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On the Inconsistency of Classical Propositional Calculus

Teodor J. Stępień and Łukasz T. Stępień

The Pedagogical University of Cracow, Krakow, ul. Podchorazych2, 30-084, Poland

Abstract: The classical propositional calculus (often called also as "zero-order logic"), is the most fundamental two-valued logical system. It is necessary to construct the classical calculus of quantifiers (often called also as "classical calculus of predicates" or "first-order logic"), which is necessary to construct the classical functional calculus. This last one is being used for formalization of the Arithmetic System. At the beginning of this paper, we introduce a notation and we repeat certain well-known notions (among others, the notions of operation of consequence, a system, consistency in the traditional sense, consistency in the absolute sense). Next, we establish that classical propositional calculus is an inconsistent theory.

Keywords: Classical propositional calculus, consistency in the traditional sense, consistency in the absolute sense.

1. Introduction

Let: $\rightarrow$, $\sim$, $\vee$, $\wedge$, $\equiv$ denote the connectives of implication, negation, disjunction, conjunction and equivalence, respectively. $\mathcal{N} = \{1,2,\dots\}$ denotes the set of all natural numbers.

Next, $At_0 = \{p_1^1, p_2^1, \dots, p_1^2, p_2^2, \dots, p_1^k, p_2^k, \dots\}$ $(k \in \mathcal{N})$ denotes the set of all propositional variables. Hence, S_0 is the set of all well-formed formulas, which are built in the usual manner from propositional variablesby means of logical connectives. $P_0(\phi)$ denotes the set of all propositional variables occuring in ϕ $(\phi \in S_0)$.

R_{S_0} denotes the set of all rules over S_0.$E(\mathfrak{M})$ is the set of all formulas valid in the matrix $\mathfrak{M}$. $\mathfrak{M}_2$ denotes the classical two-valued matrix. Z_2 is the set of all formulas valid in the matrix $\mathfrak{M}_2$ (see [10], cf. [1-7], [11-13]). Next, $S_0^0 = \{\phi \in S_0 : \phi \notin Z_2 \& \sim\phi \notin Z_2\}$.

We use $\Rightarrow, \neg, \mathbb{V}, \&, \Leftrightarrow, \forall, \exists$ as metalogical symbols.Next, r_0 denotes Modus Ponens in propositional calculus. Hence, $R_0 = \{r_0\}$. We write $X \subset Y$ for $X \subseteq Y$ and $X \neq Y$. For any $X \subseteq S_0$ and $R \subseteq R_{S_0}$, $Cn(R, X)$ is the smallest subset of S_0, containing X, and closed under the rules belonging to R, where$R \subseteq R_{S_0}$.

The couple $\langle R, X\rangle$ is called a system, whenever $R \subseteq R_{S_0}$, and $X \subseteq S_0$. Hence, $\langle R_0, Z_2\rangle$ is the system of the classical propositional calculus.

Now we repeat some well-known definitions (see [10], cf. [5, 7-9, 11]). Let $R \subseteq R_{S_0}$ and $X \subseteq S_0$. Then:

Definition 1.1 $\langle R, X\rangle \in Cns^T \Leftrightarrow (\neg\exists\alpha \in S_0)\ [\alpha \in Cn(R, X) \& \sim\alpha \in Cn(R, X)]$.

Definition 1.2 $\langle R, X\rangle \in Cns^A \Leftrightarrow Cn(R, X) \neq S_0$.

$\langle R, X\rangle \in Cns^T$ denotes that the system $\langle R, X\rangle$ is consistent in the traditional sense. $\langle R, X\rangle \in Cns^A$ denotes that the system $\langle R, X\rangle$ is consistent in the absolute sense (see [10], cf.[11]).

2. The Main Result

Theorem $\langle R_0, Z_2\rangle \notin Cns^T$. (cf. [14]).

Proof. Elementary.

Corresponding author: Łukasz T. Stępień, Ph.D., research fields: proof theory, entailment, integrable systems, nonlinear differential equations, classical and quantum field theory. E-mail: sfstepie@cyf-kr.edu.pl, lukasz.stepien@up.krakow.pl.

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