Regular and Irregular Sampling Linear Transforms in Series of Shift-Invariant Spaces

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Abstract: In this article we show that there exists an analogue of the Fourier duality technique in the setting a series of shift-invariant spaces. Really, every a series shift -invariant space \( \sum_{i=1}^{n} V_{\phi_i} \) with a stable generator \( \sum_{i=1}^{n} \phi_i \) is the range space of a bounded one-to-one linear operator \( T \) between \( L^2(0,1) \) and \( L^2(\mathbb{R}) \). We show regular and irregular sampling formulas in \( \sum_{i=1}^{n} V_{\phi_i} \) are obtained by transforming.

Key words: Shift-invariant spaces, sampling expansions, zak transform.

1. Introduction

Regular and irregular Sampling linear transforms with shift-invariant spaces are known as very important in mathematical analysis and its applications. In the recent decades, the regular and irregular Sampling linear transforms have been studied and there are many interesting papers published in this area of research (see for example [2, 3] and the references therein). The Whittaker-Shannon-Kotel’nikov sampling theorem states that any \( f \) in the classical Paley-Wiener space \( PW_\pi \), \( PW_\pi = \{ f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) : \text{ supp } \hat{f} \subseteq [-\pi, \pi] \} \), i.e., band limited to \([-\pi, \pi]\), may be reconstructed from its samples \( \{ f(n) \}_{n \in \mathbb{Z}} \) on the integers as

\[
f(t) = \sum_{n=-\infty}^{\infty} f(n) \text{sinc}(t - n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(\mathcal{W}) e^{i\mathcal{W}t} d\mathcal{W} = \left\{ \frac{n}{\sqrt{2\pi}} e^{-i\mathcal{W}t} \right\}_{n \in \mathbb{Z}} \text{, } t \in \mathbb{R}
\]

so every \( f(t_n) \) of \( f \) is the inner product in \( L^2[-\pi,\pi] \) of \( \hat{f} \) and the complex exponential \( e^{-itn}/\sqrt{2\pi} \). Recall that the Paley-Wiener space \( PW_\pi \) is a reproducing kernel Hilbert space (RKHS) whose reproducing kernel is \( k(t,s) = \text{sinc}(t-s) \). An irregular sampling formula in \( PW_\pi \) at a sequence \( \{t_n\}_{n \in \mathbb{Z}} \) of real points may be obtained by perturbating the orthonormal basis \( \{ e^{-i\mathcal{W}t}/\sqrt{2\pi} \}_{n \in \mathbb{Z}} \) in such a way that the sequence of complex exponentials \( \{ e^{-itn}/\sqrt{2\pi} \}_{n \in \mathbb{Z}} \) forms a Riesz basis for \( PW_\pi \). Furthermore the Paley-Wiener-Levinson sampling theorem states that any \( f \in PW_\pi \) can be recovered from its samples \( \{f(t_n)\}_{n \in \mathbb{Z}} \) by Lagrange-type interpolation series

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where \( G \) stands for the infinite product \( G(t) = (t - t_0) \prod_{n=1}^{\infty} \left(1 - \frac{1}{t_n} \right) \left(1 - \frac{1}{t_n} \right) \) [4].

The main aim in this paper is to show that the Fourier duality for Paley-Wiener spaces can be generalized to the case of a series shift-invariant space \( \Sigma_{i=1}^{n} V_{\phi_i} \) with a stable generator \( \Sigma_{i=1}^{n} \phi_i \). To this end, we define a bounded one-to-one linear operator \( T \) between \( L^2(0,1) \) and \( L^2(\mathbb{R}) \) as \( T: L^2(0,1) \rightarrow L^2(\mathbb{R}), F \rightarrow f \) such that \( f(t) = \langle F, K_t \rangle_{L^2(0,1)} \), where the kernel transform \( t \in \mathbb{R} \rightarrow K_t \in L^2(0,1) \) is given by the Zak transform of \( \Sigma_{i=1}^{n} \phi_i \), namely, \( K_t(x) = \sum_{i=1}^{n} Z_{\phi_i}(t, x), a.e.x \in (0, 1) \).

Recall that the Zak transform of \( f \in \mathcal{L}^2(\mathbb{R}) \) is formally defined as \((Zf)(t, w) = \Sigma_{n \in \mathbb{Z}} f(t + n) e^{-2 \pi i n w}, t, w \in \mathbb{R}\).

The series shift-invariant space \( \Sigma_{i=1}^{n} V_{\phi_i} \) coincides with the range space of \( T \).

### 2. Preliminaries

Suppose \( \Sigma_{i=1}^{n} \phi_i \in L^2(\mathbb{R}) \) be a stable generator for the series shift-invariant space
\[
\Sigma_{i=1}^{n} V_{\phi_i} = \left\{ \sum_{n \in \mathbb{Z}} \left\{ a_n \phi_i(\cdot - n) \right\} : \{a_n\} \in L^2(\mathbb{Z}) \right\} = \mathcal{L}^2(\mathbb{R})
\]
i.e., the sequence \( \{ \Sigma_{i=1}^{n} \phi_i(\cdot - n) \}_{n \in \mathbb{Z}} \) is a Riesz basis for \( \Sigma_{i=1}^{n} V_{\phi_i} \). A Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Recall that the sequence \( \{ \Sigma_{i=1}^{n} \phi_i(\cdot - n) \}_{n \in \mathbb{Z}} \) is a Riesz sequence, i.e., a Riesz basis for \( \Sigma_{i=1}^{n} V_{\phi_i} \).

If \( 0 < \| \phi \|_0 \leq \| \phi \|_{\infty} < \infty \), where \( \| \phi \|_0 \) denotes the essential infimum of the function \( \phi(w) = \Sigma_{k \in \mathbb{Z}} \Sigma_{i=1}^{n} |\phi_i(w + k)|^2 \) in \([0, 1]\), and \( \| \phi \|_{\infty} \) its essential supremum (see [3]). We assume along the section that, for every \( t \in \mathbb{R}, \) the series \( \Sigma_{n \in \mathbb{Z}} \Sigma_{i=1}^{n} |\phi_i(t - n)|^2 \) converges. Then, by using the Riesz’ subsequence theorem [7] we can choose the point wise limit \( \Sigma_{j \in \mathbb{Z}} f_j(t) = \Sigma_{n \in \mathbb{Z}} a_n \Sigma_{i=1}^{n} \phi_i(\cdot - n) \) for every \( t \in \mathbb{R} \), as the representative element of the class \( \Sigma_{n \in \mathbb{Z}} a_n \Sigma_{i=1}^{n} \phi_i(\cdot - n) \) in \( L^2(\mathbb{R}) \). Indeed, for each fixed \( t \in \mathbb{R} \) we have
\[
\Sigma_{n \in \mathbb{Z}} |f_n(t)|^2 \leq \frac{1}{\| \phi \|_0} \Sigma_{j \in \mathbb{Z}} \Sigma_{n \in \mathbb{Z}} \Sigma_{i=1}^{n} |\phi_i(t - n)|^2 \|f_j\|^2
\]
by Cauchy-Schwartz’s inequality in \( \Sigma_{j \in \mathbb{Z}} f_j(t) = \Sigma_{n \in \mathbb{Z}} a_n \Sigma_{i=1}^{n} \phi_i(\cdot - n) \), and the Riesz basis condition \( \| \phi \|_0 \Sigma_{n \in \mathbb{Z}} |a_n|^2 \leq \Sigma_{j \in \mathbb{Z}} \|f_j\|^2 \leq \| \phi \|_{\infty} \Sigma_{n \in \mathbb{Z}} |a_n|^2 \) , \( \Sigma_{j \in \mathbb{Z}} \Sigma_{n \in \mathbb{Z}} \Sigma_{i=1}^{n} \phi_i(\cdot - n) \). Inequality (2) shows that convergence in the \( L^2(\mathbb{R}) \)-norm implies pointwise convergence in \( \mathbb{R} \). The reproducing kernel of \( \Sigma_{i=1}^{n} V_{\phi_i} \) is given by \( \Sigma_{j \in \mathbb{Z}} \Sigma_{n \in \mathbb{Z}} \Sigma_{i=1}^{n} \Sigma_{i=1}^{n} \phi_j^* \phi_i(s - n) \) where \( \{ \Sigma_{i=1}^{n} \phi_i(\cdot - n) \}_{n \in \mathbb{Z}} \) denotes the dual Riesz basis of \( \{ \Sigma_{i=1}^{n} \phi_i(\cdot - n) \}_{n \in \mathbb{Z}} \). Recall that the function \( \Sigma_{i=1}^{n} \phi_i \) has Fourier transform \( \Sigma_{i=1}^{n} \phi_i^* = \Sigma_{i=1}^{n} \phi_i^* \) (see [3]).

### 3. A Linear Transform Defining a Series of Shift-Invariant Space

For every \( t \in \mathbb{R} \), consider the function \( K_t \in L^2(0,1) \) defined by the Fourier series \( K_t = \Sigma_{n \in \mathbb{Z}} \Sigma_{i=1}^{n} \phi_i(t + n) e^{-2\pi inx} \).

Notice that \( K_t(x) = Z_K(t, x) a.e.x \in (0, 1) \), where \( Z \) the Zak transform of \( \Sigma_{i=1}^{n} \phi_i \). See Refs. [2, 4, 5] for properties and uses of the Zak transform. Then, for each \( F \in L^2(0,1) \) we define \( \Sigma_{j \in \mathbb{Z}} f_j: \mathbb{R} \rightarrow \mathbb{C}, t \rightarrow \Sigma_{j \in \mathbb{Z}} f(t) = \langle F, K_t \rangle_{L^2(0,1)} \). If \( T \) the linear transform which maps \( F \in L^2(0,1) \) into \( \Sigma_{j \in \mathbb{Z}} f_j \), i.e., \( T(F) = \Sigma_{j \in \mathbb{Z}} f_j \), then we can identify the range space of \( T \) as the series shift-invariant \( \Sigma_{i=1}^{n} V_{\phi_i} \), i.e., \( T(L^2(0,1)) = \Sigma_{i=1}^{n} V_{\phi_i} \).

**Theorem 1:** Let \( T \) be mapping an invertible bounded operator between \( L^2(0,1) \) and \( \Sigma_{i=1}^{n} V_{\phi_i} \).

**Proof:** Let \( T \) is bijective since it maps the orthonormal basis \( \{ e^{-2\pi inx} \}_{n \in \mathbb{Z}} \) in \( L^2(0,1) \) into the Riesz basis \( \{ \Sigma_{i=1}^{n} \phi_i(t + n) \}_{n \in \mathbb{Z}} \) in \( \Sigma_{i=1}^{n} V_{\phi_i} \). Concerning the continuity, for \( F \in L^2(0,1) \), we have
\[ \|T(F)\|_{L^2(\mathbb{R})} = \left\| \sum_{n \in \mathbb{Z}} \langle F, e^{-2\pi i n x} \rangle_{L^2(0,1)} \phi(t + n) \right\|_{L^2(\mathbb{R})}^2 \]

\[ \leq \|\phi\| \| \sum_{n \in \mathbb{Z}} \left( \| F, e^{-2\pi i n x} \|_{L^2(0,1)}^2 \right) = \|\phi\| \| F \|_{L^2(0,1)}^2 \]

by the upper Riesz basis condition for \( \{\sum_{n \in \mathbb{Z}} \phi_i(\cdot + n)\}_{n \in \mathbb{Z}} \). Having in mind the periodicity relations of the Zak transform, \( K_t \) satisfies \( K_{t+m}(x) = e^{2\pi i m x} K_t(x) \) in \( L^2(0,1) \); \( t \in \mathbb{R} \) and \( m \in \mathbb{Z} \). Now, for \( \sum_{j \in \mathbb{Z}} f_j \subseteq \sum_{n \in \mathbb{Z}} V_{\phi_j} \) consider \( F = \sum_{j \in \mathbb{Z}} T^{-1}(f_j) \) in \( L^2(0,1) \). For every \( n \in \mathbb{Z} \) we have

\[ T\left[F(x)e^{2\pi i n x}\right](t) = (F(x)e^{2\pi i n x}, K_t(x))_{L^2(0,1)} \]

\[ = (F, K_{t-n})_{L^2(0,1)} = \sum_{j \in \mathbb{Z}} f_j(t - n) \]

Since \( T \) is a bounded invertible operator, \( \{\sum_{j \in \mathbb{Z}} f_j(t - n)\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( \sum_{n \in \mathbb{Z}} V_{\phi_j} \), if and only if \( \{F(x)e^{2\pi i n x}\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( L^2(0,1) \).

**Theorem 2:** Given \( F \in L^2(0,1) \), the following results hold:

(i) \( \{F(x)e^{2\pi i n x}\}_{n \in \mathbb{Z}} \) is a Bessel sequence in \( L^2(0,1) \) if \( F \) satisfies \( \|F\|_0 < \infty \).

(ii) \( \{F(x)e^{2\pi i n x}\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( L^2(0,1) \) if \( F \) satisfies \( 0 < \|F\|_0 \leq \|F\|_\infty < \infty \). In this case, the optimal Riesz bounds of \( \{F(x)e^{2\pi i n x}\}_{n \in \mathbb{Z}} \) are \( \|F\|_0^2 \) and \( \|F\|_\infty^2 \).

(iii) \( \{F(x)e^{2\pi i n x}\}_{n \in \mathbb{Z}} \) is a frame in \( L^2(0,1) \) if and only if \( \{F(x)e^{2\pi i n x}\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( L^2(0,1) \). Therefore we have the following corollary in \( \sum_{n \in \mathbb{Z}} V_{\phi_j} \).

**Corollary 1:** Given a \( g \in \sum_{n \in \mathbb{Z}} V_{\phi_j} \), consider \( G = T^{-1}(g) \in L^2(0,1) \). Then \( \{g(t-n)\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( \sum_{n \in \mathbb{Z}} V_{\phi_j} \), if \( 0 < \|G\|_0 \leq \|G\|_\infty < \infty \).

4. **Regular Sampling in a Series of Shift-invariant Spaces**

Regular sampling in \( \sum_{n \in \mathbb{Z}} V_{\phi_j} \) arises by considering appropriate Riesz bases in \( L^2(0,1) \). A fixed \( 0 < \epsilon < 1 \), the regular samples at \( \{n - \epsilon\}_{n \in \mathbb{Z}} \) of \( \sum_{j \in \mathbb{Z}} f_j \subseteq \sum_{n \in \mathbb{Z}} V_{\phi_j} \) are given by \( \sum_{j \in \mathbb{Z}} f_j(n - \epsilon) = (F, K_{n-\epsilon})_{L^2(0,1)} \)

\[ = \{F, K_{n-\epsilon}e^{2\pi i n x}\}_{L^2(0,1), n \in \mathbb{Z}}, \quad \text{where} \quad F = \sum_{j \in \mathbb{Z}} T^{-1}(f_j). \]

The sequence \( \{K_{n-\epsilon}(x)e^{2\pi i n x}\}_{n \in \mathbb{Z}} \) in \( L^2(0,1) \) has the biorthonormal sequence \( \{e^{2\pi i n x}/K_{1-\epsilon}(x)\}_{n \in \mathbb{Z}} \) provided \( 1/K_{1-\epsilon} \in L^2(0,1) \). Hence, stable regular sampling in \( \sum_{n \in \mathbb{Z}} V_{\phi_j} \) reduces to studying whenever \( \{K_{n-\epsilon}(x)e^{2\pi i n x}\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( L^2(0,1) \), and this depends on the function \( K_{1-\epsilon} \) as stated in Theorem 2. Expanding \( F = \sum_{j \in \mathbb{Z}} T^{-1}(f_j) \) with respect to the Riesz basis \( \{e^{2\pi i n x}/K_{1-\epsilon}(x)\}_{n \in \mathbb{Z}} \), via the invertible bounded operator \( T \), we obtain a regular sampling formula for \( \sum_{j \in \mathbb{Z}} f_j \) (see [3]).

**Lemma 1:** Given \( 0 \leq \epsilon < 1 \), there exists \( S_{1-\epsilon} \subseteq \sum_{n \in \mathbb{Z}} V_{\phi_j} \) satisfying the interpolation condition \( S_{1-\epsilon}(n-\epsilon) = \delta_{n,0} \), where \( n \in \mathbb{Z} \), if \( 1/K_{1-\epsilon} \in L^2(0,1) \). In this case \( S_{1-\epsilon} = T(1/K_{1-\epsilon}) \).

**Proof:** Assume that \( S_{1-\epsilon} \subseteq \sum_{n \in \mathbb{Z}} V_{\phi_j} \) satisfying \( S_{1-\epsilon}(n-\epsilon) = \delta_{n,0} \), where \( n \in \mathbb{Z} \). For \( F_{1-\epsilon} = T^{-1}(S_{1-\epsilon}) \) we have

\[ S_{1-\epsilon}(n-\epsilon) = (F_{1-\epsilon}, K_{1-\epsilon}^{-1})_{L^2(0,1)} = (F_{1-\epsilon}e^{2\pi i n x}, K_{1-\epsilon})_{L^2(0,1)} \]

\[ = \int_0^1 F_{1-\epsilon}(x)K_{1-\epsilon}(x)e^{-2\pi i n x} dx = \delta_{n,0} \]

which implies that \( F_{1-\epsilon}(x)K_{1-\epsilon}^{-1}(x) = 1 \) a.e. in \( (0,1) \), and \( 1/K_{1-\epsilon} \in L^2(0,1) \). Conversely, if \( 1/K_{1-\epsilon} \in L^2(0,1) \), we define \( S_{1-\epsilon} = T(1/K_{1-\epsilon}) \). For \( n \in \mathbb{Z} \) it satisfies

\[ S_{1-\epsilon}(n - \epsilon) = \left( \frac{1}{K_{1-\epsilon}}, K_{1-\epsilon}^{-1} \right)_{L^2(0,1)} \]

\[ = \left\{ 1, e^{2\pi i n x} \right\}_{L^2(0,1)} = \delta_{n,0} \]

Therefore we can characterize stable regular sampling in \( \sum_{n \in \mathbb{Z}} V_{\phi_j} \).

**Theorem 3:** Consider \( 0 \leq \epsilon < 1 \) such that \( 1/K_{1-\epsilon} \in L^2(0,1) \). The following equivalent:

(i) \( 0 < \|K_{1-\epsilon}\|_0 \leq \|K_{1-\epsilon}\|_\infty < \infty \).

(ii) There exists a Riesz basis \( S_{n}\) for \( \sum_{n \in \mathbb{Z}} V_{\phi_j} \) such that, for each \( \sum_{j \in \mathbb{Z}} f_j \subseteq \sum_{n \in \mathbb{Z}} V_{\phi_j} \), we have then pointwise expansion

\[ \sum_{j \in \mathbb{Z}} f_j(t) = \sum_{n \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} f_j(n - \epsilon)S_n(t), t \in \mathbb{R} \]

Furthermore, in this case the sampling functions are \( S_n(t) = S_{1-\epsilon}(t-n) \), where \( S_{1-\epsilon} = T(1/K_{1-\epsilon}) \). The sampling series converges in the \( L^2(\mathbb{R}) \)-norm sense,
absolutely and uniformly in subsets of $\mathbb{R}$ where $K_l$ is bounded.

**Proof:** First we prove that (i) implies (ii). Consider $S_{1-\epsilon} = T(1/K_{1-\epsilon})$. Condition (i) implies that 

$$0 < \|1/K_{1-\epsilon}\|_0 \leq \|1/K_{1-\epsilon}\|_\infty < \infty$$

and, as a consequence, Corollary 1 gives that $\{S_{1-\epsilon}(t-n)\}_{n \in \mathbb{Z}}$ is a Riesz basis for $\sum_{i=1}^n V_{\phi_i}$. For each $\sum_{i=1}^n f_j \in \sum_{i=1}^n V_{\phi_i}$, there exists $\{a_n\}_{n \in \mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ such that $\sum_{i=1}^n f_j(t) = \sum_{i=1}^n a_n S_{1-\epsilon}(t-n)$ where the convergence is also point wise for each $t \in \mathbb{R}$ since $V_{\phi}$ is a RKHS. Let $\epsilon = m - \epsilon$, and using $S_{1-\epsilon}(n-\epsilon) = \delta_{n,0}$, we have that $a_n = \sum_{i=1}^n f_j(m-\epsilon)$ for any $m \in \mathbb{Z}$. Conversely, assume that the condition (ii) holds. Putting $\sum_{i=1}^n f_j(t) = S_{1-\epsilon}(t-m)$, $m \in \mathbb{Z}$, we obtain that $S_m(t) = S_{1-\epsilon}(t-m)$ and, as a consequence, $\{S_{1-\epsilon}(t-n)\}_{n \in \mathbb{Z}}$ is a Riesz basis for $\sum_{i=1}^n V_{\phi_i}$. Since $S_{1-\epsilon} = T(1/K_{1-\epsilon})$, Corollary 1 gives condition (i).

The uniform convergence is a standard result in the setting of the RKHS theory. A straightforward calculation gives the Fourier transform of $S_{1-\epsilon}$. Indeed,

$$\hat{S}_{1-\epsilon}(W) = T\left(\frac{1}{K_{1-\epsilon}}\right)(W) = \frac{\hat{\phi}(W)}{Z_{\epsilon}(1-\epsilon, W/2\pi)} a.e.$$

in $\mathbb{R}$.

### 5. Irregular Sampling in a Series of Shift-Invariant Spaces

Usually, one may consider irregular sampling as a perturbation of the regular sampling. In the present setting, we can try to recover any function

$$\sum_{j \in \mathbb{Z}} f_j = \sum_{i=1}^n V_{\phi_i}$$

from its perturbed samples

$$\{\sum_{j \in \mathbb{Z}} f_j(n-\epsilon + \delta_n)\}_{n \in \mathbb{Z}}$$

where $0 \leq \epsilon < 1$ and $\{\delta_n\}_{n \in \mathbb{Z}}$ is a sequence $\in (-1,1)$. Since

$$\sum_{j \in \mathbb{Z}} f_j(n-\epsilon + \delta_n) = \langle F, K_{n-\epsilon + \delta_n}\rangle_{L^2(0,1)} \quad n \in \mathbb{Z},$$

where

$$F = \sum_{j \in \mathbb{Z}} T^{-1}(f_j)$$

$\in L^2(0,1)$, a challenge problem is to prove that $\{K_{n-\epsilon + \delta_n}\}_{n \in \mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$ (see [3]).

**Theorem 4:** Let $F = \sum_{k \in \mathbb{Z}} a_k e^{-2\pi i k \epsilon}$ be in $L^2(0,1)$ such that

$$0 < \|F\|_0 \leq \|F\|_\infty < \infty.$$

Let $\{F_n\}_{n \in \mathbb{Z}}$ be a sequence of functions in $L^2(0,1)$ with Fourier expansions $F_n = \sum_{k \in \mathbb{Z}} a_k(n)e^{-2\pi i k \epsilon}$, $n \in \mathbb{Z}$. Suppose that the infinite matrix $D = \{d_{n,k}\}_{n,k \in \mathbb{Z}}$ with entries $d_{n,k} = a_{n,k}(n)-a_{n,k-1}$, $n,k \in \mathbb{Z}$, satisfies $\|D\|_0 \leq \|F\|_0$. Then $\{F_n(x)e^{2\pi i n x}\}_{n \in \mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$.

**Proof:** To this end we use the following result on perturbation of Riesz bases in a Hilbert space $H$ which can be found in Ref. [7]: let $\{f_k\}_{k=1}^\infty$ be a Riesz basis for $H$ with Riesz bounds $A,B$, and let $\{g_k\}_{k=1}^\infty$ be in $H$. If $R < A$ such that

$$\sum_{k=1}^\infty \left(\sum_{j \in \mathbb{Z}} |f_j - g_j|_F^2\right)^{1/2} \leq R \sum_{j \in \mathbb{Z}} \|f_j\|^2$$

then $\{g_k\}_{k=1}^\infty$ is Riesz in $H$. For any

$$\sum_{n \in \mathbb{Z}}, \sum_{j \in \mathbb{Z}} |F_n(x)e^{2\pi i n x} - F(x)e^{2\pi i n x}, f_j|_F^2$$

we have

$$= \sum_{n \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} \left( a_k(n) - a_k \right) e^{2\pi i (n-k) x}, \sum_{j \in \mathbb{Z}} c_j e^{2\pi i j x} \right|^2$$

$$= \|Dc\|_0^2 \leq \sum_{j \in \mathbb{Z}} \|D\|_0^2 \|f_j\|^2$$

Taking into account that in our case $A = \|D\|_0^2$, we obtain that $\{F_n(x)e^{2\pi i n x}\}_{n \in \mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$.

**Theorem 5:** Given $\leq \|D\|_0^2$, we obtain that $\{F_n(x)e^{2\pi i n x}\}_{n \in \mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$. Given

$$0 < \|K_{1-\epsilon}\|_0 \leq \|K_{1-\epsilon}\|_\infty < \infty.$$
satisfies \( \|D_\Delta\|_2 < \|K_{1-}\| \). Then, there exists a Riesz basis \( \{S_n\}_{n \in \mathbb{Z}} \) for \( \sum_{i=1}^n V_{\delta} \) such that
\[
\sum_{j \in \mathbb{Z}} f_j \in \sum_{i=1}^n V_{\delta}
\]
can be expanded as
\[
\sum_{j \in \mathbb{Z}} f_j (t) = \sum_{n \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} f_j \left( n - \epsilon + \delta_n \right) S_n(t), \quad t \in \mathbb{R}.
\]
The convergence of the series is absolute and uniform in subsets of \( \mathbb{R} \) where \( \|K_{1-}\| \) is bounded. Also, it converges in the \( L^2(\mathbb{R}) \)-norm sense.

**Proof:** Applying Theorem 4 to
\[
K_{1-\delta_\epsilon}(x) = \sum_{k \in \mathbb{Z}} \phi(k - \epsilon) e^{-2\pi ik x}
\]
and
\[
K_{1-\delta_\epsilon}(x) = \sum_{k \in \mathbb{Z}} \phi(k - \epsilon + \delta_n) e^{-2\pi ik x}, \quad n \in \mathbb{Z}
\]
we obtain that \( \{K_{1-\delta_\epsilon} e^{2\pi i n x}\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( L^2(0,1) \). Denote by \( \{G_n\}_{n \in \mathbb{Z}} \) its dual Riesz basis. Now, given \( \sum_{j \in \mathbb{Z}} f_j \in \sum_{i=1}^n V_{\delta} \), we expand
\[
F = \sum_{j \in \mathbb{Z}} \phi' \left( j - \epsilon - k + \delta_j \right) - \phi' \left( j - \epsilon - k \right)
\]
to \( \{G_n\}_{n \in \mathbb{Z}} \). Therefore,
\[
F = \sum_{n \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} f_j \left( n - \epsilon + \delta_n \right) G_n \in L^2(0,1).
\]
Applying the operator \( T \), we get
\[
\sum_{j \in \mathbb{Z}} f_j = \sum_{n \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} f_j \left( n - \epsilon + \delta_n \right) T(G_n) \in L^2(\mathbb{R}).
\]
Furthermore, since \( T \) is an invertible bounded operator, the sequence \( \{S_n = T(G_n)\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( \sum_{i=1}^n V_{\delta} \).

**Theorem 6:** For any sequence \( \Delta = \{\delta_n\}_{n \in \mathbb{Z}} \in [\alpha, \beta] \) then (3) holds:
\[
\|D_\Delta\|_2 \leq \sup_{[\alpha, \beta]} \sum_{n \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \left| \phi(n - \epsilon + d_n) - \phi(n - \epsilon) \right|
\]

**Proof:** Assume that the second term in the above inequality is finite. Otherwise, the inequality trivially holds. For every \( c = \{c_k\}_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}) \) we have
\[
\|D_\Delta c\|_{l^2(\mathbb{Z})}^2 = \sum_{n \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} c_k d_n \right|^2 
\]

\[
\leq \sum_{n \in \mathbb{Z}} \sup_{[\alpha, \beta]} \sum_{j \in \mathbb{Z}} \left| \phi(n - \epsilon + \delta_n) - \phi(n - \epsilon) \right| \leq \|D_\Delta\|_2 \|c\|_{l^2(\mathbb{Z})}^2.
\]
\[
\sum_{n \in \mathbb{Z}} \left| \left( F_n(x) e^{2\pi inx} - F(x) e^{2\pi inx}, f \right) \right|^2
\]
\[
= \sum_{n \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} (a_k(n) - a_k) e^{2\pi i(n-k)x} + \sum_{j \in \mathbb{Z}} c_j e^{2\pi jx} \right)^2
\]
\[
= \sum_{n \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} (a_{n-k}(n) - a_{n-k}) c_k \right)^2
\]
\[
= \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} d_{n,k} c_k \leq \left\| \mathcal{D}_{\frac{2}{2}} f \right\|_2 \tilde{\Phi} \sum_{n \in \mathbb{Z}} |a_n|^2
\]

such that \( \|\Phi\|_0 \) denotes the essential infimum of the function. Putting \( \epsilon_0 + 1 = \| f \|_0^2 \), we get \( \{ F_n(x) e^{2\pi inx} \}_{n \in \mathbb{Z}} \) is a Riesz basis for \( L^2(0,1) \).

**References**


