New Class of Almost Unbiased Modified Ratio Cum Product Estimators with Known parameters of Auxiliary Variables

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Abstract

This manuscript deals with new class of almost unbiased ratio cum product estimators for the estimation of population mean of the study variable by using the known values auxiliary variable. The bias and mean squared error of proposed estimators are obtained. An empirical study is carried out to assess the efficiency of proposed estimators over the existing estimators with the help of some known natural populations and it shows that the proposed estimators are almost unbiased and it perform better than the existing estimators.

Keywords: Auxiliary variable, Bias, Mean squared error, Natural populations, Ratio and Product estimators, Simple random sampling.

1. Introduction

The efficiency of the estimators of the population parameters can be increased by using the prior information of the study characteristics. In literature several estimators exist with auxiliary variables involved. The commonly used the population parameters of the auxiliary variables are mean, median, coefficient of variation, coefficient of skewness, coefficient of kurtosis etc. Ratio method of estimation is extensively used because of its computational simplicity and applicability. The correlation between study and auxiliary variables are negative the product method of estimation is used. Several researchers have directed their efforts towards to get efficient estimators of population mean. These estimators are biased but the percentage relative efficiency is better than that of simple random sampling, ratio and product estimators. For this reason, we consider the problem of estimation of population mean of study variable using known values of the auxiliary variable. So we have suggested new class modified ratio cum product estimators for estimating the population mean of the study variable. To know more about historical developments of the estimation of population mean are referred to Cochran[1,2], Subramani and Master Ajith [4,5], Murthy[7,8], Subramani[14], Subramani and kumarapandiyan[15,16,17], Upadhyaya and Singh[18], Yan and Tian[19] and the references cited there in.

Consider a finite population U of size N consisting of $U_1, U_2, U_3 \dots U_N$ units. Each $U_i = (X_i, Y_i)$, ($i = 1, 2, 3, \dots, N$) has a pair of values. Here Y be the study variable and X be the auxiliary variable which is correlated with Y . If $y = \{y_1, y_2, y_3, \dots, y_n\}$, and $x = \{x_1, x_2, x_3, \dots, x_n\}$ be n sample values. Let $\bar{y}$ and $\bar{x}$ be the sample means of the study and auxiliary variables, $S_y^2 = \frac{1}{N-1} \sum_{i=1}^N (Y_i - \bar{Y})^2$, $S_x^2 = \frac{1}{N-1} \sum_{i=1}^N (X_i - \bar{X})^2$ and $S_{xy} = \frac{1}{N-1} \sum_{i=1}^N (Y_i - \bar{Y})(X_i - \bar{X})$ be the population variance and co-variance of the study variable and auxiliary variable. Similarly the coefficient of variations and

coefficient of co-variance of these variables are defined as $C_y = \frac{S_y}{\bar{Y}}$, $C_x = \frac{S_x}{\bar{X}}$ and $C_{xy} = \frac{S_{xy}}{\bar{X}\bar{Y}} = \rho C_x C_y$ where ρ is the correlation coefficient

The simple random sample mean without replacement is used only when there is no additional information of the study variable is available,. In simple random sampling without replacement the estimator of population mean $\bar{y}_{sr,s}$ is an unbiased estimator for the population mean. The auxiliary variable and study variable are positively correlated, Cochran [1] introduced ratio estimator for the estimation of population mean and it is given by

$$\hat{Y}_R = \frac{\bar{y}}{\bar{x}} \bar{X} = \hat{R} \bar{X}$$

the bias and mean squared error of ratio estimator up to first order approximations are

$$B(\hat{Y}_R) = \delta \bar{Y} [C_x^2 - \rho C_x C_y]$$

$$MSE(\hat{Y}_R) = \delta \bar{Y}^2 [C_y^2 + C_x^2 - 2\rho C_x C_y]$$

$$\text{Where } \delta = \left(\frac{1-f}{n}\right), f = \frac{n}{N}$$

Some existing modified ratio estimators with their biases and mean squared errors are in table 1

Table 1: Bias and MSE of Existing modified ratio estimators

Existing Estimators	Constants	Bias	Variance/Mean squared Error
$\hat{Y}_{MR1} = \bar{y} \left(\frac{C_x \bar{X} + \beta_1}{C_x \bar{x} + \beta_1} \right)$ Yan and Tian [19]	$\theta_1 = \frac{C_x \bar{X}}{C_x \bar{X} + \beta_1}$	$B(\hat{Y}_{MR1}) = \delta \bar{Y} [\theta_1^2 C_x^2 - \theta_1 \rho C_x C_y]$	$MSE(\hat{Y}_{MR1}) = \delta \bar{Y}^2 [C_y^2 + \theta_1^2 C_x^2 - 2\theta_1 \rho C_x C_y]$
$\hat{Y}_{MR2} = \bar{y} \left(\frac{\beta_1 \bar{X} + C_x}{\beta_1 \bar{x} + C_x} \right)$	$\theta_2 = \frac{\beta_1 \bar{X}}{\beta_1 \bar{X} + C_x}$	$B(\hat{Y}_{MR2}) = \delta \bar{Y} [\theta_2^2 C_x^2 - \theta_2 \rho C_x C_y]$	$MSE(\hat{Y}_{MR2}) = \delta \bar{Y}^2 [C_y^2 + \theta_2^2 C_x^2 - 2\theta_2 \rho C_x C_y]$
$\hat{Y}_{MR3} = \bar{y} \left(\frac{C_x \bar{X} + \rho}{C_x \bar{x} + \rho} \right)$	$\theta_3 = \frac{C_x \bar{X}}{C_x \bar{X} + \rho}$	$B(\hat{Y}_{MR3}) = \delta \bar{Y} [\theta_3^2 C_x^2 - \theta_3 \rho C_x C_y]$	$MSE(\hat{Y}_{MR3}) = \delta \bar{Y}^2 [C_y^2 + \theta_3^2 C_x^2 - 2\theta_3 \rho C_x C_y]$
$\hat{Y}_{MR4} = \bar{y} \left(\frac{\rho \bar{X} + C_x}{\rho \bar{x} + C_x} \right)$	$\theta_4 = \frac{\rho \bar{X}}{\rho \bar{X} + C_x}$	$B(\hat{Y}_{MR4}) = \delta \bar{Y} [\theta_4^2 C_x^2 - \theta_4 \rho C_x C_y]$	$MSE(\hat{Y}_{MR4}) = \delta \bar{Y}^2 [C_y^2 + \theta_4^2 C_x^2 - 2\theta_4 \rho C_x C_y]$
$\hat{Y}_{MR5} = \bar{y} \left(\frac{C_x \bar{X} + \beta_2}{C_x \bar{x} + \beta_2} \right)$ Upadhyaya and Singh [18]	$\theta_5 = \frac{C_x \bar{X}}{C_x \bar{X} + \beta_2}$	$B(\hat{Y}_{MR5}) = \delta \bar{Y} [\theta_5^2 C_x^2 - \theta_5 \rho C_x C_y]$	$MSE(\hat{Y}_{MR5}) = \delta \bar{Y}^2 [C_y^2 + \theta_5^2 C_x^2 - 2\theta_5 \rho C_x C_y]$
$\hat{Y}_{MR6} = \bar{y} \left(\frac{\beta_2 \bar{X} + C_x}{\beta_2 \bar{x} + C_x} \right)$ Upadhyaya and Singh [18]	$\theta_6 = \frac{\beta_2 \bar{X}}{\beta_2 \bar{X} + C_x}$	$B(\hat{Y}_{MR6}) = \delta \bar{Y} [\theta_6^2 C_x^2 - \theta_6 \rho C_x C_y]$	$MSE(\hat{Y}_{MR6}) = \delta \bar{Y}^2 [C_y^2 + \theta_6^2 C_x^2 - 2\theta_6 \rho C_x C_y]$

The auxiliary variable and study variable are negatively correlated, product estimator (Murthy[7]) is used, the bias and mean squared error of product estimator up to first order approximations are

$$B(\hat{Y}_R) = \delta \bar{Y} [\rho C_x C_y]$$

$$MSE(\hat{Y}_R) = \delta \bar{Y}^2 [C_y^2 + C_x^2 + 2\rho C_x C_y]$$

Existing Estimators	Constants	Bias	Variance/Mean squared Error
$\hat{Y}_{MP1} = \bar{y} \left(\frac{C_x \bar{x} + \beta_1}{C_x \bar{X} + \beta_1} \right)$	$\theta_1 = \frac{C_x \bar{X}}{C_x \bar{X} + \beta_1}$	$B(\hat{Y}_{MP1}) = \delta \bar{Y} [\theta_1 \rho C_x C_y]$	$MSE(\hat{Y}_{MP1}) = \delta \bar{Y}^2 [C_y^2 + \theta_1^2 C_x^2 + 2\theta_1 \rho C_x C_y]$
$\hat{Y}_{MP2} = \bar{y} \left(\frac{\beta_1 \bar{x} + C_x}{\beta_1 \bar{X} + C_x} \right)$	$\theta_2 = \frac{\beta_1 \bar{X}}{\beta_1 \bar{X} + C_x}$	$B(\hat{Y}_{MP2}) = \delta \bar{Y} [\theta_2 \rho C_x C_y]$	$MSE(\hat{Y}_{MP2}) = \delta \bar{Y}^2 [C_y^2 + \theta_2^2 C_x^2 + 2\theta_2 \rho C_x C_y]$
$\hat{Y}_{MP3} = \bar{y} \left(\frac{C_x \bar{x} + \rho}{C_x \bar{X} + \rho} \right)$	$\theta_3 = \frac{C_x \bar{X}}{C_x \bar{X} + \rho}$	$B(\hat{Y}_{MP3}) = \delta \bar{Y} [\theta_3 \rho C_x C_y]$	$MSE(\hat{Y}_{MP3}) = \delta \bar{Y}^2 [C_y^2 + \theta_3^2 C_x^2 + 2\theta_3 \rho C_x C_y]$
$\hat{Y}_{MP4} = \bar{y} \left(\frac{\rho \bar{x} + C_x}{\rho \bar{X} + C_x} \right)$	$\theta_4 = \frac{\rho \bar{X}}{\rho \bar{X} + C_x}$	$B(\hat{Y}_{MP4}) = \delta \bar{Y} [\theta_4 \rho C_x C_y]$	$MSE(\hat{Y}_{MP4}) = \delta \bar{Y}^2 [C_y^2 + \theta_4^2 C_x^2 + 2\theta_4 \rho C_x C_y]$
$\hat{Y}_{MP5} = \bar{y} \left(\frac{C_x \bar{x} + \beta_2}{C_x \bar{X} + \beta_2} \right)$	$\theta_5 = \frac{C_x \bar{X}}{C_x \bar{X} + \beta_2}$	$B(\hat{Y}_{MP5}) = \delta \bar{Y} [\theta_5 \rho C_x C_y]$	$MSE(\hat{Y}_{MP5}) = \delta \bar{Y}^2 [C_y^2 + \theta_5^2 C_x^2 + 2\theta_5 \rho C_x C_y]$
$\hat{Y}_{MP6} = \bar{y} \left(\frac{\beta_2 \bar{x} + C_x}{\beta_2 \bar{X} + C_x} \right)$	$\theta_6 = \frac{\beta_2 \bar{X}}{\beta_2 \bar{X} + C_x}$	$B(\hat{Y}_{MP6}) = \delta \bar{Y} [\theta_6 \rho C_x C_y]$	$MSE(\hat{Y}_{MP6}) = \delta \bar{Y}^2 [C_y^2 + \theta_6^2 C_x^2 + 2\theta_6 \rho C_x C_y]$

Some existing modified product estimators with their biases and mean squared errors are given in table 2

Table 2: Bias and MSE of Existing modified product estimators

2. Suggested Class of Estimators

In this section, new class of modified ratio cum product estimators for the population mean by using the known parameters of auxiliary variables is proposed and also derived the bias and the mean squared errors of the proposed estimators.

The proposed estimators are given by

$$\begin{aligned}
 \hat{Y}_{P1} &= \alpha_1 \lambda_1 \bar{y} \left(\frac{C_x \bar{x} + \beta_1}{C_x \bar{x} + \beta_1} \right) + (1 - \alpha_1) \gamma_1 \bar{y} \left(\frac{C_x \bar{x} + \beta_1}{C_x \bar{X} + \beta_1} \right) \\
 \hat{Y}_{P2} &= \alpha_2 \lambda_2 \bar{y} \left(\frac{\beta_1 \bar{x} + C_x}{\beta_1 \bar{x} + C_x} \right) + (1 - \alpha_2) \gamma_2 \bar{y} \left(\frac{\beta_1 \bar{x} + C_x}{\beta_1 \bar{X} + C_x} \right) \\
 \hat{Y}_{P3} &= \alpha_3 \lambda_3 \bar{y} \left(\frac{C_x \bar{x} + \rho}{C_x \bar{x} + \rho} \right) + (1 - \alpha_3) \gamma_3 \bar{y} \left(\frac{C_x \bar{x} + \rho}{C_x \bar{X} + \rho} \right) \\
 \hat{Y}_{P4} &= \alpha_4 \lambda_4 \bar{y} \left(\frac{\rho \bar{x} + C_x}{\rho \bar{x} + C_x} \right) + (1 - \alpha_4) \gamma_4 \bar{y} \left(\frac{\rho \bar{x} + C_x}{\rho \bar{X} + C_x} \right) \\
 \hat{Y}_{P5} &= \alpha_5 \lambda_5 \bar{y} \left(\frac{C_x \bar{x} + \beta_2}{C_x \bar{x} + \beta_2} \right) + (1 - \alpha_5) \gamma_5 \bar{y} \left(\frac{C_x \bar{x} + \beta_2}{C_x \bar{X} + \beta_2} \right) \\
 \hat{Y}_{P6} &= \alpha_6 \lambda_6 \bar{y} \left(\frac{\beta_2 \bar{x} + C_x}{\beta_2 \bar{x} + C_x} \right) + (1 - \alpha_6) \gamma_6 \bar{y} \left(\frac{\beta_2 \bar{x} + C_x}{\beta_2 \bar{X} + C_x} \right)
 \end{aligned}$$

Where $\lambda_i = \frac{S_y}{S_y + a_i C_y}$, $\gamma_i = \frac{S_y}{S_y + b_i C_y}$, $i = 1, 2, 3, 4, 5, 6$. Here a_i 's and b_i 's are constants. S_y and C_y are the population variance and coefficient of variation of study variable respectively. It is reasonable to assume that the values of S_y and C_y are known from the previous studies.

2.1 The Bias and Mean Squared error of the Proposed Estimator

To obtain the bias and mean squared error of the proposed estimator,
Consider

$$e_0 = \frac{\bar{y} - \bar{Y}}{\bar{Y}}, e_1 = \frac{\bar{x} - \bar{X}}{\bar{X}},$$

$$\theta_1 = \frac{C_x \bar{X}}{C_x \bar{X} + \beta_1}, \theta_2 = \frac{\beta_1 \bar{X}}{\beta_1 \bar{X} + C_x}, \theta_3 = \frac{C_x \bar{X}}{C_x \bar{X} + \rho}, \theta_4 = \frac{\rho \bar{X}}{\rho \bar{X} + C_x}, \theta_5 = \frac{C_x \bar{X}}{C_x \bar{X} + \beta_2}, \theta_6 = \frac{\beta_2 \bar{X}}{\beta_2 \bar{X} + C_x}$$

$$E(e_0) = E(e_1) = 0, E(e_0^2) = \left(\frac{1-f}{n}\right) \bar{Y}^2 C_y^2, E(e_1^2) = \left(\frac{1-f}{n}\right) \bar{X}^2 C_x^2, E(e_0 e_1) = \left(\frac{1-f}{n}\right) \rho C_x C_y$$

Substitute the values of e_0 and e_1 in the proposed class of estimators and neglecting the higher order expressions, we get

$$B(\hat{Y}_{Pi}) = E(\hat{Y}_{Pi} - \bar{Y})$$

$$B(\hat{Y}_{Pi}) = \frac{1}{2} [2\bar{Y}(P_i - 1) + P_i \{B(\hat{Y}_{MRI}) + B(\hat{Y}_{MPi})\} + Q_i \{B(\hat{Y}_{MRI}) - B(\hat{Y}_{MPi})\}]$$

Where $B(\hat{Y}_{MRI}) = \delta \bar{Y} [\theta_i^2 C_x^2 - \theta_i \rho C_x C_y]$, and $B(\hat{Y}_{MPi}) = \delta \bar{Y} [\theta_i \rho C_x C_y]$, $i=1,2,3,4$,

$$\delta = \left(\frac{1-f}{n}\right), \theta_1 = \frac{C_x \bar{X}}{C_x \bar{X} + \beta_1}, \theta_2 = \frac{\beta_1 \bar{X}}{\beta_1 \bar{X} + C_x}, \theta_3 = \frac{C_x \bar{X}}{C_x \bar{X} + \rho}, \theta_4 = \frac{\rho \bar{X}}{\rho \bar{X} + C_x}, \theta_5 = \frac{C_x \bar{X}}{C_x \bar{X} + \beta_2}, \theta_6 = \frac{\beta_2 \bar{X}}{\beta_2 \bar{X} + C_x}$$

$$\text{Where } P_i = (\alpha_i \lambda_i + (1 - \alpha_i) \gamma_i) \text{ and } Q_i = (\alpha_i \lambda_i - (1 - \alpha_i) \gamma_i)$$

Where λ_i and γ_i are as defined above. If we assume that $\alpha_i = 0$, $b_i = 0$ and $\alpha_i = 1$ then the proposed estimators are exactly equal to the estimators given in Table 1. If $\alpha_i = 0$, $b_i = 0$ and $\alpha_i = 0$ then the proposed estimators are exactly equal to the estimators given in Table 2. If we assume that $\alpha_i = B(\hat{Y}_{MRI})$, $b_i = B(\hat{Y}_{MPi})$, $\alpha_i = 1$ or $\alpha_i = 0$, then the proposed estimators are almost unbiased ratio estimators corresponding to the estimators given in Table 1 and 2.

The detailed derivation of the mean squared errors are given in the appendix and the final expression is obtained with only first order approximation in the Taylor series expansion as,

$$MSE(\hat{Y}_{Pi}) = E(\hat{Y}_{Pi} - \bar{Y})^2$$

$$MSE(\hat{Y}_{Pi}) = \frac{1}{4} [4\bar{Y}^2 (P_i - 1)^2 + (P_i + Q_i)^2 \{MSE(\hat{Y}_{MRI}) + 2\bar{Y} B(\hat{Y}_{MRI})\} \\ + (P_i - Q_i)^2 \{MSE(\hat{Y}_{MPi}) + 2\bar{Y} B(\hat{Y}_{MPi})\} \\ - 4\bar{Y} \{(P_i + Q_i) B(\hat{Y}_{MRI}) + (P_i - Q_i) B(\hat{Y}_{MPi})\} + 2(P_i^2 - Q_i^2) V(\bar{y}_{srs})]$$

$$\text{Where } B(\hat{Y}_{MRI}) = \delta \bar{Y} [\theta_i^2 C_x^2 - \theta_i \rho C_x C_y], \quad B(\hat{Y}_{MPi}) = \delta \bar{Y} [\theta_i \rho C_x C_y]$$

$$MSE(\hat{Y}_{MRI}) = \delta \bar{Y}^2 [C_y^2 + \theta_i^2 C_x^2 - 2\theta_i \rho C_x C_y]$$

$$MSE(\hat{Y}_{MPi}) = \delta \bar{Y}^2 [C_y^2 + \theta_i^2 C_x^2 + 2\theta_i \rho C_x C_y] \text{ and } V(\bar{y}_{srs}) = \delta S_y^2 = \delta \bar{Y}^2 C_y^2$$

The optimal value of α_i 's are determined by minimizing the $MSE(\hat{Y}_{pi})$ with respect to α_i . For this differentiate MSE with respect to α_i and equate to zero.

$$\frac{\partial MSE}{\partial \alpha_i} = 0, \text{ and we get the value of } \alpha_i, \text{ as}$$

$$\alpha_i = \frac{\bar{Y}^2(\gamma_i - 1)(\gamma_i - \lambda_i) + \gamma_i^2 \{MSE(\hat{\bar{Y}}_{MPi}) + 2\bar{Y}B(\hat{\bar{Y}}_{MPi})\} + \bar{Y} \{ \lambda_i B(\hat{\bar{Y}}_{MRi}) - \gamma_i B(\hat{\bar{Y}}_{MPi}) \} - \lambda_i \gamma_i V(\bar{y}_{srS})}{\bar{Y}^2(\lambda_i - \gamma_i)^2 + \lambda_i^2 \{MSE(\hat{\bar{Y}}_{MRi}) + 2\bar{Y}B(\hat{\bar{Y}}_{MRi})\} + \gamma_i^2 \{MSE(\hat{\bar{Y}}_{MPi}) + 2\bar{Y}B(\hat{\bar{Y}}_{MPi})\} - 2\lambda_i \gamma_i V(\bar{y}_{srS})}$$

Where $\lambda_i = \frac{S_y}{S_y + a_i C_y}$, $\gamma_i = \frac{S_y}{S_y + b_i C_y}$, $i = 1, 2, 3, 4, 5, 6$

3. Efficiency Comparison

If that $a_i = B(\hat{\bar{Y}}_{MRi})$, $b_i = B(\hat{\bar{Y}}_{MPi})$ then $\lambda_i = \frac{S_y}{S_y + B(\hat{\bar{Y}}_{MRi})C_y}$, $\gamma_i = \frac{S_y}{S_y + B(\hat{\bar{Y}}_{MPi})C_y}$, $i = 1, 2, 3, 4, 5, 6$.

Substitute these values in the biased estimator and taking expectation we get the proposed estimators are almost unbiased. The efficiency comparison of the mean squared errors of proposed estimators under optimum conditions with that of the existing estimators are as follows

3.1 Comparison of Proposed estimators and Modified Ratio Estimator

$$\begin{aligned} & MSE(\hat{\bar{Y}}_{MRi}) \geq MSE(\hat{\bar{Y}}_{Pi}) \\ & \delta \bar{Y}^2 [C_y^2 + \theta_i^2 C_x^2 - 2\theta_i \rho C_x C_y] \\ & \geq \bar{Y}^2 (\alpha_i \lambda_i + (1 - \alpha_i) \gamma_i - 1)^2 + \alpha_i^2 \lambda_i^2 \{MSE(\hat{\bar{Y}}_{MRi}) + 2\bar{Y}B(\hat{\bar{Y}}_{MRi})\} \\ & + (1 - \alpha_i)^2 \gamma_i^2 \{MSE(\hat{\bar{Y}}_{MPi}) + 2\bar{Y}B(\hat{\bar{Y}}_{MPi})\} \\ & - 2\bar{Y} \{ \alpha_i \lambda_i B(\hat{\bar{Y}}_{MRi}) + (1 - \alpha_i) \gamma_i B(\hat{\bar{Y}}_{MPi}) \} + 2(\alpha_i \lambda_i (1 - \alpha_i) \gamma_i V(\bar{y}_{srS})) \\ & \delta \bar{Y}^2 [C_y^2 + \theta_i^2 C_x^2 - 2\theta_i \rho C_x C_y] \\ & \geq \bar{Y}^2 (\alpha_i \lambda_i + (1 - \alpha_i) \gamma_i - 1)^2 \\ & + \delta \bar{Y}^2 (C_y^2 (\alpha_i \lambda_i + (1 - \alpha_i) \gamma_i)^2 + \theta^2 C_x^2 (3\alpha_i^2 \lambda_i^2 + (1 - \alpha_i)^2 \gamma_i^2 - 2\alpha_i \lambda_i) \\ & + 2\theta \rho C_x C_y (\alpha_i \lambda_i - (1 - \alpha_i) \gamma_i) - 2(\alpha_i^2 \lambda_i^2 - (1 - \alpha_i)^2 \gamma_i^2)) \\ & \delta \bar{Y}^2 [C_y^2 + \theta^2 C_x^2 - 2\theta \rho C_x C_y] \geq \bar{Y}^2 (P_i - 1)^2 \\ & + \delta \bar{Y}^2 \{ P_i^2 C_y^2 + \theta^2 C_x^2 (P_i^2 + (P_i + Q_i)(Q_i - 1) - 2\theta \rho C_x C_y Q_i (2P_i - 1)) \} \\ & 2\delta \theta \rho C_x C_y (Q_i (2P_i - 1) - 1) \\ & \geq (P_i - 1)^2 + \delta \{ C_y^2 (P_i^2 - 1) + \theta^2 C_x^2 (P_i^2 + (P_i + Q_i)(Q_i - 1) - 1) \} \\ & \rho \geq \frac{(P_i - 1)^2 + \delta [C_y^2 (P_i^2 - 1) + \theta^2 C_x^2 (P_i^2 + (P_i + Q_i)(P_i - 1) - 1)]}{2\delta \theta C_x C_y (Q_i (2P_i - 1) - 1)} \end{aligned}$$

3.2 Comparison of Proposed estimators and Modified Product Estimators

$$MSE(\hat{\bar{Y}}_{MPi}) \geq MSE(\hat{\bar{Y}}_{Pi})$$

$$\begin{aligned} & \delta \bar{Y}^2 \left[C_y^2 + \theta_i^2 C_x^2 + 2\theta_i \rho C_x C_y \right] \\ & \geq \bar{Y}^2 (\alpha_i \lambda_i + (1 - \alpha_i) \gamma_j - 1)^2 + \alpha_i^2 \lambda_i^2 \left\{ MSE(\hat{\bar{Y}}_{MRI}) + 2\bar{Y} B(\hat{\bar{Y}}_{MRI}) \right\} \\ & + (1 - \alpha_i)^2 \gamma_i^2 \left\{ MSE(\hat{\bar{Y}}_{MPi}) + 2\bar{Y} B(\hat{\bar{Y}}_{MPi}) \right\} \\ & - 2\bar{Y} \left\{ \alpha_i \lambda_i B(\hat{\bar{Y}}_{MRI}) + (1 - \alpha_i) \gamma_i B(\hat{\bar{Y}}_{MPi}) \right\} + 2(\alpha_i \lambda_i (1 - \alpha_i) \gamma_i V(\bar{y}_{srs})) \end{aligned}$$

$$\begin{aligned} & \delta \bar{Y}^2 \left[C_y^2 + \theta_i^2 C_x^2 + 2\theta_i \rho C_x C_y \right] \\ & \geq \bar{Y}^2 (\alpha_i \lambda_i + (1 - \alpha_i) \gamma_i - 1)^2 \\ & + \delta \bar{Y}^2 (C_y^2 (\alpha_i \lambda_i + (1 - \alpha_i) \gamma_i)^2 + \theta^2 C_x^2 (3\alpha_i^2 \lambda_i^2 + (1 - \alpha_i)^2 \gamma_i^2 - 2\alpha_i \lambda_i) \\ & + 2\theta \rho C_x C_y (\alpha_i \lambda_i - (1 - \alpha_i) \gamma_i) - 2(\alpha_i^2 \lambda_i^2 - (1 - \alpha_i)^2 \gamma_i^2)) \end{aligned}$$

$$\begin{aligned} & \delta \bar{Y}^2 \left[C_y^2 + \theta^2 C_x^2 + 2\theta \rho C_x C_y \right] \geq \bar{Y}^2 (P_i - 1)^2 \\ & + \delta \bar{Y}^2 \left\{ P_i^2 C_y^2 + \theta^2 C_x^2 (P_i^2 + (P_i + Q_i)(P_i - 1) - 2\theta \rho C_x C_y Q_i (2P_i - 1)) \right\} \end{aligned}$$

$$\begin{aligned} & 2\delta \theta \rho C_x C_y (Q_i (2P_i - 1) + 1) \\ & \geq (P_i - 1)^2 + \delta \{ C_y^2 (P_i^2 - 1) + \theta^2 C_x^2 (P_i^2 + (P_i + Q_i)(Q_i - 1) - 1) \} \end{aligned}$$

$$\rho \geq \frac{(P_i - 1)^2 + \delta \{ C_y^2 (P_i^2 - 1) + \theta^2 C_x^2 (P_i^2 - (P_i + Q_i)(Q_i - 1) - 1) \}}{2\delta \theta C_{xy} (Q_i (2P_i - 1) + 1)}$$

4. Numerical Study

In this section we consider three natural populations, population 1 (Singh and Chaudhary [11] page 177), population 2 (Khoshnevisan et.al.[6]) and Population 3 (Cochran[2] page 152) and are used to obtain the biases and mean squared errors and also used to compare the percentage relative efficiency of proposed estimator with that of the existing estimators. The computed values of constants and parameters of these populations are given below

Table 3 : Parameters and Constants of Different Populations

Constants	Population 1	Population 2	Population 3
N	34.000	20.000	49.000
n	5.000	5.000	5.000
$\bar{Y}$	199.441	19.550	127.796
$\bar{X}$	856.412	18.800	103.143
S_x	733.141	7.413	104.405
S_y	150.215	6.944	123.121
C_x	0.856	0.394	1.012
C_y	0.753	0.355	0.963
β_1	7.955	3.061	4.777
β_2	13.367	0.547	7.511
ρ	0.445	-0.919	0.982
θ_1	0.989	0.708	0.956

θ_2	1.000	0.993	0.998
θ_3	0.856	0.415	1.003
θ_4	0.998	1.023	0.990
θ_5	0.843	0.383	0.944
θ_6	1.000	0.963	0.999
λ_1	0.931	0.975	0.996
λ_2	0.929	0.960	0.988
λ_3	0.953	0.988	0.988
λ_4	0.930	0.958	0.990
λ_5	0.955	0.989	0.998
λ_6	0.929	0.961	0.988
γ_1	0.954	1.014	0.859
γ_2	0.953	1.020	0.854
γ_3	0.960	1.008	0.853
γ_4	0.953	1.020	0.855
γ_5	0.960	1.007	0.860
γ_6	0.953	1.019	0.853
α_1	0.710	-0.102	0.983
α_2	0.709	0.079	0.958
α_3	0.733	-0.535	0.955
α_4	0.709	0.093	0.962
α_5	0.736	-0.620	0.992
α_6	0.709	0.065	0.957

Table 4: Bias and MSE of Proposed Estimators

Proposed Estimators	Population 1		Population 2		Population 3	
	Bias	MSE	Bias	MSE	Bias	MSE
$\hat{Y}_{p1}$	1.78E-15	2560.326	3.01E-06	1.033955	9.88E-15	96.78642
$\hat{Y}_{p2}$	7.11E-15	2545.273	2.50E-06	1.063967	4.44E-15	90.55698
$\hat{Y}_{p3}$	4.26E-14	2730.305	3.17E-06	1.022477	5.33E-15	89.59797
$\hat{Y}_{p4}$	-1.60E-14	2548.292	2.43E-06	1.067739	-2.18E-14	92.03092
$\hat{Y}_{p5}$	-1.07E-14	2744.652	3.16E-06	1.022694	1.61E-14	97.90284
$\hat{Y}_{p6}$	-1.78E-15	2545.201	2.57E-06	1.060262	-1.55E-14	90.40969

Table 5: Bias and MSE of Existing Estimators

Existing Estimators	Population 1		Population 2		Population 3	
	Bias	MSE	Bias	MSE	Bias	MSE
$\hat{Y}_{MR1}$	14.737	4860.970	0.496	22.153	0.491	99.939
$\hat{Y}_{MR2}$	15.159	4924.565	0.825	30.698	1.491	110.643
$\hat{Y}_{MR3}$	9.895	4155.649	0.235	14.890	1.610	112.522
$\hat{Y}_{MR4}$	15.075	4911.803	0.864	31.684	1.297	107.830
$\hat{Y}_{MR5}$	9.480	4097.819	0.212	14.202	0.203	98.754
$\hat{Y}_{MR6}$	15.161	4924.872	0.787	29.728	1.510	110.929
$\hat{Y}_{MP1}$	9.664	12570.470	-0.267	1.243	21.012	10840.980
$\hat{Y}_{MP2}$	9.768	12716.740	-0.375	1.354	21.928	11320.110
$\hat{Y}_{MP3}$	8.358	10823.600	-0.157	2.641	22.033	11375.270
$\hat{Y}_{MP4}$	9.747	12687.500	-0.387	1.450	21.756	11229.150
$\hat{Y}_{MP5}$	8.234	10666.710	-0.145	2.882	20.733	10696.890
$\hat{Y}_{MP6}$	9.768	12717.440	-0.364	1.273	21.945	11328.770

Table 6: Percentage Relative Efficiency of Proposed Estimators

Proposed Estimators	Population 1		Population 2		Population 3	
	Modified Ratio	Modified Product	Modified Ratio	Modified Product	Modified Ratio	Modified Product
$\hat{Y}_{p1}$	189.857	490.971	2142.513	120.194	103.257	11200.930
$\hat{Y}_{p2}$	193.479	499.622	2885.204	127.221	122.180	12500.540
$\hat{Y}_{p3}$	152.205	396.424	1456.240	258.270	125.586	12695.900
$\hat{Y}_{p4}$	192.749	497.883	2967.437	135.808	117.167	12201.500
$\hat{Y}_{p5}$	149.302	388.636	1388.658	281.770	100.870	10926.030
$\hat{Y}_{p6}$	193.496	499.664	2803.839	120.101	122.696	12530.480

Conclusions

In this paper we have suggested a new class of the modified ratio cum product estimators for finite population mean of the study variable Y with known parameters of the auxiliary variable. The biases and mean squared errors of the proposed estimators are obtained and compared with that of some existing modified ratio and modified product estimators. Theoretically we have shown that the proposed estimator is always more efficient than other existing estimators under the optimum values of α_i and $a_i = B(\hat{\bar{Y}}_{MRi})$, $b_i = B(\hat{\bar{Y}}_{MRi})$. We have also studied the performance of the proposed estimators for certain known natural populations, it shows that the proposed estimator has less bias and mean squared error than all these existing estimators. That is the proposed estimator is more efficient than all these existing estimators. Hence we strongly recommended that the proposed estimator is more preferable than these existing estimators.

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APPENDIX

Bias and MSE of Proposed Estimators

The proposed class of ratio cum product estimators with known parameters of auxiliary variables are

$$\begin{aligned}\widehat{Y}_{P1} &= \alpha_1 \lambda_1 \bar{y} \left(\frac{C_x \bar{X} + \beta_1}{C_x \bar{x} + \beta_1} \right) + (1 - \alpha_1) \gamma_1 \bar{y} \left(\frac{C_x \bar{x} + \beta_1}{C_x \bar{X} + \beta_1} \right) \\ \widehat{Y}_{P2} &= \alpha_2 \lambda_2 \bar{y} \left(\frac{\beta_1 \bar{X} + C_x}{\beta_1 \bar{x} + C_x} \right) + (1 - \alpha_2) \gamma_2 \bar{y} \left(\frac{\beta_1 \bar{x} + C_x}{\beta_1 \bar{X} + C_x} \right) \\ \widehat{Y}_{P3} &= \alpha_3 \lambda_3 \bar{y} \left(\frac{C_x \bar{X} + \rho}{C_x \bar{x} + \rho} \right) + (1 - \alpha_3) \gamma_3 \bar{y} \left(\frac{C_x \bar{x} + \rho}{C_x \bar{X} + \rho} \right) \\ \widehat{Y}_{P4} &= \alpha_4 \lambda_4 \bar{y} \left(\frac{\rho \bar{X} + C_x}{\rho \bar{x} + C_x} \right) + (1 - \alpha_4) \gamma_4 \bar{y} \left(\frac{\rho \bar{x} + C_x}{\rho \bar{X} + C_x} \right) \\ \widehat{Y}_{P5} &= \alpha_5 \lambda_5 \bar{y} \left(\frac{C_x \bar{X} + \beta_2}{C_x \bar{x} + \beta_2} \right) + (1 - \alpha_5) \gamma_5 \bar{y} \left(\frac{C_x \bar{x} + \beta_2}{C_x \bar{X} + \beta_2} \right) \\ \widehat{Y}_{P6} &= \alpha_6 \lambda_6 \bar{y} \left(\frac{\beta_2 \bar{X} + C_x}{\beta_2 \bar{x} + C_x} \right) + (1 - \alpha_6) \gamma_6 \bar{y} \left(\frac{\beta_2 \bar{x} + C_x}{\beta_2 \bar{X} + C_x} \right)\end{aligned}$$

Where $\lambda_i = \frac{S_y}{S_y + a_i C_y}$, $\gamma_i = \frac{S_y}{S_y + b_i C_y}$, $i = 1, 2, 3, 4, 5, 6$

To obtain the bias and mean squared error of the proposed estimators,

Consider, $e_0 = \frac{\bar{y} - \bar{Y}}{\bar{Y}}$, $e_1 = \frac{\bar{x} - \bar{X}}{\bar{X}}$, $\delta = \left(\frac{1-f}{n} \right)$, $f = \frac{n}{N}$

$$\theta_1 = \frac{C_x \bar{X}}{C_x \bar{X} + \beta_1}, \theta_2 = \frac{\beta_1 \bar{X}}{\beta_1 \bar{X} + C_x}, \theta_3 = \frac{C_x \bar{X}}{C_x \bar{X} + \rho}, \theta_4 = \frac{\rho \bar{X}}{\rho \bar{X} + C_x}, \theta_5 = \frac{C_x \bar{X}}{C_x \bar{X} + \beta_2}, \theta_6 = \frac{\beta_2 \bar{X}}{\beta_2 \bar{X} + C_x}$$

Where $i = 1, 2, 3, 4, 5, 6$

$$\begin{aligned}E(e_0) &= E(e_1) = 0, E(e_0^2) = \delta \bar{Y}^2 C_y^2, E(e_1^2) = \delta \bar{X}^2 C_x^2, \\ E(e_0 e_1) &= \delta \rho C_x C_y\end{aligned}$$

Substitute these values in $\widehat{Y}_{pi}$ and neglecting the higher order expressions, we get

$$\begin{aligned}\widehat{Y}_{pi} &= \alpha_i \lambda_i \bar{Y} (1 + e_0) (1 + \theta_i e_1)^{-1} + (1 - \alpha_i) \gamma_i \bar{Y} (1 + e_0) (1 + \theta_i e_1) \\ &= \bar{Y} \{ \alpha_i \lambda_i (1 + e_0) (1 - \theta_i e_1 + \theta_i^2 e_1^2) + (1 - \alpha_i) \gamma_i (1 + e_0) (1 + \theta_i e_1) \} \\ &= \bar{Y} \{ \alpha_i \lambda_i (1 - \theta_i e_1 + \theta_i^2 e_1^2 + e_0 - \theta_i e_1 e_0) + (1 - \alpha_i) \gamma_i (1 + \theta_i e_1 + e_0 + \theta_i e_0 e_1) \} \\ \widehat{Y}_{pi} - \bar{Y} &= \bar{Y} \{ \alpha_i \lambda_i (1 - \theta_i e_1 + \theta_i^2 e_1^2 + e_0 - \theta_i e_1 e_0) + (1 - \alpha_i) \gamma_i (1 + \theta_i e_1 + e_0 + \theta_i e_0 e_1) \} \\ B(\widehat{Y}_{pi}) &= E(\widehat{Y}_{pi} - \bar{Y}) \\ &= E \{ \bar{Y} (\alpha_i \lambda_i (1 - \theta_i e_1 + \theta_i^2 e_1^2 + e_0 - \theta_i e_1 e_0) + (1 - \alpha_i) \gamma_i (1 + \theta_i e_1 + e_0 + \theta_i e_0 e_1) - 1) \} \\ &= \delta \bar{Y} \{ \alpha_i \lambda_i (1 + \theta_i^2 C_x^2 - \theta_i \rho C_x C_y) + (1 - \alpha_i) \gamma_i (1 + \theta_i \rho C_x C_y) - 1 \}\end{aligned}$$

$$B(\hat{\bar{Y}}_{Pi}) = \bar{Y}(\alpha_i \lambda_i + (1 - \alpha_i) \gamma_i - 1) + \alpha_i \lambda_i B(\hat{\bar{Y}}_{MRi}) + (1 - \alpha_i) \gamma_i B(\hat{\bar{Y}}_{MPi})$$

Take a substitution $P_i = (\alpha_i \lambda_i + (1 - \alpha_i) \gamma_i)$ and $Q_i = (\alpha_i \lambda_i - (1 - \alpha_i) \gamma_i)$ then

$$B(\hat{\bar{Y}}_{Pi}) = \bar{Y}(P_i - 1) + \frac{(P_i + Q_i)}{2} B(\hat{\bar{Y}}_{MRi}) + \frac{(P_i - Q_i)}{2} B(\hat{\bar{Y}}_{MPi})$$

$$B(\hat{\bar{Y}}_{Pi}) = \frac{1}{2} [2\bar{Y}(P_i - 1) + P_i \{B(\hat{\bar{Y}}_{MRi}) + B(\hat{\bar{Y}}_{MPi})\} + Q_i \{B(\hat{\bar{Y}}_{MRi}) - B(\hat{\bar{Y}}_{MPi})\}]$$

The mean squared errors of the proposed estimators are

$$MSE(\hat{\bar{Y}}_{Pi}) = E(\hat{\bar{Y}}_{Pi} - \bar{Y})^2$$

$$= E\{\bar{Y}^2(\alpha_i^2 \lambda_i^2 (1 - \theta_i e_1 + \theta_i^2 e_1^2 + e_0 - \theta_i e_1 e_0) + (1 - \alpha_i) \gamma_i (1 + \theta_i e_1 + e_0 + \theta_i e_0 e_1) - 1)^2\}$$

$$= E\{\bar{Y}^2(\alpha_i^2 \lambda_i^2 (1 - \theta_i e_1 + \theta_i^2 e_1^2 + e_0 - \theta_i e_1 e_0)^2 + (1 - \alpha_i)^2 \gamma_i^2 (1 + \theta_i e_1 + \theta_i e_0 e_1)^2 + 1$$

$$+ 2\alpha_i \lambda_i (1 - \alpha_i) \gamma_i (1 - \theta_i e_1 + \theta_i^2 e_1^2 + e_0 - \theta_i e_1 e_0)(1 + \theta_i e_1 + e_0 + \theta_i e_0 e_1)$$

$$- 2\alpha_i \lambda_i (1 - \theta_i e_1 + \theta_i^2 e_1^2 + e_0 - \theta_i e_1 e_0) - 2(1 - \alpha_i) \gamma_i (1 + \theta_i e_1 + e_0 + \theta_i e_0 e_1)\}$$

$$= \delta \bar{Y}^2 \{\alpha_i^2 \lambda_i^2 (1 + 3\theta_i^2 C_x^2 + C_y^2 - 4\theta_i \rho C_x C_y)$$

$$+ (1 - \alpha_i)^2 \gamma_i^2 (1 + \theta_i^2 C_x^2 + C_y^2 + 4\theta_i \rho C_x C_y) + 2\alpha_i \lambda_i (1 - \alpha_i) \gamma_i (1 + C_y^2)$$

$$- 2\alpha_i \lambda_i (1 + \theta_i^2 C_x^2 - \theta_i \rho C_x C_y) - 2(1 - \alpha_i) \gamma_i (1 + \theta_i \rho C_x C_y)\}$$

$$MSE(\hat{\bar{Y}}_{Pi}) = \bar{Y}^2(\alpha_i \lambda_i + (1 - \alpha_i) \gamma_i - 1)^2 + \alpha_i^2 \lambda_i^2 \{MSE(\hat{\bar{Y}}_{MRi}) + 2\bar{Y}B(\hat{\bar{Y}}_{MRi})\}$$

$$+ (1 - \alpha_i)^2 \gamma_i^2 \{MSE(\hat{\bar{Y}}_{MPi}) + 2\bar{Y}B(\hat{\bar{Y}}_{MPi})\}$$

$$- 2\bar{Y} \{\alpha_i \lambda_i B(\hat{\bar{Y}}_{MRi}) + (1 - \alpha_i) \gamma_i B(\hat{\bar{Y}}_{MPi})\} + 2(\alpha_i \lambda_i (1 - \alpha_i) \gamma_i V(\bar{y}_{srs}))$$

$$MSE(\hat{\bar{Y}}_{Pi}) = \bar{Y}^2(P_i - 1)^2 + \alpha_i^2 \lambda_i^2 \{MSE(\hat{\bar{Y}}_{MRi}) + 2\bar{Y}B(\hat{\bar{Y}}_{MRi})\}$$

$$+ (1 - \alpha_i)^2 \gamma_i^2 \{MSE(\hat{\bar{Y}}_{MPi}) + 2\bar{Y}B(\hat{\bar{Y}}_{MPi})\}$$

$$- 2\bar{Y} \{\alpha_i \lambda_i B(\hat{\bar{Y}}_{MRi}) + (1 - \alpha_i) \gamma_i B(\hat{\bar{Y}}_{MPi})\} + 2(\alpha_i \lambda_i (1 - \alpha_i) \gamma_i V(\bar{y}_{srs}))$$

We take a substitution $P_i = (\alpha_i \lambda_i + (1 - \alpha_i) \gamma_i)$ and $Q_i = (\alpha_i \lambda_i - (1 - \alpha_i) \gamma_i)$ then the MSE is

$$MSE(\hat{\bar{Y}}_{Pi}) = \bar{Y}^2(P_i - 1)^2 + \frac{(P_i + Q_i)^2}{4} \{MSE(\hat{\bar{Y}}_{MRi}) + 2\bar{Y}B(\hat{\bar{Y}}_{MRi})\}$$

$$+ \frac{(P_i - Q_i)^2}{4} \{MSE(\hat{\bar{Y}}_{MPi}) + 2\bar{Y}B(\hat{\bar{Y}}_{MPi})\}$$

$$- 2\bar{Y} \left\{ \frac{(P_i + Q_i)}{2} B(\hat{\bar{Y}}_{MRi}) + \frac{(P_i - Q_i)}{2} B(\hat{\bar{Y}}_{MPi}) \right\} + 2 \frac{(P_i^2 - Q_i^2)}{4} V(\bar{y}_{srs})]$$

$$MSE(\hat{\bar{Y}}_{Pi}) = \frac{1}{4} [4\bar{Y}^2(P_i - 1)^2 + (P_i + Q_i)^2 \{MSE(\hat{\bar{Y}}_{MRi}) + 2\bar{Y}B(\hat{\bar{Y}}_{MRi})\}$$

$$+ (P_i - Q_i)^2 \{MSE(\hat{\bar{Y}}_{MPi}) + 2\bar{Y}B(\hat{\bar{Y}}_{MPi})\}$$

$$- 4\bar{Y} \{(P_i + Q_i)B(\hat{\bar{Y}}_{MRi}) + (P_i - Q_i)B(\hat{\bar{Y}}_{MPi})\} + 2(P_i^2 - Q_i^2)V(\bar{y}_{srs})]$$

$$\begin{aligned} \text{Where } B(\hat{\bar{Y}}_{MRi}) &= \delta \bar{Y} [\theta_i^2 C_x^2 - \theta_i \rho C_x C_y], B(\hat{\bar{Y}}_{MPi}) = \delta \bar{Y} [\theta_i \rho C_x C_y] \\ \text{MSE}(\hat{\bar{Y}}_{MRi}) &= \delta \bar{Y}^2 [C_y^2 + \theta_i^2 C_x^2 - 2\theta_i \rho C_x C_y] \\ \text{MSE}(\hat{\bar{Y}}_{MPi}) &= \delta \bar{Y}^2 [C_y^2 + \theta_i^2 C_x^2 + 2\theta_i \rho C_x C_y] \text{ and } V(\bar{y}_{srs}) = \delta S_y^2 = \delta \bar{Y}^2 C_y^2 \\ \delta &= \left(\frac{1-f}{n}\right), \theta_1 = \frac{C_x \bar{X}}{C_x \bar{X} + \beta_1}, \theta_2 = \frac{\beta_1 \bar{X}}{\beta_1 \bar{X} + C_x}, \theta_3 = \frac{C_x \bar{X}}{C_x \bar{X} + \rho}, \theta_4 = \frac{\rho \bar{X}}{\rho \bar{X} + C_x}, \theta_5 = \frac{C_x \bar{X}}{C_x \bar{X} + \beta_2}, \theta_6 = \frac{\beta_2 \bar{X}}{\beta_2 \bar{X} + C_x} \\ \text{where, } i &= 1, 2, 3, 4, 5, 6 \end{aligned}$$

The optimal value of α_i is determined by minimize the MSE ($\hat{\bar{Y}}_{pi}$) with respect to α_i . For this differentiate MSE with respect to α_i and equate to zero.

$$\begin{aligned} \text{ie, } \frac{\partial \text{MSE}}{\partial \alpha_i} &= 0, \text{ and we get the value of } \alpha_i, \text{ as} \\ 2\bar{Y}^2(\alpha_i \lambda_i + (1 - \alpha_i) \gamma_i - 1)(\lambda_i - \gamma_i) &+ 2\alpha_i \lambda_i^2 \{ \text{MSE}(\hat{\bar{Y}}_{MRi}) + 2\bar{Y} B(\hat{\bar{Y}}_{MRi}) \} \\ &- 2(1 - \alpha_i) \gamma_i^2 \{ \text{MSE}(\hat{\bar{Y}}_{MPi}) + 2\bar{Y} B(\hat{\bar{Y}}_{MPi}) \} - 2\bar{Y} \{ \lambda_i B(\hat{\bar{Y}}_{MRi}) + \gamma_i B(\hat{\bar{Y}}_{MPi}) \} \\ &+ 2(\lambda_i(1 - 2\alpha_i) \gamma_i V(\bar{y}_{srs})) = 0 \\ (\bar{Y}^2(\gamma_i - 1)(\gamma_i - \lambda_i) + \gamma_i^2 \{ \text{MSE}(\hat{\bar{Y}}_{MPi}) &+ 2\bar{Y} B(\hat{\bar{Y}}_{MPi}) \} + \bar{Y} \{ \lambda_i B(\hat{\bar{Y}}_{MRi}) - \gamma_i B(\hat{\bar{Y}}_{MPi}) \} \\ &- \lambda_i \gamma_i V(\bar{y}_{srs})) \\ \alpha_i &= \frac{(\bar{Y}^2(\lambda_i - \gamma_i)^2 + \lambda_i^2 \{ \text{MSE}(\hat{\bar{Y}}_{MRi}) + 2\bar{Y} B(\hat{\bar{Y}}_{MRi}) \} + \gamma_i^2 \{ \text{MSE}(\hat{\bar{Y}}_{MPi}) + 2\bar{Y} B(\hat{\bar{Y}}_{MPi}) \} \\ &- 2\lambda_i \gamma_i V(\bar{y}_{srs}))}{(\bar{Y}^2(\gamma_i - 1)(\gamma_i - \lambda_i) + \gamma_i^2 \{ \text{MSE}(\hat{\bar{Y}}_{MPi}) + 2\bar{Y} B(\hat{\bar{Y}}_{MPi}) \} + \bar{Y} \{ \lambda_i B(\hat{\bar{Y}}_{MRi}) - \gamma_i B(\hat{\bar{Y}}_{MPi}) \} - \lambda_i \gamma_i V(\bar{y}_{srs}))} \end{aligned}$$

Where $\delta = \left(\frac{1-f}{n}\right)$, $i = 1, 2, 3, 4, 5, 6$