MONOMIAL ENUMERATION OF THE FULL ICOSAHEDRAL GROUP

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Abstract. From the presentation of the full icosahedral group, viewed as a Coxeter group, we construct its Gröbner-Shirshov basis and the corresponding standard monomials.

1. Introduction

The icosahedral group is the group of rotational symmetries of a regular icosahedron. The object in this paper is the full icosahedral group, which is a double covering of the icosahedral group. We note that this object is among a general class of groups, so called, the Coxeter groups. They appear naturally in geometry and algebra. In 1935, the finite Coxeter groups were classified by Coxeter in terms of Coxeter-Dynkin diagrams [10]. Contrary to the crystallographic Coxeter groups, we note that the root systems of type $H_k$ ($k = 2, 3, 4$) and the affine extensions of the Coxeter groups of type $H$ are related to quasicrystals with five-fold symmetry [9, 15].

Our approach to understanding the structure of Coxeter groups is the noncommutative Gröbner-Shirshov basis theory. The effective notion stems from Shirshov’s Composition Lemma and his algorithm [16] for Lie algebras and independently from Buchberger’s algorithm [8] of computing Gröbner bases for commutative algebras. In [2], Bokut applied Shirshov’s method to associative algebras, and Bergman mentioned the diamond lemma for ring theory [1].

The Gröbner-Shirshov bases for finite dimensional simple Lie algebras were completely determined by Bokut and Klein [4, 5, 6]. For basic classical Lie superalgebras of types $A, B, C, D$ and their universal enveloping algebras, Bokut et al. developed the corresponding theory and gave an explicit construction of Gröbner-Shirshov bases [3]. Another types of Lie superalgebras were treated in [13].

For crystallographic Coxeter groups of classical and exceptional types, their Gröbner-Shirshov bases were constructed in [7, 12, 14, 17]. In this paper, we deal with a noncrystallographic Coxeter group. By completing the relations coming from a presentation of the Coxeter group, we find a Gröbner-Shirshov basis to obtain a set of standard monomials. For the icosahedral group, its monomial enumeration follows naturally.

2010 Mathematics Subject Classification. Primary 20F55, Secondary 05E15, 68W30.

Key words and phrases. full icosahedral group, Gröbner-Shirshov basis.
2. Preliminaries

In this section, we recall a basic theory of Gröbner-Shirshov bases for associative algebras to make the paper self-contained.

Let $X$ be a set and let $\langle X \rangle$ be the free monoid of associative words on $X$. We denote the empty word by $1$ and the length (or degree) of a word $u$ by $l(u)$. A well-ordering $< \in \langle X \rangle$ is called a monomial order if $x < y$ implies $axb < ayb$ for all $a, b \in \langle X \rangle$.

Fix a monomial order $< \in \langle X \rangle$ and let $\mathbb{F}(X)$ be the free associative algebra generated by $X$ over a field $\mathbb{F}$. Given a nonzero element $p \in \mathbb{F}(X)$, we denote by $\mathbf{p}$ the maximal monomial (called the leading monomial) appearing in $p$ under the ordering $<$. Thus $p = \alpha \mathbf{p} + \sum \beta_i w_i$ with $\alpha, \beta_i \in \mathbb{F}$, $w_i \in \langle X \rangle$, $\alpha \neq 0$ and $w_i < \mathbf{p}$. If $\alpha = 1$, $p$ is said to be monic.

Let $S$ be a subset of monic elements in $\mathbb{F}(X)$, and let $I$ be the two-sided ideal of $\mathbb{F}(X)$ generated by $S$. Then we say that the algebra $A = \mathbb{F}(X)/I$ is defined by $S$.

**Definition 2.1.** Given a subset $S$ of monic elements in $\mathbb{F}(X)$, a monomial $u \in \langle X \rangle$ is said to be $S$-standard (or $S$-reduced) if $u \neq \alpha \mathbf{p}b$ for any $s \in S$ and $a, b \in \langle X \rangle$. Otherwise, the monomial $u$ is said to be $S$-reducible.

**Lemma 2.2** ([1, 2]). Every $p \in \mathbb{F}(X)$ can be expressed as

\[
p = \sum \alpha_i a_i s_i b_i + \sum \beta_j u_j,
\]

where $\alpha_i, \beta_j \in \mathbb{F}$, $a_i, b_i, u_j \in \langle X \rangle$, $s_i \in S$, $a_i \mathbf{p}b_i \leq \mathbf{p}$, $u_j \leq \mathbf{p}$ and $u_j$ are $S$-standard.

**Remark.** The term $\sum \beta_j u_j$ in the expression (2.1) is called a normal form (or a remainder) of $p$ with respect to the subset $S$ (and with respect to the monomial order $<$). In general, a normal form is not unique.

As an immediate corollary of Lemma 2.2, we obtain:

**Proposition 2.3.** The set of $S$-standard monomials spans the algebra $A = \mathbb{F}(X)/I$ defined by the subset $S$, as a vector space over $\mathbb{F}$.

Let $p$ and $q$ be monic elements in $\mathbb{F}(X)$ with leading monomials $\mathbf{p}$ and $\mathbf{q}$. We define the composition of $p$ and $q$ as follows.

**Definition 2.4.** (a) If there exist $a$ and $b$ in $\langle X \rangle$ such that $\mathbf{p}a = \mathbf{q}b = w$ with $l(\mathbf{p}) > l(b)$, then the composition of $p$ and $q$ is defined to be

\[(p, q)_w = pa - bq.\]

(b) If there exist $a$ and $b$ in $\langle X \rangle$ such that $a \neq 1$, $a\mathbf{p}b = \mathbf{q} = w$, then the composition of inclusion is defined to be

\[(p, q)_{a,b} = apb - q.\]

Let $p, q \in \mathbb{F}(X)$ and $w \in \langle X \rangle$. We define the congruence relation on $\mathbb{F}(X)$ as follows: $p \equiv q \mod (S; w)$ if and only if $p - q = \sum \alpha_i a_i s_i b_i$, where $\alpha_i \in \mathbb{F}$, $a_i, b_i \in \langle X \rangle$, $s_i \in S$, $a_i \mathbf{p}b_i \leq \mathbf{p}$ and $u_j \leq \mathbf{p}$ and $u_j$ are $S$-standard.

**Definition 2.5.** A subset $S$ of monic elements in $\mathbb{F}(X)$ is said to be closed under composition if

\[(p, q)_w \equiv 0 \mod (S; w)\]

and

\[(p, q)_{a,b} \equiv 0 \mod (S; w)\]

for all $p, q \in S$, $a, b \in \langle X \rangle$ whenever the compositions $(p, q)_w$ and $(p, q)_{a,b}$ are defined.

The following theorem is a main tool for our results in the next section.

**Theorem 2.6** ([1, 2]). Let $S$ be a subset of monic elements in $\mathbb{F}(X)$. Then the following conditions are equivalent:

(a) $S$ is closed under composition.

(b) For each $p \in \mathbb{F}(X)$, a normal form of $p$ with respect to $S$ is unique.

(c) The set of $S$-standard monomials forms a linear basis of the algebra $A = \mathbb{F}(X)/I$ defined by $S$. 

Definition 2.7. A subset $S$ of monic elements in $\mathbb{F}(X)$ is a Gröbner-Shirshov basis if $S$ satisfies one of the equivalent conditions in Theorem 2.6. In this case, we say that $S$ is a Gröbner-Shirshov basis for the algebra $A$ defined by $S$.

3. RESULTS

Let $G$ be the full icosahedral group, and take the complex field $\mathbb{C}$ as our base field. Then the group algebra $\mathbb{C}[G]$ is generated by three elements $s_i$ ($i = 1, 2, 3$) with the defining relations:

(3.1) $s_i^2 = s_i^3 = 1$,
(3.2) $(s_1 s_2)^3 = (s_1 s_3)^2 = (s_2 s_3)^5 = 1$.

Note that the relations (3.1) indicate that three elements $s_1, s_2, s_3$ are reflections (or involutions).

We define our monomial order to be the degree-lexicographic order with

$s_3 > s_2 > s_1$.

Combining the above quadratic and braid relations, we get three braid relations:

$$
\begin{align*}
    s_2 s_1 s_2 &= s_1 s_2 s_1, \\
    s_3 s_1 &= s_1 s_3, \\
    s_3 s_2 s_3 s_2 s_3 &= s_2 s_3 s_2 s_3 s_2.
\end{align*}
$$

(3.3)

These are easily checked. For instance, $s_2 s_1 s_2 = s_1 s_2 s_1$ follows from $(s_1 s_2)^3 = 1$, multiplying by $s_2, s_1, s_2$ successively from the right. Note that these relations are homogeneous.

Our method is to reduce a monomial with respect to the degree-lexicographic order. We denote by $R$ the set of relations (3.1) and (3.3), that is,

$$
R = \{ s_i^2 - 1, \quad s_i^3 - 1, \quad s_i s_j - s_j s_i \mid i, j = 1, 2, 3 \}.
$$

(3.4)

Note that the group algebra $\mathbb{C}[G]$ is isomorphic to $\mathbb{C}(s_1, s_2, s_3)/I$, where $I$ is the two-sided ideal generated by $R$ in the free associative algebra $\mathbb{C}(s_1, s_2, s_3)$.

From now on, we identify $\mathbb{C}[G]$ with $\mathbb{C}(s_1, s_2, s_3)/I$. We say that the algebra $\mathbb{C}[G]$ is defined by $R$.

A reducible monomial is called minimal if every proper submonomial is not reducible. Our strategy to find relations is considering all minimal reducible monomials until we obtain the smallest set of standard monomials. We check the relations in order of the degree of a monomial, one by one.

Lemma 3.1. The following relations hold in $\mathbb{C}[G]$:

(a) $(s_3 s_2 s_1) s_2 - s_1 (s_3 s_2 s_1) = 0$,
(b) $(s_3 s_2 s_3 s_2 s_1) s_3 - s_2 (s_3 s_2 s_3 s_2 s_1) = 0$,
(c) $(s_3 s_2 s_1)^2 (s_3 s_2)^2 - s_2 (s_3 s_2 s_1)^2 (s_3 s_2) s_3 = 0$,
(d) $(s_3 s_2 s_1)^4 - s_2 (s_3 s_2 s_1)^3 (s_3 s_2) = 0$.

Proof. (a) Using the relations in (3.3), we get that

$$
(s_3 s_2 s_1) s_2 = s_1 s_2 s_1 = s_1 (s_3 s_2 s_1).
$$

(b) In a similar way, we obtain that

$$
(s_3 s_2 s_3 s_2 s_1) s_3 = s_3 s_2 s_3 s_2 s_3 s_1 = s_2 (s_3 s_2 s_3 s_2 s_1).
$$

(c) We transform successively as follows:

$$
(s_3 s_2 s_1)^2 (s_3 s_2)^2 = s_3 s_2 s_3 s_2 s_1 (s_3 s_2)^2 = s_3 s_2 s_3 s_2 s_1 (s_3 s_2) s_3 = s_2 s_3 s_2 s_3 s_1 s_2 s_3 s_3 = s_2 s_3 s_2 s_3 s_1 s_2 s_3 s_3 = (s_3 s_2 s_1)^2 (s_3 s_2) s_3.
$$

The last equality follows from (a).

(d) We deduce that

$$
(s_3 s_2 s_1)^4 = (s_3 s_2 s_3 s_2 s_1 s_2) (s_3 s_2 s_3 s_2 s_1 s_2) = s_3 s_2 s_3 s_2 s_1 (s_3 s_2 s_3 s_2 s_1 s_2) s_2 = (s_3 s_2 s_3 s_2 s_1) (s_3 s_2 s_3 s_2 s_1 s_2) = (s_3 s_2 s_1)^2 (s_3 s_2)
$$

by the relations (3.3) and (a).

Let $s_{i,j} = s_i s_{i-1} \cdots s_j$ if $i \geq j$, and by convention $s_{i,j} = 1$ if $i < j$. 


Theorem 3.2. We denote by $\hat{R}$ the set of defining relations $R$ combined with the relations in Lemma 3.1. Then $\hat{R}$ forms a Gröbner-Shirshov basis for $\mathbb{C}[G]$. The corresponding standard monomials are of the form

$$s_{a,b}M$$

where $1 \leq a \leq 2$, $1 \leq b \leq 3$ and $M$ is one of the following 20 monomials:

- $s_1, s_3, s_2, s_3s_2s_1, s_3s_2s_1s_3, (s_3s_2s_1)(s_3s_2), (s_3s_2s_1)(s_3s_2)s_3$,
- $(s_3s_2)^2, s_3s_2s_3s_2s_1, (s_3s_2s_1)(s_3s_2)^2$,
- $(s_3s_2s_1)(s_3s_2s_3s_2s_1), (s_3s_2s_1)^2$,
- $(s_3s_2s_1)^2s_3, (s_3s_2s_1)^2(s_3s_2)$,
- $(s_3s_2s_1)^2(s_3s_2)s_3, (s_3s_2s_1)^3$,
- $(s_3s_2s_1)^3s_3, (s_3s_2s_1)^3(s_3s_2)$,
- $(s_3s_2s_1)^3(s_3s_2)s_3$.

Proof. We have two ways of proving this theorem.

(I) Counting the number of $\hat{R}$-standard monomials:

We enumerate all $\hat{R}$-standard monomials in $\mathbb{C}(s_1, s_2, s_3)$, which are in the above. The number of standard monomials is

$$2 \times 3 \times 20 = 120,$$

which is exactly equal to the order of $G$ (See [11, §2.13]), the dimension of $\mathbb{C}[G]$ as a $\mathbb{C}$-vector space. Hence, by Theorem 2.6, the set $\hat{R}$ is a Gröbner-Shirshov basis for $\mathbb{C}[G]$.

(II) Checking that $\hat{R}$ is closed under composition:

Note that the set $\hat{R}$ consists of ten polynomials. Following Theorem 2.6(a), we show that $\hat{R}$ is closed under composition. It is proved by checking that all possible compositions between any two polynomials reduce trivially.

For example, let $p = (s_3s_2s_1)s_2 - s_1(s_3s_2s_1)$ in Lemma 3.1(a) and $q = s_2s_1s_2 - s_1s_2s_1$ in (3.4), and consider the composition

$$(p, q)_w$$

with $w = s_3s_2s_1s_2s_1s_2$.

Then we check that

$$p(s_1s_2) - (s_3s_2s_1)q$$

$$= (s_3s_2s_1)(s_1s_2s_1) - s_1(s_3s_2s_1)(s_1s_2)$$

$$= s_3s_1 - s_1s_3 = 0.$$
is the longest element in $G$, which is of length 15, the number of positive roots in a root system of type $H_3$.

(3) A noncrystallographic root lattice induces a quasicrystal, using the cut and project method by a root map [9, §7].

(4) The full icosahedral group has the icosahedral group as a normal subgroup of index 2. Note that the subgroup of rotational symmetries of a regular icosahedron is isomorphic to the 5th alternating group $A_5$ of order 60. Since the elements of $A_5$ are of even degree, we have an explicit monomial expression for 60 elements of the icosahedral group.

Acknowledgements. A part of this work was presented in Microsymposium on crystals and beyond, IUCr (Montreal, 2014). The author expresses his gratitude to Jeong-Yup Lee and Robert Moody for their useful comments. This research was supported by NRF Grant # 2014R1A1A2054811 and a research grant from Seoul Women’s University(2017).

References


