On Some Bending Problems of Prismatic Shell with the Thickness Vanishing at Infinity

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Abstract: The present work is devoted to the bending problems of prismatic shell with the thickness vanishing at infinity as an exponential function. The bending equation in the zero approximation of Vekua's hierarchical models is considered. The problem is reduced to the Dirichlet boundary value problem for elliptic type partial differential equations on half-plane. The solution of the problem under consideration is constructed in the integral form.

Key words: Cusped prismatic shell, Vekua's hierarchical models, Elliptic type partial differential equations

1. Introduction

In 1955 I.Vekua raised the problem of investigation of elastic cusped prismatic shells, whose thickness on the prismatic shell entire boundary or on its part vanishes (see [17], [10]). In practice, such cusped prismatic shells, in particular, cusped plates, and cusped beams (i.e., beams whose cross-sections area vanishes at least at one end of the beam) are often encountered in spatial structures with partly fixed edges, e.g., stadium ceilings, aircraft wings, submarine wings etc., in machine-tool design, in astronautics, turbines, and in many other applied fields of engineering. Investigation of elastic cusped prismatic shells, considered as 3D ones, may occupy 3D domains with, in general, non-Lipschitz boundaries. The problem mathematically leads to the question of setting and solving of boundary value problems for even order

equations and systems of elliptic type with the order degeneration in the statical case and of initial boundary value problems for even order equations and systems of hyperbolic type with the order degeneration in the dynamical case. The cusped plates within the framework of the classical bending theory were first considered by Makhover and Mikhlin in [14], [15]. Problems for cusped plates have been investigated by Khvoles, Jaiani, Tsiskaridze, Khomasuridze, Devdariani, Uzunov, Naguleswaran, Kharibegashvili, Natroshvili, Wendland, and others (see, e.g. [10], [11], [13], [8], [9] and references therein).

At the same time Vekua introduced a new mathematical model for elastic prismatic shells which was based on expansions of the three-dimensional displacement vector fields and the strain and stress tensors in linear elasticity into orthogonal Fourier-Legendre series with respect to the variable plate thickness [18]. By taking only the first $N + 1$ terms of the expansions, he introduced the so called N -th approximation. Each of these approximations for $N = 0; 1; \dots$ can be considered as an independent

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mathematical model of plates. In particular, the approximation for $N = 0$ coincides with plane stress and plane deformation, $N = 1$ corresponds to the classical Kirchhoff plate model [10]. Works of Babuska, Gordeziani, Guliaev, Khoma, Khvoles, Meunargia, Schwab, Vashakmadze, Zhgenti, Jaiani, Tsiskarishvili, M. and G. Avalishvili, Wendland, Natroshvili, Kharibegashvili, Chinchaladze, Gilbert, and others are devoted to further analysis of I.Vekua's models (rigorous estimation of the modeling error, numerical solutions, etc.) and their generalizations (see, e.g., [2], [3], [4], [5], [6], [7], [8], [9], [12], [16]).

In what follows, σ_{ij} and e_{ij} are the stress and strain tensor, respectively, u_i are displacements, Φ_i are volume force components, ρ is the density, λ and μ are the Lamé constants, δ_{ij} is Kroneker delta. Moreover, repeated indices imply summation, the bar under one of the repeated indices means that we do not sum, indices after comma mean differentiation with respect to this value.

Let a 3D elastic body occupy a bounded region $\bar{\Omega}$ with boundary $\partial\Omega$:

$$\Omega := \{(x_1, x_2, x_3) \in R^3 : x := (x_1, x_2) \in \omega,$$

$$\bar{h}^{(-)}(x) < x_3 < \bar{h}^{(+)}(x)\},$$

where $\bar{\omega} = \omega \cup \partial\omega$ is the so-called projection of the plate $\bar{\Omega} = \Omega \cup \partial\Omega$.

We assume that $\bar{h}^{(-)}(x) \in C^2(\omega) \cap C(\bar{\omega})$ and the thickness is

$$2h(x) := \bar{h}^{(+)}(x) - \bar{h}^{(-)}(x) > 0 \text{ for } x \in \omega$$

and

$$2h(x) := \bar{h}^{(+)}(x) - \bar{h}^{(-)}(x) \geq 0 \text{ for } x \in \partial\omega,$$

i.e., the thickness may vanish on some part of the boundary.

In the zero approximation of Vekua's hierarchical methods the governing system has the following form (see, e.g. [10], [18])

$$\begin{aligned} & \mu[(hv_{\alpha 0, \beta})_{, \beta} + (hv_{\beta 0, \alpha})_{, \beta}] + \lambda \delta_{\alpha \beta} (hv_{\gamma 0, \gamma})_{, \beta} \\ & + \Phi_{\alpha 0} + (Q_{n \alpha}^{(+)} + Q_{n \alpha}^{(-)}) \\ & \times \sqrt{(h_{,1})^2 + (h_{,2})^2 + 1} = 0, \quad \alpha = 1, 2, \end{aligned} \quad (1)$$

$$\begin{aligned} & \mu(hv_{30, \beta})_{, \beta} + \Phi_{30} + (Q_{n 3}^{(+)} + Q_{n 3}^{(-)}) \\ & \times \sqrt{(h_{,1})^2 + (h_{,2})^2 + 1} = 0, \end{aligned} \quad (2)$$

where

$$v_{i0} := \frac{u_{i0}}{h}, \quad i = 1, 2, 3,$$

u_{i0} and Φ_{i0} are the zeroth moments of the 3D displacement vector and volume force components respectively, i.e.,

$$(u_{i0}, \Phi_{i0}) := \int_{\bar{h}^{(-)}(x)}^{\bar{h}^{(+)}(x)} (u_i, \Phi_i) dx_3, \quad i, j = 1, 2, 3,$$

$Q_{n i}^{(+)}$ and $Q_{n i}^{(-)}$ ($i = 1, 2, 3$) are components of the stress vectors acting on the upper and lower face surfaces with normals $\bar{n}^{(+)}$ and $\bar{n}^{(-)}$, respectively.

We consider the cusped prismatic shell with the half thickness as follows

$$h = h_0 e^{-\kappa x_2}, \quad h_0 = \text{const} > 0, \quad \kappa = \text{const} \geq 0, \quad (3)$$

$$-\infty < x_1 < +\infty, \quad 0 \leq x_2 < +\infty.$$

The upper and lower surfaces of the plate under consideration are given in Fig.1. On Fig.2 and 3 the projection of the plate on the planes $0x_1x_3$ and profile are given, respectively.

2. Cylindrical Bending

Let us consider the case when all mechanical quantities depend on one space variable x_2 . System (2) can be rewritten in the vector form as follows:

$$v_{,22} - \alpha v_{,2} = f, \quad (4)$$

Fig. 1 The upper and lower surfaces of the plate.

Fig. 2 The projection of the plate on $\alpha_1 x_3$.

Fig. 3 The profile of the plate on $\alpha_2 x_3$.

where

$$\begin{aligned} v &:= (v_{10}, v_{20}, v_{30})^T, \\ f &= (f_{10}, f_{20}, f_{30})^T, \\ f_{10} &:= -\frac{1}{\mu h_0} F_{10} e^{\alpha x_2}, \\ f_{20} &= -\frac{1}{(\lambda + 2\mu) h_0} F_{20} e^{\alpha x_2}, \\ f_{30} &= -\frac{1}{\mu h_0} F_{30} e^{\alpha x_2}. \end{aligned}$$

Problem 1. Find a solution of (4) under the following conditions

$$v(0) = v^0, \quad v^0 = \text{const},$$

$$|v(x_2)| = O(1), \quad x_2 \rightarrow \infty, \quad (5)$$

$$v \in C^2([0; +\infty[) \cup C^1([0; +\infty[).$$

The solution of Problem 1 has a form

$$\begin{aligned} v(x_2) &= -\frac{e^{\alpha x_2}}{\alpha} \int_{x_2}^{\infty} f d\xi \\ &- \frac{1}{\alpha} \int_0^{x_2} f e^{\alpha \xi} d\xi + \frac{1}{\alpha} \int_0^{\infty} f d\xi + v^0. \end{aligned}$$

The sufficient conditions for fulfilment of (5) are

$$\begin{aligned} f(x_2) &= \frac{B(x_2)}{(x_2 + c)^\beta} e^{-\alpha x_2}, \\ \beta &> 1, \quad c > 0, \quad B(x_2) \in C([0; +\infty)). \end{aligned}$$

3. Bending Problem

In view of (16) equation (2) can be rewritten as follows

$$\mu e^{-\kappa x_2} (v_{30,11} + v_{30,22} - \kappa v_{30,2}) + F_{30} = 0, \quad (6)$$

where

$$F_{30} = \Phi_{30} + (Q_{(+)} + Q_{(-)}) \sqrt{(h_{,1})^2 + (h_{,2})^2 + 1}.$$

Let us consider the following problem:

Problem. Find a solution of equation (6) under the following boundary conditions

$$v_{30}(x_1^0, x_2^0) = 0, \quad x := (x_1^0, x_2^0) \in \partial\omega, \quad (7)$$

and condition at infinity

$$v_{30} = O(e^{\kappa x_2}), \quad (8)$$

$$x_2 \rightarrow +\infty,$$

where

$$v_{30} \in C^2(\omega) \cap C(\bar{\omega}), \quad F_{30} \in C([0, \ell]),$$

$$\omega := \{(x_1, x_2) : -\infty < x_1 < +\infty; 0 < x_2 < +\infty\}.$$

Let us rewrite equation (6) in the following form

$$v_{30,11} + v_{30,22} - \kappa v_{30,2} = F, \quad (9)$$

where

$$F := -\frac{1}{\mu} F_{30} e^{\kappa x_2}, \quad \kappa > 0.$$

After introducing new independent variables $z := x_1 + ix_2$ and $\zeta := x_1 - ix_2$, equation (9) and boundary conditions (7) can be rewritten in the complex form as follows

$$\begin{aligned} L(U(z, \zeta)) &= \frac{\partial^2 U(z, \zeta)}{\partial z \partial \zeta} + A \frac{\partial U(z, \zeta)}{\partial z} \\ &+ B \frac{\partial U(z, \zeta)}{\partial \zeta} - F_1(z, \zeta) = 0, \end{aligned} \quad (10)$$

$$\begin{aligned} U(z, \zeta_0) &= 0, \quad U(z_0, \zeta) = 0, \\ z_0 &:= x_1^0 + ix_2^0, \quad \zeta_0 := x_1^0 - ix_2^0, \end{aligned} \quad (11)$$

where

$$F_1(z, \zeta) := \frac{1}{4} F\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right).$$

and

$$A := -i \frac{\kappa}{4}, \quad B := i \frac{\kappa}{4}$$

$$U(z, \zeta) := v_{30}\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right)$$

are analytic in some domain $(D^*, \bar{D}^*)$, where $z \in D^*$, $\zeta \in \bar{D}^*$.

The conjugate homogeneous equation of equation (10) has the form

$$\begin{aligned} L^*(V(z, \zeta)) &= \frac{\partial^2 V(z, \zeta)}{\partial z \partial \zeta} \\ &- A \frac{\partial V(z, \zeta)}{\partial z} - B \frac{\partial V(z, \zeta)}{\partial \zeta} = 0. \end{aligned} \quad (12)$$

As usual, under the symbols $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \zeta}$ we assume

$$\frac{\partial}{\partial z} := \frac{1}{2} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right), \quad \frac{\partial}{\partial \zeta} := \frac{1}{2} \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right).$$

It can be proved that the solution of equation (12) has the following form

$$V := R(z, \zeta; t, \tau) = e^{-i \frac{\kappa}{2}(\zeta - \tau) + i \frac{\kappa}{2}(z - t)}.$$

Using the last expression, the solution of equation (10) can be written as follows ([1], [19])

$$U(z, \zeta) = \int_{z_0}^z \int_{\zeta_0}^{\zeta} e^{i \frac{\kappa}{2}(\zeta - \tau) - i \frac{\kappa}{2}(z - t)} F_1(t, \tau) d\tau dt, \quad (13)$$

where

$$z_0 := x_1^0 + ix_2^0, \quad z_0 := x_1^0 - ix_2^0, \quad (x_1^0, x_2^0) \in \partial\omega.$$

Let at first $F_1(t, \tau) \equiv 1$. From (13) we have

$$\begin{aligned} U(z, \zeta) &= \int_{z_0}^z e^{-i \frac{\kappa}{2}(z-t)} dt \int_{\zeta_0}^{\zeta} e^{i \frac{\kappa}{2}(\zeta-\tau)} d\tau \\ &= \frac{4}{\kappa^2} [1 - e^{-i \frac{\kappa}{2}(z-z_0)}] [1 - e^{i \frac{\kappa}{2}(\zeta-\zeta_0)}]. \end{aligned} \quad (14)$$

Thus, by virtue of (14) we have the following estimate

$$|v_{30}(x_1, x_2)| \leq \frac{2}{\kappa^2} [1 + 2e^{\frac{\kappa}{2}(x_2 - x_2^0)} + e^{\kappa(x_2 - x_2^0)}]. \quad (15)$$

So, from (15) we get

$$v_{30}(x) = O(e^{\kappa x_2}), \text{ when } x_2 \rightarrow +\infty.$$

Let $F_1(t, \tau)$ be an arbitrary continuous bounded function ($|F_1(t, \tau)| \leq M = \text{const} < \infty$), consequently

$$\begin{aligned} |U(z, \zeta)| &= \left| \int_{z_0}^z \int_{\zeta_0}^{\zeta} e^{i\frac{\kappa}{2}(\zeta - \tau) - i\frac{\kappa}{2}(z - t)} \cdot F_1(t, \tau) d\tau dt \right| \\ &\leq \int_{z_0}^z \int_{\zeta_0}^{\zeta} |e^{i\frac{\kappa}{2}(\zeta - \tau) - i\frac{\kappa}{2}(z - t)}| |F_1(t, \tau)| d\tau dt \\ &\leq M \cdot \int_{z_0}^z \int_{\zeta_0}^{\zeta} e^{i\frac{\kappa}{2}(\zeta - \tau) - i\frac{\kappa}{2}(z - t)} d\tau dt. \end{aligned}$$

Therefore, (13) is a solution of the setting problem.

4. Conclusions

Bending problem of the cusped symmetric prismatic shells (plates) with the half thickness as follows

$$h = h_0 e^{-\kappa x_2}, \quad (16)$$

$$h_0 = \text{const} > 0, \quad \kappa = \text{const} \geq 0,$$

$$-\infty < x_1 < +\infty, \quad 0 \leq x_2 < +\infty.$$

is considered. The problem mathematically is reduced to the Dirichlet boundary value problem for elliptic type partial differential equations on half-plane. Using complex representations the solution of the problem under consideration is constructed in the integral form.

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