Asymmetric Multi-channel Sampling in a Series of Shift Invariant Spaces

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Abstract: We show asymmetric multi-channel sampling on a series of a shift invariant spaces $\sum_{d=1}^{m}V(\varphi(t_d))$ with a series of Riesz generators $\sum_{d=1}^{m}\varphi(t_d)$ in $L^2(\mathbb{R})$, where each channeled signal is assigned a uniform but distinct sampling rate. We use Fourier duality between $\sum_{d=1}^{m}V(\varphi(t_d))$ and $L^2[0,2\pi]$ to find conditions under which there is a stable asymmetric multi-channel sampling formula on $\sum_{d=1}^{m}V(\varphi(t_d))$.

Keywords: Shift invariant space, Multi-channel sampling, Frame Riesz basis

1. Introduction

The multi-channel sampling method goes back to the works of Shannon [18] and Fogel [7], where reconstruction of a band-limited signal from samples of the signal and its derivatives was found. Generalized sampling expansion using arbitrary multi-channel sampling on the Paley–Wiener space was introduced first by Papoulis [16]. Since Papoulis’ fundamental work, there have been many generalizations and applications of multi-channel sampling. See [1,5,6,14,17,19].

Papoulis’ result has also been extended to a general shift invariant space by using the filter banks technique (see [4,19,20]). More recently Garcia and Pérez-Villalon [8] derived stable generalized sampling in a shift invariant space. Most previous work related to multi-channel sampling has assumed that the sampling rates of all channels are the same.

S. Kang, J.M. Kim, K.H. Kwon [22] considered asymmetric multi-channel sampling in a shift invariant space $V(\varphi)$ with a suitable Riesz generator $\varphi(t)$, where each channeled signal is sampled with a uniform but distinct rate. Using Fourier duality between $\sum_{d=1}^{m}V(\varphi(t_d))$ and $L^2[0,2\pi]$ [8,9,10,22], we derive under the same considerations a stable series of shifted asymmetric multi-channel sampling formula in $\sum_{d=1}^{m}V(\varphi(t_d))$. The corresponding symmetric multi-channel sampling in $\sum_{d=1}^{m}V(\varphi(t_d))$ was handled in [9],[22], where $\sum_{d=1}^{m}\varphi(t_d)$ is a continuous series of Riesz generators satisfying

$$\sup_{\mathbb{R}}\sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}|\varphi(t_d - n)|^2 < \infty.$$ 

In this case all signals in $\sum_{d=1}^{m}V(\varphi(t_d))$ are continuous on $\mathbb{R}$ [21],[22]. We require only that the series of Riesz generators $\sum_{d=1}^{m}\varphi(t_d)$ are pointwise well defined everywhere on $\mathbb{R}$ and $\sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}|\varphi(t_d - n)|^2 < \infty$, $\sum_{d=1}^{m}\varphi(t_d) \in L^2(\mathbb{R})$. Hence we essentially allow any series of Riesz generators in $L^2(\mathbb{R})$. Hence [22] allow more general filters than the ones in [8] by asking only that the impulse responses of filters belong to $L^2(\mathbb{R})$ (or the frequency responses of filters belong to $L^2(\mathbb{R}) \cup L^\infty(\mathbb{R})$ when $\sum_{n\in\mathbb{Z}}|\hat{\varphi}(\xi + 2\pi n)| \in L^2[0,2\pi]$), whereas they belong to $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ in [9]. We give an illustrative examples (see[22]).

2. Preliminaries

We consider the notations and formulas in [22]. The
normalized Fourier transform is

\[ \mathcal{F}[\varphi](\xi) = \hat{\varphi}(\xi) \]

\[ = \int_{-\infty}^{\infty} \sum_{d=1}^{m} \varphi(t_d) \prod_{d=1}^{m} e^{-it_d \xi} \, dt_d \sum_{d=1}^{m} \varphi(t_d) \]

\[ \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \]

so that \( \frac{1}{2\pi} \mathcal{F} [ \cdot ] \) extends to a unitary operator from \( L^2(\mathbb{R}) \) onto \( L^2(\mathbb{R}) \). For any \( \sum_{d=1}^{m} \varphi(t_d) \in L^2(\mathbb{R}) \), let

\[ \sum_{d=1}^{m} C_{\varphi}(t_d) = \sum_{n \in \mathbb{Z}} \left( \varphi(t_d + n) \right)^2 \text{ and } G_{\varphi}(\xi) \]

\[ = \sum_{n \in \mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)|^2. \]

Then

\[ \sum_{d=1}^{m} C_{\varphi}(t_d) = \sum_{d=1}^{m} C_{\varphi}(t_d + 1) \in L^1[0, 1], G_{\varphi}(\xi) \]

\[ = G_{\varphi}(\xi + 2\pi) \in L^2[0, 2\pi] \]

and

\[ \left\| \sum_{d=1}^{m} \varphi(t_d) \right\|_{L^2(\mathbb{R})}^2 = \left\| \sum_{d=1}^{m} C_{\varphi}(t_d) \right\|_{L^1[0, 1]} \]

\[ = \frac{1}{2\pi} \left\| G_{\varphi}(\xi) \right\|_{L^1[0, 2\pi]}. \]

In particular, \( \sum_{d=1}^{m} C_{\varphi}(t_d) < \infty \) for a.e. \( \sum_{d=1}^{m} t_d \in \mathbb{R} \). We also let

\[ \sum_{d=1}^{m} Z_{\varphi}(t_d, \xi) = \sum_{n \in \mathbb{Z}} \varphi(t_d + n)e^{-in\xi} \]

be the Zak transform [12] of \( \sum_{d=1}^{m} \varphi(t_d) \) in \( L^2(\mathbb{R}) \). Then \( \sum_{d=1}^{m} Z_{\varphi}(t_d, \xi) \) is well defined a.e. on \( \mathbb{R}^2 \) and is quasi-periodic in the sense that

\[ \sum_{d=1}^{m} Z_{\varphi}(t_d + 1, \xi) = e^{i\xi} \sum_{d=1}^{m} Z_{\varphi}(t_d, \xi) \]

and

\[ \sum_{d=1}^{m} Z_{\varphi}(t_d, \xi + 2\pi) = \sum_{d=1}^{m} Z_{\varphi}(t_d, \xi). \]

A Hilbert space \( H \) consisting of complex valued functions on a set \( E \) is called a reproducing kernel Hilbert space (RKHS in short) if there is a series of a functions \( \sum_{d=1}^{m} \varphi(s_d, t_d) \) on \( E \times E \), called the reproducing kernel of \( H \), satisfying

(i) \( \sum_{d=1}^{m} \varphi(., t_d) \in H \) for each \( \sum_{d=1}^{m} t_d \in E \),

(ii) \( \sum_{d=1}^{m} \varphi(s_d, q(s_d, t_d)) = \sum_{d=1}^{m} \varphi(t_d), \varphi \in H \).

In an RKHS \( H \), any norm converging sequence also converges uniformly on any subset of \( E \), on which \( \| \sum_{d=1}^{m} \varphi(., t_d) \|^2_H = \sum_{d=1}^{m} \varphi(t_d, t_d) \) is bounded.

A sequence \( \{ \varphi_n : n \in \mathbb{Z} \} \) of vectors in a separable Hilbert space \( H \) is

(i) a Bessel sequence with a bound \( A + \varepsilon : \varepsilon > 0 \) if

\[ \sum_{n \in \mathbb{Z}} |(\varphi, \varphi_n)|^2 \leq (A + \varepsilon) \| \varphi \|^2, \varphi \in H, \varepsilon > 0, \]

(ii) a frame of \( H \) with bounds \( A + \varepsilon \geq A : \varepsilon > 0 \) if

\[ A \| \varphi \|^2 \leq \sum_{n \in \mathbb{Z}} |(\varphi, \varphi_n)|^2 \leq (A + \varepsilon) \| \varphi \|^2, \varphi \in H, \varepsilon \]

\[ > 0, \]

(iii) a Riesz basis of \( H \) with bounds \( A + \varepsilon \geq A : \varepsilon > 0 \) if it is complete in \( H \) and

\[ A \| c \|^2 \leq \sum_{n \in \mathbb{Z}} |c(\varphi_n)|^2 \leq (A + \varepsilon) \| c \|^2, c \]

where \( \| c \|^2 = \sum_{n \in \mathbb{Z}} |c(\varphi_n)|^2 \).

We let \( \sum_{d=1}^{m} V(\varphi(t_d)) \) be the series of the shift invariant spaces, where \( \sum_{d=1}^{m} \varphi(t_d) \) is a series of a Riesz generators, that is, \( \{\sum_{d=1}^{m} \varphi(t_d - n) : n \in \mathbb{Z}\} \) is a series of a Riesz bases of \( \sum_{d=1}^{m} V(\varphi(t_d)) \). Then

\[ \sum_{d=1}^{m} V(\varphi(t_d)) = \left\{ \sum_{d=1}^{m} (c \ast \varphi)(t_d) \right\} = \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} c(\varphi_n)(\varphi(t_d - n) : c \}

\[ = \{c(\varphi_n) : n \in \mathbb{Z}\} \in l^2 \right\}.
It is well known see [2] that $\sum_{d=1}^{m} \varphi(t_d)$ is a series of a Riesz generators if and only if there are constant $A$ such that $A \leq C_\varphi(\xi) \leq A + \varepsilon$ a.e. on $[0, 2\pi]$. In this case, $(\sum_{d=1}^{m} \varphi(t_d - n)) : n \in \mathbb{Z}$ is a series of a Riesz bases of $\sum_{d=1}^{m} V(\varphi(t_d))$ with bound $\varepsilon > 0$. We assume further that

(i) $\sum_{d=1}^{m} \varphi(t_d)$ is everywhere well defined on $\mathbb{R}$;

(ii) $\sum_{d=1}^{m} C_\varphi(t_d) < \infty$, $\sum_{d=1}^{m} t_d \in \mathbb{R}$, i.e., $(\sum_{d=1}^{m} \varphi(t_d - n)) : n \in \mathbb{Z}$ is 2-periodic for each $d = 1, m \in \mathbb{N}$.

We then formally all series of Riesz generators since for any $\sum_{d=1}^{m} \varphi(t_d) \in L^2(\mathbb{R})$, $\sum_{d=1}^{m} C_\varphi(t_d) < \infty$ a.e. so that $\sum_{d=1}^{m} \varphi(t_d)$ has an equivalent representative satisfying the above two conditions. Then for each $c \in l^2$, $\sum_{d=1}^{m} (c * \varphi)(t_d)$ converges both in $L^2(\mathbb{R})$ and absolutely for each $\sum_{d=1}^{m} t_d \in \mathbb{R}$. Hence $\sum_{d=1}^{m} V(\varphi(t_d))$ becomes an RKHS with the reproducing kernel (see [13])

$$
\sum_{d=1}^{m} q(s_d, t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} \varphi(s_d - n) \varphi(t_d - n), \text{where} \left\{ \sum_{d=1}^{m} \varphi(t_d) - n \right\} : n \in \mathbb{Z}
$$

is the series of the dual Riesz bases of $(\sum_{d=1}^{m} \varphi(t_d - n)) : n \in \mathbb{Z}$ with bounds for $\varepsilon > 0$. As in [9,10], we introduce an isomorphism $J$ from $L^2[0, 2\pi]$ onto $\sum_{d=1}^{m} V(\varphi(t_d))$ defined as:

$$
\sum_{d=1}^{m} (J F)(t_d)
$$

$$
= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} \langle F(\xi), e^{-i n \xi} \rangle_{L^2[0,2\pi]} \varphi(t_d - n)
$$

$$
= \sum_{d=1}^{m} \langle F(\xi), \frac{1}{2\pi} Z_\varphi(t_d, \xi) \rangle_{L^2[0,2\pi]}. \tag{1}
$$

We then have:

(i) $(J F)(\xi) = F(\xi) \tilde{\varphi}(\xi)$

(ii) $J(F(\xi)e^{-i n \xi}) = \sum_{d=1}^{m} (J F)(t_d - n), n \in \mathbb{Z}$.

3. Asymmetric Multi-channel Sampling

The aim of this paper is as follows (see [22]). Let $L(1+\varepsilon_1) : \varepsilon_1 \geq 0$ be $N$ LTI (linear time-invariant) systems with impulse responses $\{L_{d=1}^{m} L(1+\varepsilon_1)(t_d) : \varepsilon_1 \geq 0\}$. Develop a stable series of shifted multi-channel sampling formula for any signal $\sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d))$ using discrete sample values from $\sum_{d=1}^{m} L(1+\varepsilon_1)(t_d) : \varepsilon_1 \geq 0\}$, where each channelled signal $\sum_{d=1}^{m} L(1+\varepsilon_1)[f](t_d)$ for $\varepsilon_1 \geq 0$ is assigned with a distinct sampling rate

$$
\sum_{d=1}^{m} f(t_d) = \sum_{\varepsilon_1 = 0}^{N} \sum_{\varepsilon_1 = 0}^{m} \sum_{\varepsilon_1 = 0}^{m} L(1+\varepsilon_1)[f](\sigma_{1+\varepsilon_1})
$$

$$
+ \sum_{\varepsilon_1 = 0}^{N} \sum_{\varepsilon_1 = 0}^{m} V(\varphi(t_d)) \tag{1}
$$

where $\{\sum_{d=1}^{m} s_{d(1+\varepsilon_1)}(t_d) : \varepsilon_1 \geq 0, n \in \mathbb{Z}\}$ is a series of frames or a Riesz basis of $\sum_{d=1}^{m} V(\varphi(t_d))$, $\{1 + \varepsilon_1(1+\varepsilon_1) : \varepsilon_1 \geq 0\}$ are positive integers, and $\{\sigma_{1+\varepsilon_1} : \varepsilon_1 \geq 0\}$ are real constants. Note that the series of shifting of sampling instants is unavoidable in some uniform sampling [12] and arises naturally when we allow rational sampling periods in (1). Here, we assume that each $L(1+\varepsilon_1)$ is one of the following three types: the impulse response $\sum_{d=1}^{m} l(t_d)$ of an LTI system is such that

(i) $\sum_{d=1}^{m} l(t_d) = \sum_{d=1}^{m} \delta(t_d + a), a \in \mathbb{R}$ or

(ii) $\sum_{d=1}^{m} l(t_d) \in L^2(\mathbb{R})$ or

(iii) $\tilde{l}(\xi) \in L^\alpha(\mathbb{R}) \cup L^2(\mathbb{R})$ when $H_\varphi(\xi) = \sum_{n \in \mathbb{Z}} |\tilde{\varphi}(\xi + 2n\pi)| L^2[0, 2\pi]$. For type (i),

$$
\sum_{d=1}^{m} [f](t_d) = \sum_{d=1}^{m} f(t_d + a), f \in L^2(\mathbb{R})
$$

so that $L(1+\varepsilon_1) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is an isomorphism. In
particular, for any
\[
\sum_{d=1}^{m} f(t_d) = \sum_{d=1}^{m} (c \ast \varphi)(t_d) \in \sum_{d=1}^{m} \mathcal{V}(\varphi(t_d)),
\]
\[
\sum_{d=1}^{m} L[f](t_d) = \sum_{d=1}^{m} (c \ast \psi)(t_d)
\]
converges absolutely on \( \mathbb{R} \) since
\[
\sum_{d=1}^{m} C_{\psi}(t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} |\psi(t_d + n)|^2 < \infty, \sum_{d=1}^{m} t_d
\in \mathbb{R} , \text{where}
\]
\[
\sum_{d=1}^{m} \psi(t_d) = \sum_{d=1}^{m} L[\varphi](t_d) = \sum_{d=1}^{m} \varphi(t_d + a)
\]
For types (ii) and (iii), we have the following results (see [22]):

**Lemma 3.1.** Let \( L[\cdot] \) be an LTI system with the impulse response \( \sum_{d=1}^{m} l(t_d) \) of the type (ii) or (iii) as above and
\[
\sum_{d=1}^{m} \psi(t_d) = \sum_{d=1}^{m} L[\varphi](t_d) = \sum_{d=1}^{m} \varphi(l(t_d))
\]
Then
(a) \( \sum_{d=1}^{m} \psi(t_d) \in \mathcal{C}_{c}(\mathbb{R}) = \{ \sum_{d=1}^{m} u(t_d) \in \mathcal{C}(\mathbb{R}) \mid \lim_{d \rightarrow \infty} u(t_d) \}=0 \),
(b) \( \sup_{\mathbb{R}} \sum_{d=1}^{m} C_{\psi}(t_d) < \infty \);  
(c) for any \( \sum_{d=1}^{m} f(t_d) = \sum_{d=1}^{m} (c \ast \varphi)(t_d) \in \mathcal{S}_{c}(\mathbb{R}) \),
\( \sum_{d=1}^{m} L[f](t_d) = \sum_{d=1}^{m} (c \ast \psi)(t_d) \) converges absolutely and uniformly on \( \mathbb{R} \).

Hence \( \sum_{d=1}^{m} L[f](t_d) \in \mathcal{C}(\mathbb{R}) \).

Proof. First assume \( \sum_{d=1}^{m} l(t_d) \in \mathcal{L}^{2}(\mathbb{R}) \). Then \( \sum_{d=1}^{m} \psi(t_d) \in \mathcal{C}_{c}(\mathbb{R}) \) by the Riemann–Lebesgue lemma since \( \hat{\psi}(\xi) \supseteq \hat{\varphi}(\xi) \hat{\varphi}(\xi) \in \mathcal{L}^{1}(\mathbb{R}) \). Since
\[
\sum_{n \in \mathbb{Z}} \hat{\psi}(\xi + 2n\pi) \leq G_{\varphi}(\xi) \hat{\varphi}(\xi) \hat{\varphi}(\xi) \hat{\varphi}(\xi)
\]
\[
\left\| \sum_{n \in \mathbb{Z}} \hat{\psi}(\xi + 2n\pi) \right\|_{\mathcal{L}^{2}(\mathbb{R})} \leq \int_{0}^{2\pi} G_{\varphi}(\xi) \hat{\varphi}(\xi) d\xi
\]
\[
\leq 2\pi\left\| G_{\varphi}(\xi) \right\|_{\mathcal{L}^{2}(\mathbb{R})} \left\| \hat{\varphi}(\xi) \right\|_{\mathcal{L}^{2}(\mathbb{R})}^{2}
\]
Thus for any \( \sum_{d=1}^{m} t_d \in \mathbb{R} \), we have by the Poisson summation formula (see [13])
\[
\sum_{n \in \mathbb{Z}} \hat{\psi}(\xi + 2n\pi) \left\| \sum_{d=1}^{m} e^{it_d(\xi + 2n\pi)} \right\|_{\mathcal{L}^{2}(\mathbb{R})} \leq \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} \psi(t_d + n) \leq \sum_{d=1}^{m} e^{-in\xi} \in \mathcal{L}^{2}[0,2\pi]
\]
Therefore for any \( \sum_{d=1}^{m} t_d \in \mathbb{R} \)
\[
\sum_{d=1}^{m} C_{\varphi}(t_d) = \sum_{d=1}^{m} \left\| \sum_{n \in \mathbb{Z}} \psi(t_d + n) \right\|_{\mathcal{L}^{2}(\mathbb{R})} \leq \sum_{d=1}^{m} \left\| \sum_{n \in \mathbb{Z}} \psi(t_d + n) \right\|_{\mathcal{L}^{2}(\mathbb{R})}^{2}
\]
By Young’s inequality on the convolution product,
\( \|L[f]\|_{\mathcal{L}^{p}(\mathbb{R})} \leq \|f\|_{\mathcal{L}^{p}(\mathbb{R})} \|L\|_{\mathcal{L}^{q}(\mathbb{R})} \), so that \( L[\cdot] : \mathcal{L}^{2}(\mathbb{R}) \rightarrow \mathcal{L}^{\infty}(\mathbb{R}) \) is a bounded linear operator. Hence for any
\[
\sum_{d=1}^{m} f(t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} c(n) \varphi(t_d - n)
\]
\( \leq \sum_{d=1}^{m} c(n) \varphi(t_d - n) \)
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which converges absolutely and uniformly on $R$ by (b). Now assume that $H_\varphi (\xi ) \in L^2 [0,2\pi ]$. The case $\hat{l}(\xi ) \in L^2 (\mathbb{R})$ is reduced to type (ii). So let $\hat{l}(\xi ) \in L^2 (\mathbb{R})$. Then $\hat{\varphi}(\xi ) \in L^2 (\mathbb{R}) \cap L^1 (\mathbb{R})$ so that $\hat{\psi}(\xi ) = \hat{\varphi}(\xi ) \hat{l}(\xi ) \in L^2 (\mathbb{R}) \cap L^1 (\mathbb{R})$ and so $\psi(\xi ) \in C_c(\mathbb{R}) \cap L^2 (\mathbb{R})$. Since
\[
\sum_{n \in \mathbb{Z}} | \hat{\psi}(\xi + 2\pi n)| \leq \|l\|_{L^\infty (\mathbb{R})} \|H_\varphi (\xi )\|_{L^2 [0,2\pi ]},
\]
we have again by the Poisson summation formula
\[
\sum_{d=1}^{m} C_d (t_d) = \frac{1}{2\pi} \left( \sum_{n \in \mathbb{Z}} \hat{\psi}(\xi + 2\pi n) \right) \Bigg|_{\xi = l(t_d + 2\pi n)} \leq \frac{1}{2\pi} \|l\|_{L^\infty (\mathbb{R})} \|H_\varphi (\xi )\|_{L^2 [0,2\pi ]},
\]
so that $\sup \sum_{d=1}^{m} C_d (t_d) < \infty$. For any $f \in L^2 (\mathbb{R})$,
\[
\sum_{d=1}^{m} \|L[f] (t_d)\|_{L^2 (\mathbb{R})} = \|f \ast l\|_{L^\infty (\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}} \|\hat{\psi}(\xi) \hat{l}(\xi)\|_{L^2 (\mathbb{R})} \leq \|\hat{l}\|_{L^\infty (\mathbb{R})} \|f\|_{L^2 (\mathbb{R})}.
\]
Hence $L[\cdot ] : L^2 (\mathbb{R}) \to L^2 (\mathbb{R})$ is a bounded linear operator so that for any
\[
\sum_{d=1}^{m} f(t_d) = \sum_{d=1}^{m} (\epsilon * \varphi)(t_d) \in \sum_{d=1}^{m} L[V(\varphi(t_d))] \subset \sum_{d=1}^{m} L[f](t_d) = \sum_{d=1}^{m} (\epsilon * \psi t_d) \text{ converges in } L^2 (\mathbb{R}).
\]
By (b), $d^m \epsilon \psi t_d$ also converges absolutely and uniformly on $\mathbb{R}$. By Lemma 3.1(b), $\sum_{d=1}^{m} C_d (t_d) \in L^2 (\mathbb{R})$. However, $\sum_{d=1}^{m} (\epsilon * \psi)(t_d)$ may not converge in $L^2 (\mathbb{R})$ unless $\{\sum_{d=1}^{m} \psi(t_d - n) : n \in \mathbb{Z}\}$ is a Bessel sequence.

Lemma 3.1(b) improves Lemma 1 in [9], in which the proof uses $\sum_{d=1}^{m} l(t_d) \in L^2 (\mathbb{R}) \cap L^1 (\mathbb{R})$, $\sup \sum_{d=1}^{m} C_d (t_d) < \infty$, and the integral version of Minkowski inequality. Note that the condition $H_\varphi (\xi ) \in L^2 [0,2\pi ]$ implies $\sum_{d=1}^{m} \varphi(t_d) \in L^2 (\mathbb{R}) \cap C_c(\mathbb{R})$ and $\sup \sum_{d=1}^{m} C_d (t_d) < \infty$. (see [13]). Note also that $H_\varphi (\xi ) \in L^2 [0,2\pi ]$ if $\varphi(\xi ) = 0(1 + |\xi |)^{-(1+\epsilon_2)}(1 + \epsilon_2)_{(1+\epsilon_2)} > 1, \epsilon_1 \geq 0$, which holds e.g. for $\sum_{d=1}^{m} \varphi_n(t_d) = \sum_{d=1}^{m} \varphi_n (\psi_n(t_d))$ the cardinal B-spline of degree $n (\geq 1)$, where
\[
\varphi_0 = \sum_{d=1}^{m} \chi_{[0,1]}(t_d).
\]
We have as a consequence of Lemma 3.1: Let $L[\cdot ]$ be an LTI system with impulse response $\sum_{d=1}^{m} l(t_d)$ of type (i) or (ii) or (iii) as above and $\sum_{d=1}^{m} \psi(t_d) = \sum_{d=1}^{m} L[\varphi](t_d)$. Then for any
\[
\sum_{d=1}^{m} f(t_d) = \sum_{d=1}^{m} (\mathcal{F} \varphi(t_d)) = \sum_{d=1}^{m} V(\varphi(t_d)), F(\xi ) \in L^2 [0,2\pi ]
\]
we have
\[
\sum_{d=1}^{m} L[f](t_d) = \sum_{d=1}^{m} \left( \langle \xi , \frac{1}{2\pi} Z_\varphi (\xi) \rangle \right) \left( 1 + \epsilon_1 \right), \epsilon_1 \geq 0.
\]
Then we have by (2)
\[
L_{(1+\epsilon_1)} [f](\sigma(1+\epsilon_1) + (1 + \epsilon_2)(1+\epsilon_1)n) = (F(\xi ), \frac{1}{2\pi} Z_\varphi (\xi) \sigma(1+\epsilon_1) + (1 + \epsilon_2)(1+\epsilon_1)n) \in L^2 [0,2\pi ]
\]
for any $\sum_{d=1}^{m} f(t_d) = \sum_{d=1}^{m} (\mathcal{F} \varphi(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d))$ and $\epsilon_1 \geq 0$. Then by (3) and the isomorphism $\mathcal{F}$ from $L^2 [0,2\pi ]$ onto $\sum_{d=1}^{m} V(\varphi(t_d))$, the sampling expansion (1) is equivalent to
\[
F(\xi ) = \sum_{\epsilon_1 = 0}^{N} \left( F(\xi ), \mathcal{G}_{(1+\epsilon_1)}(\xi) e^{-i(1+\epsilon_2)(1+\epsilon_1)n}\right) L^2 [0,2\pi ] S_{(1+\epsilon_1),n}(\xi),
\]
Let $F(\xi) \in L^2[0,2\pi]$, where $\{S_{(1+\varepsilon_i),n}(\xi) : \varepsilon_i \geq 0, n \in \mathbb{Z}\}$ is a series of frames or a Riesz basis of $L^2[0,2\pi]$. This observation leads us to consider the problem when is $\{g_{(1+\varepsilon_i)}(\xi)e^{-(1+\varepsilon_i)(1+\varepsilon_i)n\xi} : \varepsilon_i \geq 0, n \in \mathbb{Z}\}$ a series of frames or a Riesz basis of $L^2[0,2\pi]$. Note that
\[
\left\{ g_{(1+\varepsilon_i)}(\xi)e^{-(1+\varepsilon_i)(1+\varepsilon_i)n\xi} : \varepsilon_i \geq 0, n \in \mathbb{Z}\right\} =
\left\{ g_{(1+\varepsilon_i),m_{(1+\varepsilon_i)}}(\xi)e^{-(1+\varepsilon_i)n\xi} : \varepsilon_i \geq 0, 1 \leq m_{(1+\varepsilon_i)} \leq \frac{(1+\varepsilon_i)}{(1+\varepsilon)^2(n_{(1+\varepsilon)})}, n \in \mathbb{Z}\right\}
\]
where $(1+\varepsilon_i) = lcm((1+\varepsilon_i),(1+\varepsilon_i)) : \varepsilon_i \geq 0$ and
\[
g_{(1+\varepsilon_i),m_{(1+\varepsilon_i)}}(\xi) = g_{(1+\varepsilon_i)}(\xi)e^{(1+\varepsilon_i)(m_{(1+\varepsilon_i)})-1}\xi}
\]
for $\varepsilon_i \geq 0$. Let $D$ be the unitary operator from $L^2[0,2\pi]$ onto $L^2(I(1+\varepsilon_i))$, where $=[0, \frac{2\pi}{(1+\varepsilon_i)}]$, defined by
\[
DF = \left[F(\xi + (k-1)\frac{2\pi}{(1+\varepsilon_i)})_{k=1}^{(1+\varepsilon_i)}\right], F(\xi) \in L^2[0,2\pi].
\]
We also let $G(\xi) = \begin{bmatrix} Dg_{1,1}(\xi), & \cdots, & Dg_{1,(1+\varepsilon_i)}(\xi), & \cdots, & Dg_{N,1}(\xi), & \cdots, & Dg_{N,(1+\varepsilon_i)}(\xi) \end{bmatrix}$

Thus $\lambda_m(\xi), \lambda_M(\xi)$ be the smallest and the largest eigenvalues of the positive semi-definite $(1+\varepsilon_i) \times (1+\varepsilon_i)$ matrix $G(\xi) * G(\xi)^T$, respectively.

**Lemma 3.2.** Let $\alpha_G = \|\lambda_m(\xi)\|_0$ and $\beta_G = \|\lambda_M(\xi)\|_\infty$ be the essential infimum of $\lambda_m(\xi)$ and the essential supremum of $\lambda_M(\xi)$ respectively. Then $\{g_{(1+\varepsilon_i)}(\xi)e^{-(1+\varepsilon_i)(1+\varepsilon_i)n\xi} : \varepsilon_i \geq 0, n \in \mathbb{Z}\}$ is
(a) a Bessel sequence in $L^2[0,2\pi]$ if and only if $\beta_G < \infty$ or equivalently $\{Z_{\theta_{(1+\varepsilon_i)}}(\sigma_{(1+\varepsilon_i)}) : \varepsilon_i \geq 0\} \in L^\infty[0,2\pi]$, (b) a frame of $L^2[0,2\pi]$ if and only if $0 < \alpha_G \leq \beta_G < \infty$.

(c) a Riesz basis of $L^2[0,2\pi]$ if and only if $0 < \alpha_G \leq \beta_G < \infty$ and
\[
\sum_{\varepsilon_i=0}^{N} \frac{(1+\varepsilon_2)}{(1+\varepsilon)^2(n_{(1+\varepsilon)})} = 1.
\]

**Proof.** Since $\{g_{(1+\varepsilon_i)}(\xi)e^{-(1+\varepsilon_i)(1+\varepsilon_i)n\xi} : \varepsilon_i \geq 0, n \in \mathbb{Z}\}$ is a Bessel sequence or a series of frames or a Riesz basis of $L^2[0,2\pi]$ if and only if
\[
\left\{ g_{(1+\varepsilon_i),m_{(1+\varepsilon_i)}}(\xi)e^{-(1+\varepsilon_i)n\xi} : \varepsilon_i \geq 0, 1 \leq m_{(1+\varepsilon_i)} \leq \frac{(1+\varepsilon_i)}{(1+\varepsilon)^2(n_{(1+\varepsilon)})}, n \in \mathbb{Z}\right\}
\]
is a Bessel sequence or a series of frames or a Riesz basis of $L^2[0,2\pi]$ respectively, all of the conclusions follow from Lemma 3 in [9].

Note that in [9], the authors use the Fourier transform
\[
\hat{f}(\xi) = \int_{-\infty}^{\infty} \sum_{d=1}^{m} f(d) e^{-2\pi i d\xi} dt_d
\]
so that they use $L^2[0,2\pi]$ instead of $L^2[0,2\pi]$. Assume that $0 < \alpha_G \leq \beta_G < \infty$ so that $\{g_{(1+\varepsilon_i)}(\xi)e^{-(1+\varepsilon_i)(1+\varepsilon_i)n\xi} : \varepsilon_i \geq 0, n \in \mathbb{Z}\}$ or equivalently
\[
\left\{ g_{(1+\varepsilon_i),m_{(1+\varepsilon_i)}}(\xi)e^{-(1+\varepsilon_i)n\xi} : \varepsilon_i \geq 0, 1 \leq m_{(1+\varepsilon_i)} \leq \frac{(1+\varepsilon_i)}{(1+\varepsilon)^2(n_{(1+\varepsilon)})}, n \in \mathbb{Z}\right\}
\]
is a series of frames of $L^2[0,2\pi]$. Then we can show easily (see in [9]) that
\[
\left\{ g_{(1+\varepsilon_i),m_{(1+\varepsilon_i)}}(\xi)e^{-(1+\varepsilon_i)n\xi} : \varepsilon_i \geq 0, 1 \leq m_{(1+\varepsilon_i)} \leq \frac{(1+\varepsilon_i)}{(1+\varepsilon)^2(n_{(1+\varepsilon)})}, n \in \mathbb{Z}\right\}
\]
has a series of dual frames of the form
\[
\left\{ S_{(1+\varepsilon_i),m_{(1+\varepsilon_i)}}(\xi)e^{-(1+\varepsilon_i)n\xi} : \varepsilon_i \geq 0, 1 \leq m_{(1+\varepsilon_i)} \leq \frac{(1+\varepsilon_i)}{(1+\varepsilon)^2(n_{(1+\varepsilon)})}, n \in \mathbb{Z}\right\}
\]
which is a series of frames expansion \( (22) \). We first discuss the sampling expansion \( (1) \), where frames is the pseudo-inverse of matrix with entries in \( \mathbb{C} \) is the identity matrix.

In particular, when we choose \( B(\xi) = 0 \) in \( (5) \), we have the canonical a series of dual frames of the frames

\[
\left\{ g_{(1+\epsilon_1),m(1+\epsilon_1)}(\xi)e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m(1+\epsilon_1) \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)} \right\},
\]

We are now ready to give the following results (see [22]). We first discuss the sampling expansion \( (1) \), which is a series of frames expansion in \( \sum_{d=1}^{m} V(\phi(t_d)) \).

**Theorem 3.3.** Let \( \alpha_G \) and \( \beta_G \) be the same as in Lemma 3.2. Assume \( \beta_G \) is finite. Then the following are all equivalent.

(a) There is a series of frames

\[
\sum_{d=1}^{m} s_{(1+\epsilon_1),m(1+\epsilon_1)}(t_d - (1+\epsilon_2)n) : \epsilon_1 \geq 0, 1 \leq m(1+\epsilon_1) \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)} , n \in \mathbb{Z} \}
\]

of \( \sum_{d=1}^{m} V(\phi(t_d)) \) for which

\[
\sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\phi(t_d)) \quad (6)
\]

(b) There is a series of frames

\[
\sum_{d=1}^{m} s_{(1+\epsilon_1),n}(t_d) : \epsilon_1 \geq 0, n \in \mathbb{Z} \}
\]

of \( \sum_{d=1}^{m} V(\phi(t_d)) \) for which

\[
\sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\phi(t_d)) . \quad (7)
\]

(c) \( 0 < \alpha_G \).

Proof. Assume \( \beta_G \) is finite. Then by Lemma 3.2

\[
\left\{ g_{(1+\epsilon_1),m(1+\epsilon_1)}(\xi)e^{-i((1+\epsilon_2)(1+\epsilon_1)n\xi)} : \epsilon_1 \geq 0, n \in \mathbb{Z} \right\}
\]

is a Bessel sequence in \( L^2[0,2\pi] \). First (a) implies (b) trivially. Assume (b). Applying the isomorphism \( J^{-1} \) to (7) gives by (3)

\[
F(\xi) = \sum_{m(1+\epsilon_1)}^{(1+\epsilon_2)} \sum_{n \in \mathbb{Z}} (F(\xi),\hat{\sigma}(\xi)e^{-i((1+\epsilon_2)(1+\epsilon_1)n\xi}) \xi^m \phi(2\pi) \mathcal{S}_{d(1+\epsilon_1),n}(\xi),
\]
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The Nyquist sampling rate for a series of frames of $L^2[0,2\pi]$ is $1$. Then the Bessel sequence $\left\{g_{(1+\epsilon_2)}(\xi)e^{-i(1+\epsilon_2)\xi}, 0 \leq \epsilon_2 \leq N, n \in \mathbb{Z}\right\}$ is in fact a series of dual frames of the sampling expansion $\left\{f_{(1+\epsilon_2)}(\xi), 0 \leq \epsilon_2 \leq N, n \in \mathbb{Z}\right\}$ (see [2]).

Hence (c) must hold by Lemma 3.2. Finally assume that

$F(\xi) = \sum_{\epsilon_2=0}^{N} \sum_{m_{(1+\epsilon_2)}} \sum_{n \in \mathbb{Z}} \langle F(\xi), g_{(1+\epsilon_2)}(\xi)e^{-i(1+\epsilon_2)n\xi}, f_{(1+\epsilon_2)}(\xi)e^{-i(1+\epsilon_2)n\xi} \rangle_{L^2[0,2\pi]} S_{(1+\epsilon_2)}(\xi)m_{(1+\epsilon_2)}$

where $S_{(1+\epsilon_2)}(\xi)$'s are given by (5). Then the sampling expansion (6) comes from (8) by applying the isomorphism $J$ since

$\langle F(\xi), g_{(1+\epsilon_2)}(\xi)e^{-i(1+\epsilon_2)n\xi}, f_{(1+\epsilon_2)}(\xi)e^{-i(1+\epsilon_2)n\xi} \rangle_{L^2[0,2\pi]} = (F(\xi), g_{(1+\epsilon_2)}(\xi)e^{-i(1+\epsilon_2)n\xi}, f_{(1+\epsilon_2)}(\xi)e^{-i(1+\epsilon_2)n\xi})_{L^2[0,2\pi]}$

Note that when $0 < \alpha_G \leq \beta_G < \infty$, the sampling series (6) converges not only in $L^2(\mathbb{R})$ but also uniformly on any subset of $\mathbb{R}$, which $\text{Sd}_{d=1}^{m} C_{\varphi}(\varphi(t_d))$ is bounded. Moreover since $\alpha_G > 0$, the rank of $G(\xi)$ is $(1+\epsilon_2)$ a.e. so that $1$

$\leq \sum_{\epsilon_2=0}^{N} \frac{1}{(1+\epsilon_2)(1+\epsilon_2)}$

which means that the total sampling rate

$\sum_{\epsilon_2=0}^{N} \frac{1}{(1+\epsilon_2)(1+\epsilon_2)}$

of the sampling expansion (6) must be at least 1, the Nyquist sampling rate for signals in $\text{Sd}_{d=1}^{m} V(\varphi(t_d))$. In the extreme case we have:

**Theorem 3.4.** Let $\alpha_G$ and $\beta_G$ be the same as in Lemma 3.2. Then there is a series of Riesz bases

$\left\{S_{(1+\epsilon_1)}(\xi)m_{(1+\epsilon_2)}(t_d) : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_2)} \leq \psi_{(1+\epsilon_2)}(t_d), 0 \leq \epsilon_2 \leq N, n \in \mathbb{Z}\right\}$

for which

$\sum_{d=1}^{m} f(t_d)$

$= \sum_{\epsilon_2=0}^{N} \sum_{m_{(1+\epsilon_2)}} \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} (L_{(1+\epsilon_2)}(f)\sigma_{(1+\epsilon_2)} + (1+\epsilon_2)(m_{(1+\epsilon_2)} - 1) + (1+\epsilon_2)n S_{(1+\epsilon_2)}(\xi)m_{(1+\epsilon_2)}(t_d))$

$\sum_{d=1}^{m} f(t_d) \leq \sum_{d=1}^{m} V(\varphi(t_d))$
Lemma 3.2. As the a series of the dual Riesz bases of $0 \leq \epsilon_1 \leq \infty$ for $0 \leq \epsilon_1, k \leq N$ and $n \in \mathbb{Z}$.

Proof. Assume $0 < \alpha_G \leq \beta_G < \infty$ and

$$F(\xi) = \sum_{\epsilon_1=0}^{N} \sum_{m(1+\epsilon_1)=1}^{m(1+\epsilon_1)} \sum_{n \in \mathbb{Z}} \langle F(\xi), g(1+\epsilon_1),m(1+\epsilon_1), \xi \rangle e^{-i(1+\epsilon_2)n\xi}$$

is a series of Riesz bases of $L^2[0,2\pi]$. Then we have

$$F(\xi) \in L^2[0,2\pi],$$

(10)

where

$$S_{(1+\epsilon_1),m(1+\epsilon_1)}(\xi)e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m(1+\epsilon_1)$$

and

$$J(S_{(1+\epsilon_1),m(1+\epsilon_1)}(\xi)) = \sum_{d=1}^{m} S_{(1+\epsilon_1),m(1+\epsilon_1),t_d}(\xi)$$

Applying the isomorphism $J$ to (10) gives (9), where

$$F(\xi) = \sum_{\epsilon_1=0}^{N} \sum_{m(1+\epsilon_1)=1}^{m(1+\epsilon_1)} \sum_{n \in \mathbb{Z}} \langle F(\xi), g(1+\epsilon_1),m(1+\epsilon_1), \xi \rangle e^{-i(1+\epsilon_2)n\xi}$$

which is a series of Riesz bases expansions on $L^2[0,2\pi]$. Then

$$g(1+\epsilon_1),m(1+\epsilon_1), \xi \rangle e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m(1+\epsilon_1)$$

must be a series of Riesz bases of $L^2[0,2\pi]$ so that

$$0 < \alpha_G \leq \beta_G < \infty \quad \text{and} \quad \sum_{\epsilon_1=0}^{N} \sum_{m(1+\epsilon_1)=1}^{m(1+\epsilon_1)} \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)} = 1$$

Lemma 3.2. As the a series of the dual Riesz bases of

$$\sum_{\epsilon_1=0}^{N} \sum_{m(1+\epsilon_1)=1}^{m(1+\epsilon_1)} \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)} = 1$$

by

$$g(1+\epsilon_1),m(1+\epsilon_1), \xi \rangle e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m(1+\epsilon_1)$$

must be of the form

$$\sum_{d=1}^{m} J^{-1}(S_{(1+\epsilon_1),m(1+\epsilon_1),t_d}(\xi)) : \epsilon_1 \geq 0, 1 \leq m(1+\epsilon_1)$$

must be of the form
where
\[
\left\{ S_{(1+\epsilon_1),m(1+\epsilon_1)}(\xi) : \epsilon_1 \geq 0, \, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1 + \epsilon_2)}{(1 + \epsilon_2)(1 + \epsilon_1)}, \epsilon \in \mathbb{Z} \right\}
\]
satisfy (5) with \( B(\xi) = 0 \). Hence
\[
\sum_{d=1}^{m} S_{(1+\epsilon_1),m(1+\epsilon_1)} n(t_d)
\]
\[= J \left( S_{(1+\epsilon_1),m(1+\epsilon_1)}(\xi)e^{-i(1+\epsilon_2)n\xi} \right)
= \sum_{d=1}^{m} S_{(1+\epsilon_1),m(1+\epsilon_1)} (t_d - (1 + \epsilon_2)n), \epsilon_1 \geq 0, \, n \in \mathbb{Z}.
\]
Finally, we have
\[
\sum_{d=1}^{m} s_{k,mk} (t_d) = \sum_{\epsilon_1=0}^{N} \sum_{m(1+\epsilon_1)=1} \sum_{n \in \mathbb{Z}} L_{(1+\epsilon_1)}[s_{k,mk}](\sigma_{(1+\epsilon_1)}(1 + \epsilon_2))n + (1 + \epsilon_2)(m_{(1+\epsilon_1)} - 1) + (1 + \epsilon_2)n) S_{(1+\epsilon_1),m(1+\epsilon_1)} (t_d - (1 + \epsilon_2)n)
\]
so that
\[
L_{(1+\epsilon_1)}[s_{k,mk}](\sigma_{(1+\epsilon_1)} + (1 + \epsilon_2)(1+\epsilon_1)(m_{(1+\epsilon_1)} - 1 + 1 + \epsilon_2)n=1 + 1 + \epsilon_2 n) = \delta_{d1} \delta_{d1}(1 + \epsilon_2)n, 0 \leq \delta_{d1} \delta_{d1}(1 + \epsilon_2)n.
\]
When \( N = 1 \), write \( L_{1}[] \), \( \sum_{d=1}^{m} l_{1}(t_d) = \sigma_{1}(1 + \epsilon_2) \), and \( d=1m \psi_{1}(t_d) as L[f], \)
\[
\sum_{d=1}^{m} l_{1}(t_d) = \sigma + (1 + \epsilon_2), \text{ and } \sum_{d=1}^{m} \psi_{1}(t_d)
\]
Corollary 3.5. (Cf. Theorem 3.1 in [11].) Let \( N = 1 \). Then there is a series of Riesz bases \{\sum_{d=1}^{m} s_{n}(t_d) : n \in \mathbb{Z} \} of \sum_{d=1}^{m} V(\varphi(t_d)) such that
\[
\sum_{d=1}^{m} f(t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} L[f](\sigma + (1 + \epsilon_2)n) s_{n}(t_d), \sum_{d=1}^{m} f(t_d)
\]
e \sum_{d=1}^{m} V(\varphi(t_d))
\]
if and only if \( \epsilon_2 = 0 \) and
\[
0 < \|Z_{\psi}(\sigma, \xi)\|_{0} \leq \|Z_{\psi}(\sigma, \xi)\|_{\infty} \cdot (12)
\]
In this case, we also have
(i) \( \sum_{d=1}^{m} s_{n}(t_d) = \sum_{d=1}^{m} s(t_d - n), n \in \mathbb{Z} \),
(ii) \( \delta(\xi) = \frac{\varphi(\xi)}{Z_{\psi}(\sigma, \xi)} \),
(iii) \( L[s](\sigma + n) = \delta_{n,0}, n \in \mathbb{Z} \). (13)
Proof. Note that for \( \epsilon_2 = 0, G_{\xi} = \frac{1}{2\pi} Z_{\psi}(\sigma, \xi) \) and
\[
\lambda_{m}(\xi) = \lambda_{M}(\xi) = \frac{1}{2\pi} Z_{\psi}(\sigma, \xi) \cdot \frac{1}{2\pi} Z_{\psi}(\sigma, \xi) \cdot \frac{1}{2\pi} Z_{\psi}(\sigma, \xi) \cdot \frac{1}{2\pi} Z_{\psi}(\sigma, \xi)
\]
so that
\[
0 < \alpha_{G} \leq \beta_{G} < \infty \text{ if and only if (12) holds. Therefore, everything except (13) follows from Theorem 3.4. Finally applying (11) to } \sum_{d=1}^{m} \varphi(t_d) \text{ gives}
\]
\[
\sum_{d=1}^{m} \varphi(t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} \psi(\sigma + n)s(t_d - n)
\]
from which we have (13) by taking the Fourier transform. When \( \sum_{d=1}^{m} l_{1}(t_d) = \sum_{d=1}^{m} \delta(t_d) \) so that \( L[-] \) is the identity operator, Corollary 3.5 reduces to a series of regular shifted sampling on \( \sum_{d=1}^{m} V(\varphi(t_d)) \) (see Theorem 3.3 in [13]).

Remark 3.6. In (1), we may allow rational sampling periods. If \( (1 + \epsilon_2)(1+\epsilon_1) = p_{1}(1+\epsilon_1)q_{1}(1+\epsilon_1) \), where \( p_{1}(1+\epsilon_1) \) and \( q_{1}(1+\epsilon_1) \) are coprime positive integers, then
\[
\left[L_{(1+\epsilon_1)}[f](\sigma_{1}(1+\epsilon_1) + (1 + \epsilon_2)(1+\epsilon_1)n)n \in \mathbb{Z} \right]
\]
\[= \left[L_{(1+\epsilon_1)}[f](\sigma_{1}(1+\epsilon_1) + (1 + \epsilon_2)(1+\epsilon_1)(k - 1) + p_{1}(1+\epsilon_1)n-k \leq q_{1}(1+\epsilon_1), n \in \mathbb{Z} \right]
\]
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Hence the case of rational sampling periods \( \{(1 + \epsilon_1)_{(1+\epsilon_2)}^{N}\}_{\epsilon_1=0} \) can be reduced to the case of integer sampling periods \( \{p_{(1+\epsilon_1)}^{N}\}_{\epsilon_1=1} \) by extending the number of LTI systems involved. For example when \( N = 1 \), we have:

**Corollary 3.7.** Let \( N = 1 \) and \( q \geq 2 \) be an integer. Assume \( Z_{\psi}(\sigma_{(1+\epsilon_1)}, \xi) \in L^m[0, 2\pi] \), \( 0 \leq \epsilon_1 \leq q - 1 \), where \( \sigma_{(1+\epsilon_1)} = \sigma + \frac{1}{q-1} (\epsilon_1) \). Then the following are all equivalent.

(a) There is a series of frames \( \{\Sigma_{d=1}^{m} s_n(t_d) : n \in \mathbb{Z}\} \) of \( \Sigma_{d=1}^{m} V(\varphi(t_d)) \) for which

\[
\sum_{d=1}^{m} f(t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} L[f] (\sigma
+ \frac{1}{q-1} n) s_n(t_d), \sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d)).
\]

(b) There is a series of frames \( \{\sum_{d=1}^{m} s_{(1+\epsilon_1)} (t_d - n) : 0 \leq \epsilon_1 \leq q - 1, n \in \mathbb{Z}\} \) of \( \sum_{d=1}^{m} V(\varphi(t_d)) \) for which

\[
\sum_{d=1}^{m} f(t_d) = \sum_{d=1}^{m} \sum_{d=1}^{q-1} \sum_{n \in \mathbb{Z}} L[f] (\sigma_{(1+\epsilon_1)}
+ n)s_{(1+\epsilon_1)} (t_d - n), \sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d)).
\]

(c) \( \left\| \sum_{\epsilon_1=0}^{\epsilon_1=q} \left| Z_{\psi}(\sigma_{(1+\epsilon_1)}, \xi) \right| \right\|_0 > 0 \).

Proof. Since

\[
\{L[f] (\sigma + \frac{1}{q-1} n) : n \in \mathbb{Z}\} = \{L[f] (\sigma_{(1+\epsilon_1)} + n) : 0 \leq \epsilon_1 \leq q - 1, n \in \mathbb{Z}\},
\]

we have a series of shifted symmetric multi-channel sampling for \( q \) LTI systems \( \{L_{(1+\epsilon_1)} [\cdot] : 0 \leq \epsilon_1 \leq q - 1\} \) with \( L_{(1+\epsilon_1)} [\cdot] = L [\cdot], 0 \leq \epsilon_1 \leq q - 1 \). Then

\[
g_{(1+\epsilon_1)} (\xi) = \frac{1}{2\pi} Z_{\psi}(\sigma_{(1+\epsilon_1)}, \xi), 0 \leq \epsilon_1 \leq q - 1
\]

and

\[
G(\xi) = \frac{1}{(2\pi)^2} \sum_{\epsilon_1=0}^{q-1} \left| Z_{\psi}(\sigma_{(1+\epsilon_1)}, \xi) \right|^2.
\]

Hence \( \alpha_\xi > 0 \) if and only if \( \left\| \sum_{\epsilon_1=0}^{q-1} \left| Z_{\psi}(\sigma_{(1+\epsilon_1)}, \xi) \right| \right\|_0 > 0 \). Therefore, Corollary 3.7 is a consequence of Theorem 3.3.

**4. Example**

Let \( \varphi_0 = \sum_{d=1}^{m} \chi_{(0,1)}(t_d) \) be the Haar scaling function and

\[
\sum_{d=1}^{m} \varphi_1(t_d) = \sum_{d=1}^{m} (\varphi_0 * \varphi_0)(t_d) = \sum_{d=1}^{m} \chi_{(0,1)}(t_d)
+ \sum_{d=1}^{m} (2 - t_d) \chi_{(1,2)}(t_d)
\]

a B-spline of degree 1. Then \( \sum_{d=1}^{m} \varphi_1(t_d) \) is a continuous series of Riesz generators [3], [22] and \( \sup_{n=1}^{m} C_{\varphi_1}(t_d) = \sup_{n=1}^{m} \sum_{d=1}^{m} | \varphi_1(t_d + n) |^2 \) < \( \infty \). First we take

\( N = 2, \sigma_1 = \sigma_2 = 0, (1 + \epsilon_2)_1 = 1, (1 + \epsilon_2)_2 = 2 \), and two LTI systems \( L_1[f] \) and \( L_2[f] \) with impulse responses \( \sum_{d=1}^{m} l_1(t_d) = \sum_{d=1}^{m} \chi_{[-1,0]}(t_d) \) and \( \sum_{d=1}^{m} l_2(t_d) = \sum_{d=1}^{m} \chi_{[-1,0]}(t_d) \). Then it’s easy to see that

\[
g_1(\xi) = \frac{1}{2\pi} Z_{\psi_1}(0, \xi) = \frac{1}{2\pi} \sum_{n \in \mathbb{R}} \psi_1(n) e^{-in\xi} = \frac{1}{16\pi} (1 + 3e^{-i\xi}),
\]

where

\[
\psi_1(n) = \begin{cases} 
1, & e^{-in\xi} \leq 1,
0, & e^{-in\xi} > 1.
\end{cases}
\]
where
\[ \sum_{d=1}^{m} \psi((1+\epsilon_1)(t_d)) = \sum_{d=1}^{m} L(1+\epsilon_1)[\psi](t_d). \]

Hence
\[ g_{1,1}(\xi) = g_1(\xi), \quad g_{1,2}(\xi) = g_1(\xi)e^{i\xi}, \quad g_{2,1}(\xi) = g_2(\xi) \]
so that (see (4))
\[ G(\xi) = [Dg_{1,1}, Dg_{1,2}, Dg_{2,1}]^T \]
\[ = \frac{1}{16\pi} \begin{bmatrix} 1 + 3e^{-i\xi} & 1 - 3e^{-i\xi} \\ 3 + e^{i\xi} & 3 - e^{i\xi} \\ 3 + e^{-i\xi} & 3 - e^{-i\xi} \end{bmatrix} \]

And
\[ G(\xi)^*G(\xi) = \frac{1}{(16\pi)^2} \begin{bmatrix} 30 + 18\cos\xi & 8 + 6i\sin\xi \\ 8 - 6i\sin\xi & 30 + 18\cos\xi \end{bmatrix} \]

The eigenvalues of \( G(\xi)^*G(\xi) \) are
\[ \frac{1}{(16\pi)^2} [30 + 18\cos\xi \pm \sqrt{100 - 36\cos^2\xi}] \]
so that
\[ \frac{1}{(16\pi)^2} \leq \alpha_G = \|\lambda_m(\xi)\|_0 < \beta_G = \|\lambda_M(\xi)\|_\infty \leq \frac{58}{(16\pi)^2} \]

Hence by Theorem 3.3, there is a series of frames \( \{\sum_{d=1}^{m} s_{1+\epsilon_1}(t_d - 2n) : 0,1,2,3 \text{ and } n \in \mathbb{Z} \} \) of the space of linear splines \( \sum_{d=1}^{m} V(\varphi_1(t_d)) \) for which the following series of asymmetric multi-channel sampling expansions holds:
\[ f \in \sum_{d=1}^{m} V(\varphi_1(t_d)), \]
which converges in \( L^2(\mathbb{R}) \) and absolutely and uniformly on \( \mathbb{R} \).

We now take \( N = 1 \) and \( \sum_{d=1}^{m} l(t_d) = \sum_{d=1}^{m} \delta(t_d) \) so that \( L[-] \) is the identity operator. Let \( q \geq 1 \) be an integer and \( 0 \leq \sigma < \frac{1}{q} \). Note first that for any fixed
\[ \sum_{d=1}^{m} t_d \text{ in } \mathbb{R}, \sum_{d=1}^{m} \varphi_1(t_d, \xi) \]
\[ = \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} \left( \varphi_1(t_d + n)e^{-in\xi} \right) \in C[0,2\pi] \]
since \( \sum_{d=1}^{m} \varphi_1(t_d) \) has compact support. Hence
\[ \sum_{d=1}^{m} \|Z_{\varphi_1}(t_d, \cdot)\|_{L^\infty[0,2\pi]} < \infty \text{ for each } \sum_{d=1}^{m} t_d \text{ in } \mathbb{R}. \]

Since \( Z_{\varphi_1}(\sigma, \xi) = \sigma + (1 - \sigma)e^{-i\xi} \) for
\[ 0 \leq \sigma < 1, \|Z_{\varphi_1}(\sigma, \xi)\|_0 = 2|\sigma - \frac{1}{2}| \]
and
\[ \|Z_{\varphi_1}(\sigma, \xi)\|_\infty = 1. \]

Therefore, by Corollary 3.5, for any \( \sigma \) with \( 0 \leq \sigma < 1 \), there is a series of Riesz bases
\[ \{\sum_{d=1}^{m} s(t_d - n) : n \in \mathbb{Z} \} \text{ of } \sum_{d=1}^{m} V(\varphi_1(t_d)) \] such that
\[ \sum_{d=1}^{m} f(t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} f(\sigma + n)s(t_d - n), \sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi_1(t_d)) \]
if and only if \( \sigma \neq \frac{1}{2} \). On the other hand, by Corollary 3.7, for any \( q \geq 2 \) and any \( \sigma \) with
\[ 0 \leq \sigma < \frac{1}{q}, \text{ there is a series of frames } \}
\[ \{\sum_{d=1}^{m} s_{1+\epsilon_1}(t_d - n) : 0 \leq \epsilon_1 \leq q - 1, n \in \mathbb{Z} \} \]
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such that

\[ \sum_{d=1}^{m} f(t_d) = \sum_{\epsilon_1=0}^{q-1} \sum_{d=1}^{m} f(\sigma + \frac{1}{q-1}(\epsilon_1) + n)s_{(1+\epsilon_1)}(t_d - n), \]

\[ \in \sum_{d=1}^{m} V(\phi_1(t_d)). \]

As an example we show the following Corollary (see [22])

**Corollary 3.7.** Assume

\[ Z_{\psi}(2 - \epsilon, \xi) \in L^\infty[0,2\pi], 0 \leq \epsilon_1 \leq q - 1 \]

then the following are all equivalent.

(a) There is a series of frames \( \{\sum_{d=1}^{m} s_n(t_d) : n \in \mathbb{Z}\} \) of \( \sum_{d=1}^{m} V(\phi(t_d)) \) for which

\[ \sum_{d=1}^{m} f(t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} \frac{1}{2\pi} Z_{\psi}(2 - \epsilon, \xi), \]

\[ \epsilon \in \sum_{d=1}^{m} V(\phi(t_d)). \]

(b) There is a series of frames

\[ \{\sum_{d=1}^{m} s_{(1+\epsilon_1)}(t_d - n) : \epsilon_1 > 0, n \in \mathbb{Z}\} \]

of

\[ \sum_{d=1}^{m} V(\phi(t_d)) \]

for which

\[ \sum_{d=1}^{m} f(t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} \frac{1}{2\pi} Z_{\psi}(2 - \epsilon, \xi), \]

\[ \epsilon \in \sum_{d=1}^{m} V(\phi(t_d)). \]

(c) \[ \left\| \sum_{\epsilon_1 \geq 0} |Z_{\psi}(2 - \epsilon, \xi)|_0 \right\| > 0. \]

**Proof.** Since

\[ \{L[f](2 - \epsilon)\} = \{L[f](n - \epsilon) : n \in \mathbb{Z}\}. \]

Now we have \( \{L_{(1+\epsilon)}[\cdot] : \epsilon_1 > 0\} \) with

\[ L_{(1+\epsilon)}[\cdot] = L[\cdot], \epsilon_1 > 0. \]

Then

\[ g_{(1+\epsilon)}(\xi) = \frac{1}{(2\pi)^2} \sum_{\epsilon_1 \geq 0} Z_{\psi}(2 - \epsilon, \xi), \epsilon_1 > 0 \quad \text{and} \quad G(\xi)^*G(\xi) = \frac{1}{(2\pi)^2} \sum_{\epsilon_1 \geq 0} |Z_{\psi}(2 - \epsilon, \xi)|^2. \]

There for \( \alpha_g > 0 \) if and only if

\[ \left\| \sum_{\epsilon_1 \geq 0} |Z_{\psi}(2 - \epsilon, \xi)|_0 \right\| > 0. \]

**References**


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