On New Subclass of Multivalent Functions Involving a Generalized Al-Oboudi Differential Operator

Bitrus Sambo, Oladejo Semiu Oladipupo and Ayuba Sanda

Department of Mathematics, Gombe State University, P.M.B.127, Gombe, Nigeria

Abstract: The main aim of this paper is to introduce and investigate a new subclass of multivalent functions in the open unit disc U which is defined by a generalized Al-Oboudi differential operator. Coefficient inequalities, extreme points, Hadamard product and integral mean inequalities for fractional derivative of functions in this class are given.

Key words: Multivalent functions, extreme points, Hadamard product, fractional derivative, normalize.

1. Introduction

Let A denote the class of functions, $f(z)$ is normalized by:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad (1.1)$$

which are analytic in the open unit disc, $U = \{z: |z| < 1\}$.

For $f(z) \in A$, Al-Oboudi [1] introduced and investigated the following operator:

$$\begin{aligned} D_{\delta}^0 f(z) &= f(z) \\ D_{\delta}^1 f(z) &= (1 - \delta)f(z) + \delta z f'(z) \\ D_{\delta}^n f(z) &= D_{\delta}(D_{\delta}^{n-1} f(z)) \quad (n \in \mathbb{N}) \end{aligned} \quad (1.2)$$

If f is given by Eq. (1.1), then from Eq. (1.2).

$$D_{\delta}^n f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\delta]^n a_j z^j \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}) \quad (1.3)$$

Note:

when $\delta = 1$, we get Salagean Operator [7].

Let A_p denote the class of functions of the form:

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j \quad (p = 1, 2, 3, \dots) \quad (1.4)$$

which are analytic and p -valent in the open unit disc, $U = \{z: |z| < 1\}$.

We can write the following equalities for the functions $f \in A_p$:

$$\begin{aligned} D_{\delta,p}^0 f(z) &= f(z) \\ D_{\delta,p}^1 f(z) &= \frac{\delta}{p} z f'(z) + \frac{1}{p} (1 - \delta) f(z) + \frac{(p-1)}{p} (1 - \delta) z^p \\ D_{\delta,p}^n f(z) &= D_{\delta,p}(D_{\delta,p}^{n-1} f(z)) \quad (n \in \mathbb{N}) \end{aligned} \quad (1.5)$$

If f is given by Eq. (1.4), then from Eq. (1.5), we see that:

$$D_{\delta,p}^n f(z) = z^p + \sum_{j=p+1}^{\infty} \left[\frac{1 + (j-1)\delta}{p} \right]^n a_j z^j \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, p \in \mathbb{N}) \quad (1.6)$$

Remarks:

(1) If $p = 1$, $D_{\delta,p}^n f(z) = D_{\delta}^n f(z)$ is introduced by Al-Oboudi [1].

(2) If $p = 1$, $\delta = 1$, $D_{\delta,p}^n f(z) = D^n f(z)$ is defined by Salagean [7].

Definition 1.1 Let $C_{\delta,p}^{m,n}(\lambda)$ denote the subclass of A_p consisting of functions f satisfying the inequalities:

$$\Re \left(\frac{D_{\delta,p}^m f(z)}{D_{\delta,p}^n f(z)} \right) > \lambda \quad (1.7)$$

For some $0 \leq \lambda < 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $D_{\delta,p}^n f(z)$ as defined in Eq. (1.6) and for all $z \in U$.

Remarks:

If $p = 1$, the class $C_{\delta,p}^{m,n}(\lambda)$ reduces to the class

$S_{m,n,\delta}(\alpha)$ investigated by Eker and Guney [3].

2. Results and Discussions

Theorem 2.1 Let $f(z) \in A_p$ satisfy:

$$\sum_{j=p+1}^{\infty} \Psi_p(m, n, j, \delta, \lambda) |a_j| \leq 2(1 - \lambda) \quad (2.1)$$

Where:

$$\begin{aligned} \Psi_p(m, n, j, \delta, \lambda) = & \left| \left[\frac{1 + (j-1)\delta}{p} \right]^m - (1 \right. \\ & + \lambda) \left[\frac{1 + (j-1)\delta}{p} \right]^n \left| \right. \\ & + \left[\frac{1 + (j-1)\delta}{p} \right]^m + (1 \\ & - \lambda) \left[\frac{1 + (j-1)\delta}{p} \right]^n \end{aligned} \quad (2.2)$$

For some $0 \leq \lambda < 1, m \in \mathbb{N}, n \in \mathbb{N}_0, \delta \geq 0$, then $f \in C_{\delta,p}^{m,n}(\lambda)$.

Proof: Suppose that Eq. (2.1) holds true for some $0 \leq \lambda < 1, m \in \mathbb{N}, n \in \mathbb{N}_0, \delta \geq 0$. For $f(z) \in A_p$,

Let us define:

$$\frac{D_{\delta,p}^m f(z)}{D_{\delta,p}^n f(z)} - \lambda = F(z)$$

By the definition of the class,

$$\Rightarrow F(z) < \frac{1+z}{1-z}$$

There exists a Schwarz function $w(z)$, with $w(0) = 0$ and $|w(z)| < 1$, such that:

$$F(z) = \frac{1+w(z)}{1-w(z)} \quad (2.3)$$

This implies that:

$$w(z) = \frac{F(z) - 1}{F(z) + 1}$$

$$|w(z)| = \left| \frac{F(z) - 1}{F(z) + 1} \right| < 1$$

We note that:

$$\begin{aligned} \left| \frac{F(z) - 1}{F(z) + 1} \right| &= \left| \frac{\frac{D_{\delta,p}^m f(z)}{D_{\delta,p}^n f(z)} - \lambda - 1}{\frac{D_{\delta,p}^m f(z)}{D_{\delta,p}^n f(z)} - \lambda + 1} \right| = \left| \frac{D_{\delta,p}^m f(z) - (1 + \lambda) D_{\delta,p}^n f(z)}{D_{\delta,p}^m f(z) + (1 - \lambda) D_{\delta,p}^n f(z)} \right| \\ &= \left| \frac{z^p + \sum_{j=p+1}^{\infty} \left[\frac{1+(j-1)\delta}{p} \right]^m a_j z^j - (1 + \lambda) \left\{ z^p + \sum_{j=p+1}^{\infty} \left[\frac{1+(j-1)\delta}{p} \right]^n a_j z^j \right\}}{z^p + \sum_{j=p+1}^{\infty} \left[\frac{1+(j-1)\delta}{p} \right]^m a_j z^j + (1 - \lambda) \left\{ z^p + \sum_{j=p+1}^{\infty} \left[\frac{1+(j-1)\delta}{p} \right]^n a_j z^j \right\}} \right| \\ &= \left| \frac{-\lambda z^p + \sum_{j=p+1}^{\infty} \left\{ \left[\frac{1+(j-1)\delta}{p} \right]^m - (1 + \lambda) \left[\frac{1+(j-1)\delta}{p} \right]^n \right\} a_j z^j}{(2 - \lambda) z^p + \sum_{j=p+1}^{\infty} \left\{ \left[\frac{1+(j-1)\delta}{p} \right]^m + (1 - \lambda) \left[\frac{1+(j-1)\delta}{p} \right]^n \right\} a_j z^j} \right| \\ &= \left| \frac{\lambda - \sum_{j=p+1}^{\infty} \left\{ \left[\frac{1+(j-1)\delta}{p} \right]^m - (1 + \lambda) \left[\frac{1+(j-1)\delta}{p} \right]^n \right\} a_j z^{j-p}}{(2 - \lambda) + \sum_{j=p+1}^{\infty} \left\{ \left[\frac{1+(j-1)\delta}{p} \right]^m + (1 - \lambda) \left[\frac{1+(j-1)\delta}{p} \right]^n \right\} a_j z^{j-p}} \right| \\ &\leq \frac{\lambda + \sum_{j=p+1}^{\infty} \left\{ \left| \left[\frac{1+(j-1)\delta}{p} \right]^m - (1 + \lambda) \left[\frac{1+(j-1)\delta}{p} \right]^n \right| \right\} |a_j| |z|^{j-p}}{(2 - \lambda) - \sum_{j=p+1}^{\infty} \left\{ \left[\frac{1+(j-1)\delta}{p} \right]^m + (1 - \lambda) \left[\frac{1+(j-1)\delta}{p} \right]^n \right\} |a_j| |z|^{j-p}} \\ &< \frac{\lambda + \sum_{j=p+1}^{\infty} \left\{ \left| \left[\frac{1+(j-1)\delta}{p} \right]^m - (1 + \lambda) \left[\frac{1+(j-1)\delta}{p} \right]^n \right| \right\} |a_j|}{(2 - \lambda) - \sum_{j=p+1}^{\infty} \left\{ \left[\frac{1+(j-1)\delta}{p} \right]^m + (1 - \lambda) \left[\frac{1+(j-1)\delta}{p} \right]^n \right\} |a_j|} \end{aligned} \quad (2.4)$$

The last expression is bounded above by 1 if:

$$\begin{aligned}
& \sum_{j=p+1}^{\infty} \left\{ \left| \left[\frac{1+(j-1)\delta}{p} \right]^m - (1+\lambda) \left[\frac{1+(j-1)\delta}{p} \right]^n \right| \right\} |a_j| \\
& \quad + \sum_{j=p+1}^{\infty} \left\{ \left[\frac{1+(j-1)\delta}{p} \right]^m + (1-\lambda) \left[\frac{1+(j-1)\delta}{p} \right]^n \right\} |a_j| \leq 2(1-\lambda) \\
\Rightarrow & \sum_{j=p+1}^{\infty} \left\{ \left| \left[\frac{1+(j-1)\delta}{p} \right]^m - (1+\lambda) \left[\frac{1+(j-1)\delta}{p} \right]^n \right| + \left[\frac{1+(j-1)\delta}{p} \right]^m + (1-\lambda) \left[\frac{1+(j-1)\delta}{p} \right]^n \right\} |a_j| \\
& \leq 2(1-\lambda) \tag{2.5}
\end{aligned}$$

which is equivalent to the condition in Eq. (2.1). This completes the proof of Theorem 2.1.

3. Extreme Points

In view of Theorem 2.1, we now introduce the subclass:

$$\bar{C}_{\delta,p}^{m,n}(\lambda) \subset C_{\delta,p}^{m,n}(\lambda) \tag{3.1}$$

which consists of functions:

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j \quad (a_j \geq 0)$$

where Taylor-Maclaurin coefficients satisfy inequality Eq. (2.1). Now, let us determine the extreme points of the class $\bar{C}_{\delta,p}^{m,n}(\lambda)$.

Theorem 3.1

$$f_p(z) = z^p, f_j(z) = z^p + \frac{2(1-\lambda)}{\Psi_p(m,n,j,\delta,\lambda)} z^j$$

($j \geq p+1$) where $\Psi_p(m,n,j,\delta,\lambda)$ is as given Eq.(2.2), then $f \in \bar{C}_{\delta,p}^{m,n}(\lambda)$ if and only if

It can be expressed in the form:

$$f(z) = \eta_p f_p(z) + \sum_{j=p+1}^{\infty} \eta_j f_j(z) \tag{3.2}$$

where $\eta_j \geq 0$ and $\eta_p = 1 - \sum_{j=p+1}^{\infty} \eta_j$.

Proof: Assume that:

$$f(z) = \eta_p f_p(z) + \sum_{j=p+1}^{\infty} \eta_j f_j(z)$$

Then,

$$f(z) = (1 - \sum_{j=p+1}^{\infty} \eta_j) z^p$$

$$+ \sum_{j=p+1}^{\infty} \eta_j (z^p + \frac{2(1-\lambda)}{\Psi_p(m,n,j,\delta,\lambda)} z^j) \tag{3.3}$$

$$f(z) = z^p + \sum_{j=p+1}^{\infty} \eta_j (\frac{2(1-\lambda)}{\Psi_p(m,n,j,\delta,\lambda)} z^j)$$

Thus,

$$\begin{aligned}
& \sum_{j=p+1}^{\infty} \Psi_p(m,n,j,\delta,\lambda) \eta_j \frac{2(1-\lambda)}{\Psi_p(m,n,j,\delta,\lambda)} \\
& = 2(1-\lambda) \sum_{j=p+1}^{\infty} \eta_j
\end{aligned}$$

$$= 2(1-\lambda)(1 - \eta_p) \leq 2(1-\lambda)$$

which shows that f satisfies the condition in Eq. (2.1) and therefore, $f \in \bar{C}_{\delta,p}^{m,n}(\lambda)$.

Conversely, suppose that $f \in \bar{C}_{\delta,p}^{m,n}(\lambda)$, since:

$$a_j \leq \frac{2(1-\lambda)}{\Psi_p(m,n,j,\delta,\lambda)}, j \geq p+1$$

We may set:

$$\eta_j = \frac{\Psi_p(m,n,j,\delta,\lambda)}{2(1-\lambda)} a_j$$

$$\eta_p = 1 - \sum_{j=p+1}^{\infty} \eta_j$$

Then we obtain from:

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j$$

$$\begin{aligned}
f(z) &= (\eta_p + \sum_{j=p+1}^{\infty} \eta_j) z^p \\
&\quad + \sum_{j=p+1}^{\infty} \eta_j \frac{2(1-\lambda)}{\Psi_p(m, n, j, \delta, \lambda)} z^j \\
\Rightarrow f(z) &= \eta_p z^p + \sum_{j=p+1}^{\infty} \eta_j \frac{2(1-\lambda)}{\Psi_p(m, n, j, \delta, \lambda)} z^j \\
&= \eta_p z^p + \sum_{j=p+1}^{\infty} \eta_j f_j(z)
\end{aligned}$$

This completes the proof of Theorem 3.1.

Corollary 3.2: The extreme points of the class $\bar{C}_{\delta, p}^{m, n}(\lambda)$ are given by:

$$\text{Let } f_p(z) = z^p, f_j(z) = z^p + \frac{2(1-\lambda)}{\Psi_p(m, n, j, \delta, \lambda)} z^j \quad (j \geq p+1)$$

where $\Psi_p(m, n, j, \delta, \lambda)$ is as given Eq. (2.2).

Theorem 3.3: Let the functions,

$$\begin{aligned}
f(z) &= z^p + \sum_{j=p+1}^{\infty} a_j z^j \quad (a_j \geq 0) \\
g(z) &= z^p + \sum_{j=p+1}^{\infty} b_j z^j \quad (b_j \geq 0)
\end{aligned}$$

be in the class $\bar{C}_{\delta, p}^{m, n}(\lambda)$, for $0 \leq t \leq 1$, then the function $h(z)$ defined by:

$$\begin{aligned}
h(z) &= (1-t)f(z) + tg(z) \\
&= z^p + \sum_{j=p+1}^{\infty} c_j z^j \quad (3.4) \\
(1-t)a_j + tb_j &\geq 0
\end{aligned}$$

is also in the class $\bar{C}_{\delta, p}^{m, n}(\lambda)$.

Proof: suppose that each of the functions f, g is in the class $\bar{C}_{\delta, p}^{m, n}(\lambda)$, then making use of Eq.(2.1)

We see that:

$$\begin{aligned}
&\sum_{j=p+1}^{\infty} \Psi_p(m, n, j, \delta, \lambda) c_j \\
&= (1-t) \sum_{j=p+1}^{\infty} \Psi_p(m, n, j, \delta, \lambda) a_j \\
&\quad + t \sum_{j=p+1}^{\infty} \Psi_p(m, n, j, \delta, \lambda) b_j \\
&\leq (1-t)2(1-\lambda) + t2(1-\lambda) \\
&= 2(1-\lambda) \quad (3.5)
\end{aligned}$$

Which completes the proof of Theorem 3.3.

4. Integral Mean Inequalities for Fractional Derivatives

We will make use of the definitions of fractional derivatives by Owa [6], Srivastava and Owa [8], see also Eker and Owa [4].

Definition 4.1: The fractional derivative of order l is defined, for functions f by:

$$D_z^l = \frac{1}{\Gamma(1-l)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^l} dt, \quad (0 \leq l < 1) \quad (4.1)$$

where f is an analytic function in a simply connected region of the z -plane containing the origin, and the multiplicity of $(z-t)^{-l}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Definition 4.2: Under the hypothesis of definition 4.1, the fractional derivative of order $q+l$ is defined, for function f , by:

$$D_z^{q+l} f(z) = \frac{d^q}{dz^q} D_z^l f(z) \quad (0 \leq l < 1, q \in \mathbb{N}_0) \quad (4.2)$$

It readily follows from 4.2 that

$$D_z^l z^k = \frac{\Gamma(k+1)}{\Gamma(k-l+1)} z^{k-l} \quad (0 \leq l < 1, k \in \mathbb{N}) \quad (4.3)$$

Further, we need the concept of subordination between analytic functions and a subordination theorem of Littlewoods in our investigation.

Definition 4.3: Two functions f and g analytic in U , and write:

$$f < g, (z \in U)$$

If there exist a Schwarz function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1$, such that:

$$f(z) = g(w(z)), (z \in U)$$

In particular, if the function $g(z)$ is univalent in U , the above subordination is equivalent to:

$$f(0) = g(0), f(U) = g(U) [2]$$

In 1925, Littlewoods [5], proved the following

$$\sum_{j=p+1}^{\infty} (j-q)_{q+1} a_j = \frac{2(1-\lambda)\Gamma(k+1)\Gamma(2+p-l-q)}{\Psi_p(m, n, j, \delta, \lambda)\Gamma(k+1-l-q)\Gamma(p+1-q)} \quad (4.5)$$

For some $j \geq q, 0 \leq l < 1, (j-q)_{q+1}$ denotes the Pochhammer symbol defined by:

$$(j-q)_{q+1} = (j-q)(j-q+1) \dots j.$$

Also let the function:

$$f_k(z) = z^p + \frac{2(1-\lambda)}{\Psi_p(m, n, j, \delta, \lambda)} z^k \quad (k \geq p+1) \quad (4.6)$$

If there exist an analytic function $w(z)$ given by:

$$(w(z))^{k-p} = \frac{\Psi_p(m, n, j, \delta, \lambda)}{2(1-\lambda)} \frac{\Gamma(k+1-l-q)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} (j-q)_{q+1} \Phi(j) a_j z^{j-p} \quad (4.7)$$

$$k \geq q$$

$$\Phi(j) = \frac{\Gamma(j-q)}{\Gamma(k+1-l-q)} \quad (0 \leq l < 1, j \geq p+1) \quad (4.8)$$

then for $\sigma > 0$ and $z = re^{i\theta} (0 < r < 1)$:

$$\int_0^{2\pi} |D_z^{q+l} f(z)|^\sigma d\theta \leq \int_0^{2\pi} |D_z^{q+l} f_k(z)|^\sigma d\theta \quad (4.9)$$

Proof: Let

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j$$

By means of Eq. (4.3) and definition 4.1, we have:

$$\begin{aligned} D_z^{q+l} f(z) &= \frac{\Gamma(p+1)z^{p-l-q}}{\Gamma(p+1-l-q)} \left[1 + \sum_{j=p+1}^{\infty} \frac{\Gamma(j+1)\Gamma(p+1-l-q)}{\Gamma(p+1)\Gamma(j+1-l-q)} a_j z^{j-p} \right] \\ &= \frac{\Gamma(p+1)z^{p-l-q}}{\Gamma(p+1-l-q)} \left[1 + \sum_{j=p+1}^{\infty} \frac{\Gamma(p+1-l-q)}{\Gamma(p+1)} (j-q)_{q+1} \Phi(j) a_j z^{j-p} \right] \end{aligned} \quad (4.10)$$

where:

$$\Phi(j) = \frac{\Gamma(j-q)}{\Gamma(j+1-l-q)}, 0 \leq l < 1, j \geq p+1$$

Since Φ is decreasing function of j , we get:

subordination theorem.

Lemma 4.4: If $f(z)$ and $g(z)$ are analytic in U with $f(z) < g(z)$, then for $\sigma > 0$ and $z = re^{i\theta} (0 < r < 1)$:

$$\int_0^{2\pi} |f(z)|^\sigma d\theta \leq \int_0^{2\pi} |g(z)|^\sigma d\theta \quad (4.4)$$

Theorem 4.5: Let $f \in \bar{C}_{\delta,p}^{m,n}(\lambda)$ and suppose that

$$0 < \Phi(j) \leq \Phi(p+1) = \frac{\Gamma(p+1-q)}{\Gamma(2+p-l-q)}$$

Similarly, from Eqs. (2.1) and (4.3) and definition 4.1:

$$D_Z^{q+l} f_k(z) = \frac{\Gamma(p+1)z^{k-l-q}}{\Gamma(p+1-l-q)} \left[1 + \frac{2(1-\lambda)\Gamma(k+1)\Gamma(p+1-l-q)}{\Psi_p(m, n, j, \delta, \lambda)\Gamma(p+1)\Gamma(k+1-l-q)} z^{k-p} \right] \quad (4.11)$$

for $\sigma > 0$ and $z = re^{i\theta}$ ($0 < r < 1$), we show that:

$$\begin{aligned} \int_0^{2\pi} \left| 1 + \sum_{j=p+1}^{\infty} \frac{\Gamma(j+1)\Gamma(p+1-l-q)}{\Gamma(p+1)\Gamma(j+1-l-q)} a_j z^{j-p} \right|^\sigma d\theta \\ \leq \int_0^{2\pi} \left| 1 + \frac{2(1-\lambda)\Gamma(k+1)\Gamma(p+1-l-q)}{\Psi_p(m, n, j, \delta, \lambda)\Gamma(p+1)\Gamma(k+1-l-q)} z^{k-p} \right|^\sigma d\theta \end{aligned} \quad (4.12)$$

So, by applying lemma 4.4, it is enough to show that:

$$\begin{aligned} 1 + \sum_{j=p+1}^{\infty} \frac{\Gamma(j+1)\Gamma(p+1-l-q)}{\Gamma(p+1)\Gamma(j+1-l-q)} a_j z^{j-p} \\ < 1 + \frac{2(1-\lambda)\Gamma(k+1)\Gamma(p+1-l-q)}{\Psi_p(m, n, j, \delta, \lambda)\Gamma(p+1)\Gamma(k+1-l-q)} z^{k-p} S \end{aligned}$$

$$j \geq p+1$$

If the above subordination holds true, then we have analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$, such that:

$$\begin{aligned} 1 + \sum_{j=p+1}^{\infty} \frac{\Gamma(j+1)\Gamma(p+1-l-q)}{\Gamma(p+1)\Gamma(j+1-l-q)} a_j z^{j-p} \\ = 1 + \frac{2(1-\lambda)\Gamma(k+1)\Gamma(p+1-l-q)}{\Psi_p(m, n, j, \delta, \lambda)\Gamma(p+1)\Gamma(k+1-l-q)} w(z)^{k-p} \end{aligned}$$

By the condition of the Theorem, we define the function $w(z)$ by:

$$w(z)^{k-p} = \frac{\Psi_p(m, n, j, \delta, \lambda)\Gamma(k+1-l-q)}{2(1-\lambda)\Gamma(k+1)} \sum_{j=p+1}^{\infty} (j-q)_{q+1} \Phi(j) a_j z^{j-p} \quad (4.13)$$

which readily yields $w(0) = 0$. For such a function $w(z)$, we have:

$$\begin{aligned} |w(z)|^{k-p} &\leq \frac{\Psi_p(m, n, j, \delta, \lambda)\Gamma(k+1-l-q)}{2(1-\lambda)\Gamma(k+1)} \sum_{j=p+1}^{\infty} (j-q)_{q+1} \Phi(j) a_j |z|^{j-p} \\ &\leq |z| \frac{\Psi_p(m, n, j, \delta, \lambda)\Gamma(k+1-l-q)}{2(1-\lambda)\Gamma(k+1)} \Phi(p+1) \sum_{j=p+1}^{\infty} (j-q)_{q+1} a_j \leq |z| < 1 \end{aligned}$$

where $\Psi_p(m, n, j, \delta, \lambda)$ is as in Eq. (2.2). By means of the hypothesis of the Theorem, thus, the theorem is proved.

As a special case $q = 0$, we have the following from Theorem 4.5.

Corollary 4.6: Let $f \in \bar{C}_{\delta, p}^{m, n}(\lambda)$ and suppose that

$$\begin{aligned} \sum_{j=p+1}^{\infty} j a_j \\ = \frac{2(1-\lambda)\Gamma(k+1)\Gamma(2+p-l)}{\Psi_p(m, n, j, \delta, \lambda)\Gamma(k+1-l)\Gamma(p+1)} \quad (j \geq p+1) \end{aligned} \quad (4.14)$$

if there exists analytic function $w(z)$ define by:

$$(w(z))^{k-p} = \frac{\Psi_p(m, n, j, \delta, \lambda) \Gamma(k+1-l)}{2(1-\lambda) \Gamma(k+1)} \sum_{j=p+1}^{\infty} j \Phi(j) a_j z^{j-p} \quad (4.15)$$

With,

$$\Phi(j) = \frac{\Gamma(j)}{\Gamma(j+1-l)} \quad (0 \leq l < 1, j \geq p+1)$$

Then $\sigma > 0$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |D_Z^l f(z)|^\sigma d\theta \leq \int_0^{2\pi} |D_Z^l f_k(z)|^\sigma d\theta \quad (4.16)$$

Letting $q = 1$, we have the following from Theorem 4.5:

Corollary 4.7: Let $f \in \bar{C}_{\delta,p}^{m,n}(\lambda)$ and suppose that:

$$\sum_{j=p+1}^{\infty} j(j-1)a_j = \frac{2(1-\lambda)\Gamma(k+1)\Gamma(p+1-l)}{\Psi_p(m, n, j, \delta, \lambda)\Gamma(k-l)\Gamma(p)} \quad (j \geq p+1) \quad (4.17)$$

If there exists analytic function $w(z)$ define by:

$$(w(z))^{k-p} = \frac{\Psi_p(m, n, j, \delta, \lambda) \Gamma(k-l)}{2(1-\lambda) \Gamma(k+1)} \sum_{j=p+1}^{\infty} j(j-1)\Phi(j) a_j z^{j-p} \quad (4.18)$$

With,

$$\Phi(j) = \frac{\Gamma(j-1)}{\Gamma(j-l)}, \quad (0 \leq l < 1, j \geq p+1) \quad (4.19)$$

Then $\sigma > 0$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |D_Z^{1+l} f(z)|^\sigma d\theta \leq \int_0^{2\pi} |D_Z^{1+l} f_k(z)|^\sigma d\theta \quad (4.20)$$

5. Conclusion

In this paper, using a generalized Al-Oboudi differential operator, we defined a new subclass of P-valent functions and established its properties. Results obtained provide new properties of certain subclasses of Multivalent functions.

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