Gamma(p+1)\Gamma(k+1-l-q)} z^{k-p} \right] \]  

(4.11)

for \( \sigma > 0 \) and \( z = re^{i\theta} \) (\( 0 < r < 1 \)), we show that:

\[
\int_{0}^{2\pi} \left| 1 + \sum_{j=p+1}^{\infty} \frac{\Gamma(j+1)\Gamma(p+1-l-q)}{\Gamma(p+1)\Gamma(j+1-l-q)} a_{j}z^{j-p} \right|^{\sigma} d\theta 
\leq \int_{0}^{2\pi} \left| 1 + \frac{2(1-\lambda)\Gamma(k+1)\Gamma(p+1-l-q)}{\Psi_{p}(m,n,j,\delta,\lambda)\Gamma(p+1)\Gamma(k+1-l-q)} z^{k-p} \right|^{\sigma} d\theta 
\]

(4.12)

So, by applying lemma 4.4, it is enough to show that:

\[
1 + \sum_{j=p+1}^{\infty} \frac{\Gamma(j+1)\Gamma(p+1-l-q)}{\Gamma(p+1)\Gamma(j+1-l-q)} a_{j}z^{j-p} < 1 + \frac{2(1-\lambda)\Gamma(k+1)\Gamma(p+1-l-q)}{\Psi_{p}(m,n,j,\delta,\lambda)\Gamma(p+1)\Gamma(k+1-l-q)} z^{k-p} S 
\]

\[ j \geq p + 1 \]

If the above subordination holds true, then we have analytic function \( w(z) \) with \( w(0) = 0 \) and \( |w(z)| < 1 \), such that:

\[
1 + \sum_{j=p+1}^{\infty} \frac{\Gamma(j+1)\Gamma(p+1-l-q)}{\Gamma(p+1)\Gamma(j+1-l-q)} a_{j}z^{j-p} = 1 + \frac{2(1-\lambda)\Gamma(k+1)\Gamma(p+1-l-q)}{\Psi_{p}(m,n,j,\delta,\lambda)\Gamma(p+1)\Gamma(k+1-l-q)} w(z)^{k-p} 
\]

By the condition of the Theorem, we define the function \( w(z) \) by:

\[
w(z)^{k-p} = \frac{\Psi_{p}(m,n,j,\delta,\lambda)\Gamma(k+1-l-q)}{2(1-\lambda)\Gamma(k+1)} \sum_{j=p+1}^{\infty} (j-q)\Phi(j)a_{j}z^{j-p} 
\]

(4.13)

which readily yields \( w(0) = 0 \). For such a function \( w(z) \), we have:

\[
|w(z)|^{k-p} \leq \frac{\Psi_{p}(m,n,j,\delta,\lambda)\Gamma(k+1-l-q)}{2(1-\lambda)\Gamma(k+1)} \sum_{j=p+1}^{\infty} (j-q)\Phi(j)a_{j}|z|^{j-p} \leq |z| \left| \frac{\Psi_{p}(m,n,j,\delta,\lambda)\Gamma(k+1-l-q)}{2(1-\lambda)\Gamma(k+1)} \Phi(p+1) \sum_{j=p+1}^{\infty} (j-q)\Phi(j)a_{j} \right| < 1 
\]

where \( \Psi_{p}(m,n,j,\delta,\lambda) \) is as in Eq. (2.2). By means of the hypothesis of the Theorem, thus, the theorem is proved.

As a special case \( q = 0 \), we have the following from Theorem 4.5.

Corollary 4.6: Let \( f \in \mathfrak{L}_{\delta,p}^{m,n}(\lambda) \) and suppose that

\[ \sum_{j=p+1}^{\infty} j a_{j} = \frac{2(1-\lambda)\Gamma(k+1)\Gamma(2+p-l)}{\Psi_{p}(m,n,j,\delta,\lambda)\Gamma(k+1-l)\Gamma(p+1)} \]

(4.14)

\[ (j \geq p + 1) \]
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if there exists analytic function \( w(z) \) define by:
\[
(w(z))^{k-p} = \Psi_p(m,n,j,\delta,\lambda) \frac{\Gamma(k+1-l)}{2(1-\lambda)} \frac{\Gamma(k+1)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} j\Phi(j) a_j z^{j-p}
\]
(4.15)

With,
\[
\Phi(j) = \frac{\Gamma(j)}{\Gamma(j+1-l)} (0 \leq l < 1, j \geq p + 1)
\]

Then \( \sigma > 0 \) and \( z = re^{i\theta} \) \( (0 < r < 1) \),
\[
\int_0^{2\pi} |D_z^{l}f(z)|^\sigma \, d\theta \leq \int_0^{2\pi} |D_z^{l+f}(z)|^\sigma \, d\theta \quad (4.16)
\]

Letting \( q = 1 \), we have the following from Theorem 4.5:

Corollary 4.7: Let \( f \in \mathcal{C}_{\delta,p}^{m,n}(\lambda) \) and suppose that:
\[
\sum_{j=p+1}^{\infty} j(j-1)a_j = \frac{2(1-\lambda)\Gamma(k+1)\Gamma(p+1-l)}{\Psi_p(m,n,j,\delta,\lambda)\Gamma(k-l)\Gamma(p)}
\]
(4.17)

If there exists analytic function \( w(z) \) define by:
\[
(w(z))^{k-p} = \Psi_p(m,n,j,\delta,\lambda) \frac{\Gamma(k-l)}{2(1-\lambda)} \frac{\Gamma(k+1)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} j(j-1) a_j z^{j-p}
\]
(4.18)

With,
\[
\Phi(j) = \frac{\Gamma(j-1)}{\Gamma(j-l)} (0 \leq l < 1, j \geq p + 1)
\]
(4.19)

Then \( \sigma > 0 \) and \( z = re^{i\theta} \) \( (0 < r < 1) \),
\[
\int_0^{2\pi} |D_z^{l+f}(z)|^\sigma \, d\theta \leq \int_0^{2\pi} |D_z^{l+f}(z)|^\sigma \, d\theta
\]
(4.20)

5. Conclusion

In this paper, using a generalized Al-Oboudi differential operator, we defined a new subclass of \( P \)-valent functions and established its properties. Results obtained provide new properties of certain subclasses of Multivalent functions.

References