Asymmetric Multi-channel Sampling in a Series of Shift Invariant Spaces

Adam Zakria

University of Kordofan , Faculty of Science, Department of Mathematics, Sudan, adammath2020@gmail.com

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Abstract: We show asymmetric multi-channel sampling on a series of a shift invariant spaces $\sum_{d=1}^m V(\varphi(t_d))$ with a series of Riesz generators $\sum_{d=1}^m \varphi(t_d)$ in $L^2(\mathbb{R})$, where each channeled signal is assigned a uniform but distinct sampling rate. We use Fourier duality between $\sum_{d=1}^m V(\varphi(t_d))$ and $L^2[0, 2\pi]$ to find conditions under which there is a stable asymmetric multi-channel sampling formula on $\sum_{d=1}^m V(\varphi(t_d))$.

Keywords: Shift invariant space , Multi-channel sampling , Frame Riesz basis

1. Introduction

The multi-channel sampling method goes back to the works of Shannon [18] and Fogel [7], where reconstruction of a band-limited signal from samples of the signal and its derivatives was found. Generalized sampling expansion using arbitrary multi-channel sampling on the Paley–Wiener space was introduced first by Papoulis [16]. Since Papoulis' fundamental work, there have been many generalizations and applications of multi-channel sampling. See [1,5,6,14,17,19].

Papoulis' result has also been extended to a general shift invariant space by using the filter banks technique (see [4,19,20]). More recently Garcia and Pérez-Villalon [8] derived stable generalized sampling in a shift invariant space. Most previous work related to multi-channel sampling has assumed that the sampling rates of all channels are the same.

S. Kang , J.M. Kim, K.H. Kwon [22] considered asymmetric multi-channel sampling in a shift invariant space $V(\varphi)$ with a suitable Riesz generator $\varphi(t)$, where each channeled signal is sampled with a

uniform but distinct rate. Using Fourier duality between $\sum_{d=1}^m V(\varphi(t_d))$ and $L^2[0, 2\pi]$ [8,9,10,22], we derive under the same considerations a stable series of shifted asymmetric multi-channel sampling formula in $\sum_{d=1}^m V(\varphi(t_d))$. The corresponding symmetric multi-channel sampling in $\sum_{d=1}^m V(\varphi(t_d))$ was handled in [9],[22], where $\sum_{d=1}^m \varphi(t_d)$ is a continuous series of Riesz generators satisfying

$\sup_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \sum_{d=1}^m |\varphi(t_d - n)|^2 < \infty$. In this case all signals in $\sum_{d=1}^m V(\varphi(t_d))$ are continuous on $\mathbb{R}$ [21],[22]. We require only that the series of Riesz generators $\sum_{d=1}^m \varphi(t_d)$ are pointwise well defined everywhere on $\mathbb{R}$ and $\sum_{n \in \mathbb{Z}} \sum_{d=1}^m |\varphi(t_d - n)|^2 < \infty, \sum_{d=1}^m t_d \in \mathbb{R}$. Hence we essentially allow any series of Riesz generators in $L^2(\mathbb{R})$. Hence [22] allow more general filters than the ones in [8] by asking only that the impulse responses of filters belong to $L^2(\mathbb{R})$ (or the frequency responses of filters belong to $L^2(\mathbb{R}) \cup L^\infty(\mathbb{R})$ when $\sum_{n \in \mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)| \in L^2[0, 2\pi]$), whereas they belong to $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ in [9]. We give an illustrative examples (see[22]).

2. Preliminaries

We consider the notations and formulas in [22]. The

Corresponding author: Adam Zakria, University of Kordofan , Faculty of Science, Department of Mathematics, Sudan. E-mail: adammath2020@gmail.com

normalized Fourier transform is

$$\begin{aligned}\mathcal{F}[\varphi](\xi) &= \hat{\varphi}(\xi) \\ &= \int_{-\infty}^{\infty} \sum_{d=1}^m \varphi(t_d) \prod_{d=1}^m e^{-it_d \xi} dt_d, \sum_{d=1}^m \varphi(t_d) \\ &\in L^2(\mathbb{R}) \cap L^1(\mathbb{R})\end{aligned}$$

so that $\frac{1}{\sqrt{2\pi}} \mathcal{F}[\cdot]$ extends to a unitary operator from

$L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. For any $\sum_{d=1}^m \varphi(t_d) \in L^2(\mathbb{R})$, let

$$\begin{aligned}\sum_{d=1}^m C_{\varphi}(t_d) &= \sum_{n \in \mathbb{Z}} \sum_{d=1}^m |\varphi(t_d + n)|^2 \text{ and } G_{\varphi}(\xi) \\ &= \sum_{n \in \mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)|^2.\end{aligned}$$

Then

$$\begin{aligned}\sum_{d=1}^m C_{\varphi}(t_d) &= \sum_{d=1}^m C_{\varphi}(t_d + 1) \in L^1[0, 1], G_{\varphi}(\xi) \\ &= G_{\varphi}(\xi + 2\pi) \in L^2[0, 2\pi]\end{aligned}$$

and

$$\begin{aligned}\left\| \sum_{d=1}^m \varphi(t_d) \right\|_{L^2(\mathbb{R})}^2 &= \left\| \sum_{d=1}^m C_{\varphi}(t_d) \right\|_{L^1[0, 1]} \\ &= \frac{1}{2\pi} \|G_{\varphi}(\xi)\|_{L^1[0, 2\pi]}.\end{aligned}$$

In particular, $\sum_{d=1}^m C_{\varphi}(t_d) < \infty$ for a.e. $\sum_{d=1}^m t_d \in \mathbb{R}$. We also let

$$\sum_{d=1}^m Z_{\varphi}(t_d, \xi) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^m \varphi(t_d + n) e^{-in\xi}$$

be the Zak transform [12] of $\sum_{d=1}^m \varphi(t_d)$ in $L^2(\mathbb{R})$. Then $\sum_{d=1}^m Z_{\varphi}(t_d, \xi)$ is well defined a.e. on $\mathbb{R}^2$ and is quasi-periodic in the sense that

$$\sum_{d=1}^m Z_{\varphi}(t_d + 1, \xi) = e^{i\xi} \sum_{d=1}^m Z_{\varphi}(t_d, \xi)$$

and

$$\sum_{d=1}^m Z_{\varphi}(t_d, \xi + 2\pi) = \sum_{d=1}^m Z_{\varphi}(t_d, \xi).$$

A Hilbert space H consisting of complex valued

functions on a set E is called a reproducing kernel Hilbert space (RKHS in short) if there is a series of a functions $\sum_{d=1}^m q(s_d, t_d)$ on $E \times E$, called the reproducing kernel of H , satisfying

- (i) $\sum_{d=1}^m q(\cdot, t_d) \in H$ for each $\sum_{d=1}^m t_d \in E$,
- (ii) $\sum_{d=1}^m \langle f(s_d), q(s_d, t_d) \rangle = \sum_{d=1}^m f(t_d)$, $f \in H$.

In an RKHS H , any norm converging sequence also converges uniformly on any subset of E , on which $\|\sum_{d=1}^m q(\cdot, t_d)\|_H^2 = \sum_{d=1}^m q(t_d, t_d)$ is bounded.

A sequence $\{\varphi_n : n \in \mathbb{Z}\}$ of vectors in a separable Hilbert space H is

- (i) a Bessel sequence with a bound $A + \epsilon : \epsilon > 0$ if

$$\sum_{n \in \mathbb{Z}} |\langle \varphi, \varphi_n \rangle|^2 \leq (A + \epsilon) \|\varphi\|^2, \varphi \in H, \epsilon > 0,$$

- (ii) a frame of H with bounds $A + \epsilon \geq A : \epsilon > 0$ if

$$A \|\varphi\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle \varphi, \varphi_n \rangle|^2 \leq (A + \epsilon) \|\varphi\|^2, \varphi \in H, \epsilon > 0,$$

- (iii) a Riesz basis of H with bounds $A + \epsilon \geq A : \epsilon > 0$ if it is complete in H and

$$\begin{aligned}A \|c\|^2 &\leq \left\| \sum_{n \in \mathbb{Z}} c(n) \varphi_n \right\|^2 \leq (A + \epsilon) \|c\|^2, c \\ &= \{c(n)\}_{n \in \mathbb{Z}} \in l^2, \epsilon > 0,\end{aligned}$$

where $\|c\|^2 = \sum_{n \in \mathbb{Z}} |c(n)|^2$.

We let $\sum_{d=1}^m V(\varphi(t_d))$ be the series of the shift invariant spaces, where $\sum_{d=1}^m \varphi(t_d)$ is a series of a Riesz generators, that is, $\{\sum_{d=1}^m \varphi(t_d - n) : n \in \mathbb{Z}\}$ is a series of a Riesz bases of $\sum_{d=1}^m V(\varphi(t_d))$. Then

$$\begin{aligned}\sum_{d=1}^m V(\varphi(t_d)) &= \left\{ \sum_{d=1}^m (c * \varphi)(t_d) \right. \\ &= \sum_{n \in \mathbb{Z}} \sum_{d=1}^m c(n) \varphi(t_d - n) : c \\ &= \{c(n)\}_{n \in \mathbb{Z}} \in l^2 \left. \right\}.\end{aligned}$$

It is well known see [2] that $\sum_{d=1}^m \varphi(t_d)$ is a series of a Riesz generators if and only if there are constant A such that $A \leq G_\varphi(\xi) \leq A + \epsilon$ a.e. on $[0, 2\pi]$. In this case, $\{\sum_{d=1}^m \varphi(t_d - n) : n \in \mathbb{Z}\}$ is a series of a Riesz bases of $\sum_{d=1}^m V(\varphi(t_d))$ with bound $\epsilon > 0$. We assume further that

- (i) $\sum_{d=1}^m \varphi(t_d)$ is everywhere well defined on $\mathbb{R}$;
- (ii)

$\sum_{d=1}^m C_\varphi(t_d) < \infty, \sum_{d=1}^m t_d \in \mathbb{R}, i.e., \{\sum_{d=1}^m \varphi(t_d - n) : n \in \mathbb{Z}\} \in l^2$ for each $d=1, m, t_d \in \mathbb{R}$

We then allow essentially all series of Riesz generators since for any $\sum_{d=1}^m \varphi(t_d) \in L^2(\mathbb{R}), \sum_{d=1}^m C_\varphi(t_d) < \infty$ a.e. so that $\sum_{d=1}^m \varphi(t_d)$ has an equivalent representative satisfying the above two conditions. Then for each $c \in l^2, \sum_{d=1}^m (c * \varphi)(t_d)$ converges both in $L^2(\mathbb{R})$ and absolutely for each $\sum_{d=1}^m t_d \in \mathbb{R}$. Hence $\sum_{d=1}^m V(\varphi(t_d))$ becomes an RKHS with the reproducing kernel (see [13])

$$\begin{aligned} & \sum_{d=1}^m q(s_d, t_d) \\ &= \sum_{n \in \mathbb{Z}} \sum_{d=1}^m \tilde{\varphi}(s_d - n) \overline{\varphi(t_d - n)}, \text{ where } \left\{ \sum_{d=1}^m \tilde{\varphi}(t_d - n) : n \in \mathbb{Z} \right\} \end{aligned}$$

is the series of the dual Riesz bases of $\{\sum_{d=1}^m \varphi(t_d - n) : n \in \mathbb{Z}\}$ with bounds for $\epsilon > 0$. As in [9,10], we introduce an isomorphism J from $L^2[0, 2\pi]$ onto $\sum_{d=1}^m V(\varphi(t_d))$ defined as:

$$\begin{aligned} & \sum_{d=1}^m (J F)(t_d) \\ &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \sum_{d=1}^m \langle F(\xi), e^{-in\xi} \rangle_{L^2[0, 2\pi]} \varphi(t_d - n) \\ &= \sum_{d=1}^m \langle F(\xi), \frac{1}{2\pi} Z_\varphi(t_d, \xi) \rangle_{L^2[0, 2\pi]}. \end{aligned}$$

We then have:

- (i) $(J F)(\xi) = F(\xi) \hat{\varphi}(\xi)$
- (ii) $J(F(\xi) e^{-in\xi}) = \sum_{d=1}^m (J F)(t_d - n), n \in \mathbb{Z}$.

3. Asymmetric Multi-channel Sampling

The aim of this paper is as follows (see [22]). Let $\{L_{(1+\epsilon_1)}[\cdot] : \epsilon_1 \geq 0\}$ be N LTI (linear time-invariant) systems with impulse responses $\{\sum_{d=1}^m L_{(1+\epsilon_1)}(t_d) : \epsilon_1 \geq 0\}$. Develop a stable series of shifted multi-channel sampling formula for any signal $\sum_{d=1}^m f(t_d) \in \sum_{d=1}^m V(\varphi(t_d))$ using discrete sample values from $\{\sum_{d=1}^m L_{(1+\epsilon_1)}(t_d) : \epsilon_1 \geq 0\}$, where each channelled signal $\sum_{d=1}^m L_{(1+\epsilon_1)}[f](t_d)$ for $\epsilon_1 \geq 0$ is assigned with a distinct sampling rate

$$\begin{aligned} \sum_{d=1}^m f(t_d) &= \sum_{\epsilon_1=0}^N \sum_{n \in \mathbb{Z}} \sum_{d=1}^m L_{(1+\epsilon_1)}[f](\sigma_{(1+\epsilon_1)} \\ &\quad + (1 + \epsilon_2)_{(1+\epsilon_1)} n) s_{d(1+\epsilon_1), n}(t_d), \\ \sum_{d=1}^m f(t_d) &\in \sum_{d=1}^m V(\varphi(t_d)) \end{aligned} \quad (1)$$

where $\{\sum_{d=1}^m s_{d(1+\epsilon_1), n}(t_d) : \epsilon_1 \geq 0, n \in \mathbb{Z}\}$ is a series of frames or a Riesz basis of $\sum_{d=1}^m V(\varphi(t_d))$,

$\{(1 + \epsilon_2)_{(1+\epsilon_1)} : \epsilon_1 \geq 0\}$ are positive integers, and $\{\sigma_{(1+\epsilon_1)} : \epsilon_1 \geq 0\}$ are real constants. Note that the series of shifting of sampling instants is unavoidable in some uniform sampling [12] and arises naturally when we allow rational sampling periods in (1). Here, we assume that each $L_{(1+\epsilon_1)}[\cdot]$ is one of the following three types: the impulse response $\sum_{d=1}^m l(t_d)$ of an LTI system is such that

- (i) $\sum_{d=1}^m l(t_d) = \sum_{d=1}^m \delta(t_d + a), a \in \mathbb{R}$ or

- (ii) $\sum_{d=1}^m l(t_d) \in L^2(\mathbb{R})$ or

- (iii) $\hat{l}(\xi) \in L^\infty(\mathbb{R}) \cup L^2(\mathbb{R})$ when

$$H_\varphi(\xi) = \sum_{n \in \mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)| \in L^2[0, 2\pi]. \text{ For type (i),}$$

type (i),

$$\sum_{d=1}^m L[f](t_d) = \sum_{d=1}^m f(t_d + a), f \in L^2(\mathbb{R})$$

so that $L[\cdot] : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is an isomorphism. In

particular, for any

$$\sum_{d=1}^m f(t_d) = \sum_{d=1}^m (\mathbf{c} * \varphi)(t_d) \in \sum_{d=1}^m V(\varphi(t_d)),$$

$$\sum_{d=1}^m L[f](t_d) = \sum_{d=1}^m (\mathbf{c} * \psi)(t_d)$$

converges absolutely on $\mathbb{R}$ since

$$\sum_{d=1}^m C_\psi(t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^m |\psi(t_d + n)|^2 < \infty, \sum_{d=1}^m t_d$$

$$\in \mathbb{R}, \text{ where}$$

$$\sum_{d=1}^m \psi(t_d) = \sum_d L[\varphi](t_d) = \sum_{d=1}^m \varphi(t_d + a)$$

For types (ii) and (iii), we have the following results (see [22]):

Lemma 3.1. Let $L[\cdot]$ be an LTI system with the impulse response $\sum_{d=1}^m l(t_d)$ of the type (ii) or (iii) as above and

$$\sum_{d=1}^m \psi(t_d) = \sum_{d=1}^m L[\varphi](t_d) = \sum_{d=1}^1 (\varphi * l)(t_d)$$

Then

- (a) $\sum_{d=1}^m \psi(t_d) \in C_\infty(\mathbb{R}) = \{\sum_{d=1}^m u(t_d) \in C(\mathbb{R}) : \lim_{d \rightarrow \infty} \sum_{d=1}^m u(t_d) = 0\}$,
- (b) $\sup_{\mathbb{R}} \sum_{d=1}^m C_\psi(t_d) < \infty$;
- (c) for any $\sum_{d=1}^m f(t_d) = \sum_{d=1}^m (\mathbf{c} * \varphi)(t_d) \in \sum_{d=1}^m V(\varphi(t_d))$,
 $\sum_{d=1}^m L[f](t_d) = \sum_{d=1}^m (\mathbf{c} * \psi)(t_d)$ converges absolutely and uniformly on $\mathbb{R}$.

Hence $\sum_{d=1}^m L[f](t_d) \in C(\mathbb{R})$.

Proof. First assume $\sum_{d=1}^m l(t_d) \in L^2(\mathbb{R})$. Then $\sum_{d=1}^m \psi(t_d) \in C_\infty(\mathbb{R})$ by the Riemann–Lebesgue lemma since $\hat{\psi}(\xi) = \hat{\varphi}(\xi) \hat{l}(\xi) \in L^1(\mathbb{R})$. Since

$$\sum_{n \in \mathbb{Z}} |\hat{\psi}(\xi + 2n\pi)| \leq G_\varphi(\xi)^{\frac{1}{2}} G_l(\xi)^{\frac{1}{2}},$$

$$\left\| \sum_{n \in \mathbb{Z}} \hat{\psi}(\xi + 2n\pi) \right\|_{L^2[0, 2\pi]}^2 \leq \int_0^{2\pi} G_\varphi(\xi) G_l(\xi) d\xi$$

$$\leq 2\pi \|G_\varphi(\xi)\|_{L^\infty(\mathbb{R})} \|l\|_{L^2(\mathbb{R})}^2.$$

Thus for any $\sum_{d=1}^m t_d$ in $\mathbb{R}$, we have by the Poisson summation formula (see [13])

$$\sum_{n \in \mathbb{Z}} \hat{\psi}(\xi + 2n\pi) \prod_{d=1}^m e^{it_d(\xi + 2n\pi)}$$

$$= \sum_{n \in \mathbb{Z}} \sum_{d=1}^m \psi(t_d + n) e^{-in\xi} \text{ in } L^2[0, 2\pi]$$

Therefore for any $\sum_{d=1}^m t_d$ in $\mathbb{R}$

$$\sum_{d=1}^m C_\psi(t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^m |\psi(t_d + n)|^2$$

$$= \frac{1}{2\pi} \left\| \sum_{n \in \mathbb{Z}} \sum_{d=1}^m \psi(t_d + n) e^{-in\xi} \right\|_{L^2[0, 2\pi]}^2$$

$$= \frac{1}{2\pi} \left\| \sum_{n \in \mathbb{Z}} \hat{\psi}(\xi + 2n\pi) \prod_{d=1}^m e^{it_d(\xi + 2n\pi)} \right\|_{L^2[0, 2\pi]}^2$$

$$\leq \|G_\varphi(\xi)\|_{L^\infty(\mathbb{R})} \|l\|_{L^2(\mathbb{R})}^2.$$

By Young's inequality on the convolution product, $\|L[f]\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})} \|l\|_{L^2(\mathbb{R})}$ so that $L[\cdot] : L^2(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ is a bounded linear operator. Hence for any

$$\sum_{d=1}^m f(t_d) = \sum_{d=1}^m (\mathbf{c} * \varphi)(t_d)$$

$$= \sum_{n \in \mathbb{Z}} \sum_{d=1}^m c(n) \varphi(t_d - n)$$

$$\in \sum_{d=1}^m V(\varphi(t_d)),$$

$$\sum_{d=1}^m L[f](t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^m c(n) L[\varphi(t_d - n)]$$

$$= \sum_{n \in \mathbb{Z}} \sum_{d=1}^m c(n) \psi(t_d - n),$$

which converges absolutely and uniformly on $\mathbb{R}$ by (b). Now assume that $H_\varphi(\xi) \in L^2[0, 2\pi]$. The case $\hat{l}(\xi) \in L^2(\mathbb{R})$ is reduced to type (ii). So let $\hat{l}(\xi) \in L^\infty(\mathbb{R})$. Then $\hat{\varphi}(\xi) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ so that $\hat{\psi}(\xi) = \hat{\varphi}(\xi)\hat{l}(\xi) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and so $\psi(\xi) \in C_\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. Since

$$\sum_{n \in \mathbb{Z}} |\hat{\psi}(\xi + 2n\pi)| \leq \|l\|_{L^\infty(\mathbb{R})} H_\varphi(\xi),$$

we have again by the Poisson summation formula

$$\begin{aligned} \sum_{d=1}^m C_\psi(t_d) &= \frac{1}{2\pi} \left\| \sum_{n \in \mathbb{Z}} \hat{\psi}(\xi + 2n\pi) \prod_{d=1}^m e^{it_d(\xi + 2n\pi)} \right\|_{L^2[0, 2\pi]}^2 \\ &\leq \|l\|_{L^\infty(\mathbb{R})}^2 \|H_\varphi(\xi)\|_{L^2[0, 2\pi]}^2 \end{aligned}$$

so that $\sup_{\mathbb{R}} \sum_{d=1}^m C_\psi(t_d) < \infty$. For any $f \in L^2(\mathbb{R})$,

$$\begin{aligned} \sum_{d=1}^m \|L[f](t_d)\|_{L^2(\mathbb{R})} &= \|f * l\|_{L^2(\mathbb{R})} \\ &= \frac{1}{\sqrt{2\pi}} \|\hat{f}(\xi)\hat{l}(\xi)\|_{L^2(\mathbb{R})} \\ &\leq \|\hat{l}\|_{L^\infty(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}. \end{aligned}$$

Hence $L[\cdot] : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is a bounded linear operator so that for any

$\sum_{d=1}^m f(t_d) = \sum_{d=1}^m (\mathbf{c} * \varphi)(t_d) \in \sum_{d=1}^m V(\varphi(t_d))$, $\sum_{d=1}^m L[f](t_d) = \sum_{d=1}^m (\mathbf{c} * \psi)(t_d)$ converges in $L^2(\mathbb{R})$. By (b), $\sum_{d=1}^m \psi(t_d)$ also converges absolutely and uniformly on $\mathbb{R}$.

By Lemma 3.1(b), $\sum_{d=1}^m \psi(t_d) \in L^2(\mathbb{R})$. However, $\sum_{d=1}^m (\mathbf{c} * \psi)(t_d)$ may not converge in $L^2(\mathbb{R})$ unless $\{\sum_{d=1}^m \psi(t_d - n) : n \in \mathbb{Z}\}$ is a Bessel sequence.

Lemma 3.1(b) improves Lemma 1 in [9], in which the proof uses $\sum_{d=1}^m l(t_d) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$,

$\sup_{\mathbb{R}} \sum_{d=1}^m C_\varphi(t_d) < \infty$, and the integral version of Minkowski inequality. Note that the condition

$$F(\xi) = \sum_{\epsilon_1=0}^N \sum_{n \in \mathbb{Z}} \langle F(\xi), \overline{g_{(1+\epsilon_1)}(\xi)} e^{-i(1+\epsilon_2)(1+\epsilon_1)n\xi} \rangle_{L^2[0, 2\pi]} S_{(1+\epsilon_1), n}(\xi),$$

$H_\varphi(\xi) \in L^2[0, 2\pi]$ implies $\sum_{d=1}^m \varphi(t_d) \in L^2(\mathbb{R}) \cap C_\infty(\mathbb{R})$ and $\sup_{\mathbb{R}} \sum_{d=1}^m C_\varphi(t_d) < \infty$. (see [13]). Note also that $H_\varphi(\xi) \in L^2[0, 2\pi]$ if $\hat{\varphi}(\xi) = O((1 + |\xi|)^{-(1+\epsilon_2)}), (1 + \epsilon_2)_{(1+\epsilon_1)} > 1, \epsilon_1 \geq 0$, which holds e.g. for $\sum_{d=1}^m \varphi_n(t_d) = \sum_{d=1}^m (\varphi_0 * \varphi_{n-1})(t_d)$ the cardinal B-spline of degree $n (\geq 1)$, where

$\varphi_0 = \sum_{d=1}^m \chi_{[0,1)}(t_d)$. We have as a consequence of Lemma 3.1: Let $L[\cdot]$ be an LTI system with impulse response $\sum_{d=1}^m l(t_d)$ of type (i) or (ii) or (iii) as above and $\sum_{d=1}^m \psi(t_d) = \sum_{d=1}^m L[\varphi](t_d)$. Then for any

$$\begin{aligned} \sum_{d=1}^m f(t_d) &= \sum_{d=1}^m (JF)(t_d) \in \sum_{d=1}^m V(\varphi(t_d)), F(\xi) \\ &\in L^2[0, 2\pi] \\ \sum_{d=1}^m L[f](t_d) & \end{aligned} \quad (2)$$

$$= \sum_{d=1}^m \langle (\xi), \frac{1}{2\pi} \overline{Z_\psi(t_d, \xi)} \rangle_{L^2[0, 2\pi]}$$

since $L[\cdot]$ is a bounded linear operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ or $L^\infty(\mathbb{R})$ and $\{\sum_{d=1}^m \psi(t_d - n) : n \in \mathbb{Z}\} \in \ell_2, d=1, m, t_d \in \mathbb{R}$. Let $d=1, m, \psi, 1+\epsilon_1(t_d) = \sum_{d=1}^m L_{(1+\epsilon_1)}[\varphi](t_d)$ and

$$g_{(1+\epsilon_1)}(\xi) = \frac{1}{2\pi} Z_{\psi_{(1+\epsilon_1)}}(\sigma_{(1+\epsilon_1)}, \xi), \epsilon_1 \geq 0.$$

Then we have by (2)

$$\begin{aligned} L_{(1+\epsilon_1)}[f](\sigma_{(1+\epsilon_1)} + (1 + \epsilon_2)_{(1+\epsilon_1)}n) &= \langle F(\xi), \frac{1}{2\pi} Z_{\psi_{(1+\epsilon_1)}}(\sigma_{(1+\epsilon_1)} \\ &\quad + (1 + \epsilon_2)_{(1+\epsilon_1)}n, \xi) \rangle_{L^2[0, 2\pi]} \\ &= \langle F(\xi), \overline{g_{(1+\epsilon_1)}(\xi)} e^{-i(1+\epsilon_2)(1+\epsilon_1)n\xi} \rangle_{L^2[0, 2\pi]} \end{aligned} \quad (3)$$

for any $\sum_{d=1}^m f(t_d) = \sum_{d=1}^m (JF)(t_d) \in \sum_{d=1}^m V(\varphi(t_d))$ and $\epsilon_1 \geq 0$. Then by (3) and the isomorphism J from $L^2[0, 2\pi]$ onto $\sum_{d=1}^m V(\varphi(t_d))$, the sampling expansion (1) is equivalent to

$F(\xi) \in L^2[0, 2\pi]$, where $\{S_{(1+\epsilon_1),n}(\xi): \epsilon_1 \geq 0, n \in \mathbb{Z}\}$ is a series of frames or a Riesz basis of $L^2[0, 2\pi]$. This observation leads us to consider the problem when is $\{\overline{g_{(1+\epsilon_1)}(\xi)} e^{-i(1+\epsilon_2)(1+\epsilon_1)n\xi}: \epsilon_1 \geq 0, n \in \mathbb{Z}\}$ a series of frames or a Riesz basis of $L^2[0, 2\pi]$. Note that

$$\begin{aligned} & \left\{ \overline{g_{(1+\epsilon_1)}(\xi)} e^{-i(1+\epsilon_2)(1+\epsilon_1)n\xi}: \epsilon_1 \geq 0, n \in \mathbb{Z} \right\} = \\ & \left\{ \overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi}: \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \right. \\ & \quad \left. \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z} \right\} \end{aligned}$$

where $(1+\epsilon_2) = \text{lcm}\{(1+\epsilon_2)_{(1+\epsilon_1)}: \epsilon_1 \geq 0\}$ and

$g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi) = g_{(1+\epsilon_1)}(\xi) e^{i(1+\epsilon_2)(1+\epsilon_1)(m_{(1+\epsilon_1)}-1)\xi}$ for $\epsilon_1 \geq 0$. Let D be the unitary operator from $L^2[0, 2\pi]$ onto $L^2(I)^{(1+\epsilon_2)}$, where $I = [0, \frac{2\pi}{(1+\epsilon_2)}]$, defined by

$$DF = \left[F\left(\xi + (k-1)\frac{2\pi}{(1+\epsilon_2)}\right) \right]_{k=1}^{(1+\epsilon_2)}, F(\xi) \in L^2[0, 2\pi].$$

We also let

$$G(\xi) = \begin{bmatrix} Dg_{1,1}(\xi), \dots, Dg_{1,\frac{(1+\epsilon_2)}{(1+\epsilon_2)_1}}(\xi), \dots, \\ Dg_{N,1}(\xi), \dots, Dg_{N,\frac{(1+\epsilon_2)}{(1+\epsilon_2)_N}}(\xi) \end{bmatrix}^T \quad (4)$$

be a $\left(\sum_{\epsilon_1=0}^N \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}} \right)$

$\times (1+\epsilon_2)$ matrix on I and $\lambda_m(\xi), \lambda_M(\xi)$

be the smallest and the largest

eigenvalues of the positive semi-definite $(1+\epsilon_2) \times (1+\epsilon_2)$ matrix $G(\xi) * G(\xi)$, respectively.

Lemma 3.2. Let $\alpha_G = \|\lambda_m(\xi)\|_0$ and $\beta_G = \|\lambda_M(\xi)\|_\infty$ be the essential infimum of $\lambda_m(\xi)$ and the essential supremum of $\lambda_M(\xi)$ respectively. Then

$\{\overline{g_{(1+\epsilon_1)}(\xi)} e^{-i(1+\epsilon_2)(1+\epsilon_1)n\xi}: \epsilon_1 \geq 0, n \in \mathbb{Z}\}$ is

(a) a Bessel sequence in $L^2[0, 2\pi]$ if and only if $\beta_G < \infty$ or equivalently

$$\{Z_{\psi_{(1+\epsilon_1)}}(\sigma_{(1+\epsilon_1)}, \xi): \epsilon_1 \geq 0\} \in L^\infty[0, 2\pi],$$

(b) a frame of $L^2[0, 2\pi]$ if and only if $0 < \alpha_G \leq \beta_G < \infty$,

(c) a Riesz basis of $L^2[0, 2\pi]$ if and only if $0 < \alpha_G \leq \beta_G < \infty$ and

$$\sum_{\epsilon_1=0}^N \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}} = 1.$$

Proof. Since $\{\overline{g_{(1+\epsilon_1)}(\xi)} e^{-i(1+\epsilon_2)(1+\epsilon_1)n\xi}: \epsilon_1 \geq 0, n \in \mathbb{Z}\}$ is a Bessel sequence or a series of frames or a Riesz basis of $L^2[0, 2\pi]$ if and only if

$$\begin{aligned} & \left\{ \overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi}: \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \right. \\ & \quad \left. \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z} \right\} \end{aligned}$$

is a Bessel sequence or a series of frames or a Riesz basis of $L^2[0, 2\pi]$ respectively, all of the conclusions follow from Lemma 3 in [9].

Note that in [9], the authors use the Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \sum_{d=1}^m f(t_d) \prod_{d=1}^m e^{-2\pi i t_d \xi} dt_d \text{ so that}$$

they use $L^2[0, 2\pi]$ instead of $L^2[0, 2\pi]$.

Assume that $0 < \alpha_G \leq \beta_G < \infty$ so that

$\{\overline{g_{(1+\epsilon_1)}(\xi)} e^{-i(1+\epsilon_2)(1+\epsilon_1)n\xi}: \epsilon_1 \geq 0, n \in \mathbb{Z}\}$ or equivalently

$$\begin{aligned} & \left\{ \overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi}: \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \right. \\ & \quad \left. \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z} \right\} \end{aligned}$$

is a series of frames of $L^2[0, 2\pi]$. Then we can show easily (see in [9]) that

$$\begin{aligned} & \left\{ \overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi}: \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \right. \\ & \quad \left. \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z} \right\} \end{aligned}$$

has a series of dual frames of the form

$$\begin{aligned} & \left\{ S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi) e^{-i(1+\epsilon_2)n\xi}: \epsilon_1 \geq 0, 1 \right. \\ & \quad \left. \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \right. \\ & \quad \left. \in \mathbb{Z} \right\} \end{aligned}$$

with

$S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi) \in L^\infty[0,2\pi]$ for $\epsilon_1 \geq 0$ and $1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)}$ satisfying

$$\begin{aligned} & \left[DS_{1,1}(\xi), \dots, DS_{1,\frac{(1+\epsilon_2)}{(1+\epsilon_2)_1}}(\xi), \dots, \right. \\ & \quad \left. DS_{N,1}(\xi), \dots, DS_{N,\frac{(1+\epsilon_2)}{(1+\epsilon_2)_N}}(\xi) \right] \\ & = \frac{(1+\epsilon_2)}{2\pi} [G(\xi)^\dagger \\ & \quad + B(\xi)(I - G(\xi)G(\xi)^\dagger)] \end{aligned} \quad (5)$$

where

$$G(\xi)^\dagger = [G(\xi)^*G(\xi)]^{-1}G(\xi)^*$$

is the pseudo-inverse of $G(\xi)$, $B(\xi)$ is any

$$(1+\epsilon_2) \times \sum_{\epsilon_1=0}^N \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}$$

matrix with entries in $L^\infty(I)$, and I is the

$$\begin{aligned} & \left(\sum_{\epsilon_1=0}^N \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}} \right) \\ & \times \left(\sum_{\epsilon_1=0}^N \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}} \right) \text{identity matrix.} \end{aligned}$$

In particular, when we choose $B(\xi) = 0$ in (5), we have the canonical a series of dual frames of the frames

$$\begin{aligned} & \left\{ \overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \right. \\ & \quad \left. \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z} \right\}. \end{aligned}$$

We are now ready to give the following results (see [22]). We first discuss the sampling expansion (1), which is a series of frames expansion in $\sum_{d=1}^m V(\varphi(t_d))$.

Theorem 3.3. Let α_G and β_G be the same as in Lemma 3.2. Assume $\beta_G < \infty$. Then the following are all equivalent.

(a) There is a series of frames

$$F(\xi) = \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}} \sum_{n \in \mathbb{Z}} \langle F(\xi), \overline{g_j(\xi)} e^{-i(1+\epsilon_2)(1+\epsilon_1)n\xi} \rangle_{L^2[0,2\pi]} S_{d(1+\epsilon_1),n}(\xi),$$

$$\begin{aligned} & \left\{ \sum_{d=1}^m S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(t_d - (1+\epsilon_2)n) : \epsilon_1 \geq 0, 1 \right. \\ & \quad \left. \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z} \right\} \end{aligned}$$

of $\sum_{d=1}^m V(\varphi(t_d))$ for which

$$\begin{aligned} & \sum_{d=1}^m f(t_d) \\ & = \sum_{\epsilon_1=0}^N \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}} \sum_{n \in \mathbb{Z}} \sum_{d=1}^m L_{(1+\epsilon_1)}[f](\sigma_{(1+\epsilon_1)} \\ & \quad + (1+\epsilon_2)_{(1+\epsilon_1)}(m_{(1+\epsilon_1)} - 1) \\ & \quad + (1+\epsilon_2)n) S_{d(1+\epsilon_1),m_{(1+\epsilon_1)}}(t_d - (1+\epsilon_2)n) \\ & \quad , \sum_{d=1}^m f(t_d) \in \sum_{d=1}^m V(\varphi(t_d)) \end{aligned} \quad (6)$$

(b) There is a series of frames

$$\sum_{d=1}^m S_{(1+\epsilon_1),n}(t_d) : \epsilon_1 \geq 0, n \in \mathbb{Z}$$

of $\sum_{d=1}^m V(\varphi(t_d))$ for which

$$\begin{aligned} & \sum_{d=1}^m f(t_d) = \sum_{\epsilon_1=0}^N \sum_{n \in \mathbb{Z}} \sum_{d=1}^m L_{(1+\epsilon_1)}[f](\sigma_{(1+\epsilon_1)} \\ & \quad + (1+\epsilon_2)_{(1+\epsilon_1)}n) S_{d(1+\epsilon_1),n}(t_d), \\ & \quad \sum_{d=1}^m f(t_d) \in \sum_{d=1}^m V(\varphi(t_d)). \end{aligned} \quad (7)$$

(c) $0 < \alpha_G$.

Proof. Assume $\beta_G < \infty$. Then by Lemma 3.2

$\{\overline{g_{(1+\epsilon_1)}(\xi)} e^{-i(1+\epsilon_2)(1+\epsilon_1)n\xi} : \epsilon_1 \geq 0, n \in \mathbb{Z}\}$ is a Bessel sequence in $L^2[0,2\pi]$. First (a) implies (b) trivially. Assume (b). Applying the isomorphism $\mathcal{J}^{-1}$ to (7) gives by (3)

$$F(\xi) \in L^2[0, 2\pi],$$

where $\{\sum_{d=1}^m s_{d(1+\epsilon_1),n}(t_d) : 0 \leq \epsilon_1 \leq N, n \in \mathbb{Z}\}$ is a series of frames of $L^2[0, 2\pi]$. Then the Bessel sequence $\{\overline{g_{(1+\epsilon_1)}(\xi)} e^{-i(1+\epsilon_2)(1+\epsilon_1)n\xi} : 0 \leq \epsilon_1 \leq N, n \in \mathbb{Z}\}$ is in fact a series of dual frames of $\{\sum_{d=1}^m s_{d(1+\epsilon_1),n}(t_d) : 0 \leq \epsilon_1 \leq N, n \in \mathbb{Z}\}$ (see [2]). Hence (c) must hold by Lemma 3.2. Finally assume

$$F(\xi) = \sum_{\epsilon_1=0}^N \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)}} \sum_{n \in \mathbb{Z}} \langle F(\xi), \overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi} \rangle_{L^2[0,2\pi]} S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi) e^{-i(1+\epsilon_2)n\xi}, F(\xi) \in L^2[0, 2\pi] \quad (8)$$

where $S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)$'s are given by (5). Then the sampling expansion (6) comes from (8) by applying the isomorphism $\mathcal{J}$ since

$$\begin{aligned} & \langle F(\xi), g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi) e^{-i(1+\epsilon_2)n\xi} \rangle_{L^2[0,2\pi]} \\ &= \langle F(\xi), \frac{1}{2\pi} Z_{\psi_J}(\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)(1+\epsilon_1)(m_{(1+\epsilon_1)} - 1) + (1+\epsilon_2)n, \xi) \rangle_{L^2[0,2\pi]} \\ &= L_{(1+\epsilon_1)}[f](\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)(1+\epsilon_1)(m_{(1+\epsilon_1)} - 1) + (1+\epsilon_2)n) \\ & \text{for } \sum_{d=1}^m (\mathcal{J}F)(t_d) = \sum_{d=1}^m f(t_d) \end{aligned}$$

Note that when $0 < \alpha_G \leq \beta_G < \infty$, the sampling series (6) converges not only in $L^2(\mathbb{R})$ but also uniformly on any subset of $\mathbb{R}$, on which $\sum_{d=1}^m C_\varphi(t_d)$ is bounded. Moreover since $\alpha_G > 0$, the rank of

$G(\xi)$ is $(1+\epsilon_2)$ a. e. so that 1

$$\leq \sum_{\epsilon_1=0}^N \frac{1}{(1+\epsilon_2)(1+\epsilon_1)},$$

which means that the total sampling rate

$$\sum_{\epsilon_1=0}^N \frac{1}{(1+\epsilon_2)(1+\epsilon_1)}$$

of the sampling expansion (6) must be at least 1, the Nyquist sampling rate for signals in $\sum_{d=1}^m V(\varphi(t_d))$. In the extreme case we have:

Theorem 3.4. Let α_G and β_G be the same as in Lemma 3.2. Then there is a series of Riesz bases

$$\left\{ \sum_{d=1}^m S_{(1+\epsilon_1),m_{(1+\epsilon_1)},n}(t_d) : \epsilon_1 \geq 0, 1 \leq \right.$$

(c). Then $0 < \alpha_G \leq \beta_G < \infty$ that

$$\left\{ \overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq \right.$$

$m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)}, n \in \mathbb{Z} \}$ is a series of frames of $L^2[0, 2\pi]$. Then we have a frame expansion on $L^2[0, 2\pi]$

$$m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)}, n \in \mathbb{Z} \} \quad \text{of } \sum_{d=1}^m V(\varphi(t_d))$$

for which

$$\begin{aligned} & \sum_{d=1}^m f(t_d) \\ &= \sum_{\epsilon_1=0}^N \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)}} \sum_{n \in \mathbb{Z}} \sum_{d=1}^m (L_{(1+\epsilon_1)}[f](\sigma_{(1+\epsilon_1)} \\ &+ (1+\epsilon_2)(1+\epsilon_1)(m_{(1+\epsilon_1)} - 1) \\ &+ (1+\epsilon_2)n) S_{(1+\epsilon_1),m_{(1+\epsilon_1)},n}(t_d) \end{aligned}$$

$$, \sum_{d=1}^m f(t_d) \in \sum_{d=1}^m V(\varphi(t_d)) \quad (9)$$

if and only if $0 < \alpha_G \leq \beta_G < \infty$

and $\sum_{\epsilon_1=0}^N \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)} = 1$. In this case, we also have

(i) $\sum_{d=1}^m S_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(t_d - (1 + \epsilon_2)n) : \epsilon_1 \geq 0,$

$$1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)}, \text{ and } n \in \mathbb{Z}$$

(ii)

$$L_{(1+\epsilon_1)}[s_{k, m_k}](\sigma_{(1+\epsilon_1)} + (1 + \epsilon_2)_{(1+\epsilon_1)}(m_{(1+\epsilon_1)} - 1) + (1 + \epsilon_2)n) = \delta_{(1+\epsilon_1), k} \delta_{n, 0}$$

for $0 \leq \epsilon_1, k \leq N$ and $n \in \mathbb{Z}$.

Proof. Assume $0 < \alpha_G \leq \beta_G < \infty$ and

$$F(\xi) = \sum_{\epsilon_1=0}^N \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)}} \sum_{n \in \mathbb{Z}} \langle F(\xi), \overline{g_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi} \rangle_{L^2[0, 2\pi]} S_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(\xi) e^{-i(1+\epsilon_2)n\xi}$$

$$, F(\xi) \in L^2[0, 2\pi], \quad (10)$$

where

$$\left\{ S_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(\xi) e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)}, n \in \mathbb{Z} \right\}$$

is the dual of

$$\left\{ \overline{g_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)}, n \in \mathbb{Z} \right\}$$

Applying the isomorphism $\mathcal{J}$ to (10) gives (9), where

$$F(\xi) = \sum_{\epsilon_1=0}^N \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)}} \sum_{n \in \mathbb{Z}} (\langle F(\xi), \overline{g_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi} \rangle_{L^2[0, 2\pi]} \mathcal{J}^{-1}(S_{(1+\epsilon_1), m_{(1+\epsilon_1), n}}(\xi)))$$

$$, F(\xi) \in L^2[0, 2\pi]$$

which is a series of Riesz bases expansions on $L^2[0, 2\pi]$. Then

$$\left\{ \overline{g_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)}, n \in \mathbb{Z} \right\}$$

must be a series of Riesz bases of $L^2[0, 2\pi]$ so that

$$0 < \alpha_G \leq \beta_G < \infty \text{ and } \sum_{\epsilon_1=0}^N \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)} = 1 \text{ by}$$

Lemma 3.2. As the a series of the dual Riesz bases of

$$\sum_{\epsilon_1=0}^N \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)} = 1$$

Then by Lemma 3.2,

$$\left\{ \overline{g_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)}, n \in \mathbb{Z} \right\}$$

is a series of Riesz bases of $L^2[0, 2\pi]$. Then we have

$$\sum_{d=1}^m S_{(1+\epsilon_1), m_{(1+\epsilon_1), n}}(t_d) = \mathcal{J}((\xi) e^{-i(1+\epsilon_2)n\xi})$$

$$= \sum_{d=1}^m S_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(t_d - (1 + \epsilon_2)n)$$

and

$\mathcal{J}(S_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(\xi)) = \sum_{d=1}^m S_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(t_d)$. Conversely assume that the series of Riesz bases expansions (9) holds on $\sum_{d=1}^m V(\varphi(t_d))$. Applying the isomorphism $\mathcal{J}^{-1}$ to (9) gives

$$\left\{ \overline{g_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)}, n \in \mathbb{Z} \right\}$$

$$\left\{ \sum_{d=1}^m \mathcal{J}^{-1}(S_{(1+\epsilon_1), m_{(1+\epsilon_1), n}}(t_d)) : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)}, n \in \mathbb{Z} \right\}$$

must be of the form

$$\left\{ S_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(\xi) e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)}, n \in \mathbb{Z} \right\}$$

where

$$\left\{ S_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(\xi) : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)} \right\}$$

satisfy (5) with $B(\xi) = 0$. Hence

$$\begin{aligned} & \sum_{d=1}^m S_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(t_d) \\ &= \mathcal{J} \left(S_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(\xi) e^{-i(1+\epsilon_2)n\xi} \right) \\ &= \sum_{d=1}^m S_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(t_d - (1+\epsilon_2)n), \epsilon_1 \geq 0, n \in \mathbb{Z}. \end{aligned}$$

Finally, we have

$$\begin{aligned} \sum_{d=1}^m s_{k, m_k}(t_d) &= \sum_{\epsilon_1=0}^N \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)(1+\epsilon_1)}} \sum_{n \in \mathbb{Z}} \sum_{d=1}^m (L_{(1+\epsilon_1)}[s_{k, m_k}](\sigma_{(1+\epsilon_1)} \\ &\quad + (1+\epsilon_2)_{(1+\epsilon_1)}(m_{(1+\epsilon_1)} - 1) \\ &\quad + (1+\epsilon_2)n) S_{(1+\epsilon_1), m_{(1+\epsilon_1)}}(t_d \\ &\quad - (1+\epsilon_2)n)) \end{aligned}$$

so

that

$$L_{(1+\epsilon_1)}[s_{k, m_k}](\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}(m_{(1+\epsilon_1)} - 1) + (1+\epsilon_2)n) = \delta_{1+\epsilon_1, k} \delta_{n, 0}.$$

When $N = 1$, write $L_1[\cdot], \sum_{d=1}^m l_1(t_d), \sigma_1, (1+\epsilon_2)$, and $d=1$ as $\psi_1(t_d)$ as $L[\cdot]$,

$$\sum_{d=1}^m l(t_d), \sigma, (1+\epsilon_2), \text{ and } \sum_{d=1}^m \psi(t_d)$$

Corollary 3.5. (Cf. Theorem 3.1 in [11].) Let $N = 1$. Then there is a series of Riesz bases $\{\sum_{d=1}^m s_n(t_d) : n \in \mathbb{Z}\}$ of $\sum_{d=1}^m V(\varphi(t_d))$ such that

$$\begin{aligned} \sum_{d=1}^m f(t_d) &= \sum_{n \in \mathbb{Z}} \sum_{d=1}^m L[f](\sigma \\ &\quad + (1+\epsilon_2)n) s_n(t_d), \sum_{d=1}^m f(t_d) \quad (11) \end{aligned}$$

$$\in \sum_{d=1}^m V(\varphi(t_d))$$

if and only if $\epsilon_2 = 0$ and

$$0 < \|Z_\psi(\sigma, \xi)\|_0 \leq \|Z_\psi(\sigma, \xi)\|_\infty. \quad (12)$$

In this case, we also have

$$(i) \sum_{d=1}^m s_n(t_d) = \sum_{d=1}^m s(t_d - n), n \in \mathbb{Z},$$

$$(ii) \hat{s}(\xi) = \frac{\hat{\varphi}(\xi)}{Z_\psi(\sigma, \xi)},$$

$$(iii) L[s](\sigma + n) = \delta_{n, 0}, n \in \mathbb{Z}. \quad (13)$$

Proof. Note that for $\epsilon_2 = 0, G(\xi) = \frac{1}{2\pi} Z_\psi(\sigma, \xi)$

and $\lambda_m(\xi) = \lambda_M(\xi) = \left(\frac{1}{2\pi}\right)^2 |Z_\psi(\sigma, \xi)|^2$ so that

$0 < \alpha_G \leq \beta_G < \infty$ if and only if (12) holds. Therefore, everything except (13) follows from Theorem 3.4.

Finally applying (11) to $\sum_{d=1}^m \varphi(t_d)$ gives

$$\sum_{d=1}^m \varphi(t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^m \psi(\sigma + n) s(t_d - n)$$

from which we have (13) by taking the Fourier transform. When $\sum_{d=1}^m l(t_d) = \sum_{d=1}^m \delta(t_d)$ so that $L[\cdot]$ is the identity operator, Corollary 3.5 reduces to a series of regular shifted sampling on $\sum_{d=1}^m V(\varphi(t_d))$ (see Theorem 3.3 in [13]).

Remark 3.6. In (1), we may allow rational sampling periods. If $(1+\epsilon_2)_{(1+\epsilon_1)} = \frac{p_{(1+\epsilon_1)}}{q_{(1+\epsilon_1)}}$,

where $p_{(1+\epsilon_1)}$ and $q_{(1+\epsilon_1)}$ are coprime positive integers, then

$$\begin{aligned} & [L_{(1+\epsilon_1)}[f](\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}n) : n \in \mathbb{Z}] \\ &= [L_{(1+\epsilon_1)}[f](\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}(k-1) + \\ &\quad p_{1+\epsilon_1}n : 1 \leq k \leq q_{1+\epsilon_1}, n \in \mathbb{Z}] \end{aligned}$$

Hence the case of rational sampling periods $\{(1 + \epsilon_2)_{(1+\epsilon_1)}\}_{\epsilon_1=0}^N$ can be reduced to the case of integer sampling periods $\{p_{(1+\epsilon_1)}\}_{\epsilon_1=0}^N = 1$ by extending the number of LTI systems involved. For example when $N = 1$, we have:

Corollary 3.7. Let $N = 1$ and $q \geq 2$ be an integer. Assume $Z_\psi(\sigma_{(1+\epsilon_1)}, \xi) \in L^\infty[0, 2\pi]$, $0 \leq \epsilon_1 \leq q - 1$,

where $\sigma_{(1+\epsilon_1)} = \sigma + \frac{1}{q-1}(\epsilon_1)$. Then the following are all equivalent.

(a) There is a series of frames $\{\sum_{d=1}^m s_n(t_d) : n \in \mathbb{Z}\}$ of $\sum_{d=1}^m V(\varphi(t_d))$ for which

$$\begin{aligned} \sum_{d=1}^m f(t_d) &= \sum_{n \in \mathbb{Z}} \sum_{d=1}^m L[f](\sigma + \frac{1}{q-1}n) s_n(t_d), \sum_{d=1}^m f(t_d) \\ &\in \sum_{d=1}^m V(\varphi(t_d)). \end{aligned}$$

(b) There is a series of frames $\{\sum_{d=1}^m s_{(1+\epsilon_1)}(t_d - n) : 0 \leq \epsilon_1 \leq q - 1, n \in \mathbb{Z}\}$ of $\sum_{d=1}^m V(\varphi(t_d))$ for which

$$\begin{aligned} \sum_{d=1}^m f(t_d) &= \sum_{d=1}^m \sum_{\epsilon_1=0}^{q-1} \sum_{n \in \mathbb{Z}} L[f](\sigma_{(1+\epsilon_1)} + n) s_{(1+\epsilon_1)}(t_d - n), \sum_{d=1}^m f(t_d) \\ &\in \sum_{d=1}^m V(\varphi(t_d)). \end{aligned}$$

$$(c) \left\| \sum_{\epsilon_1=0}^q |Z_\psi(\sigma_{(1+\epsilon_1)}, \xi)| \right\|_0 > 0.$$

Proof. Since

$\{L[f](\sigma + \frac{1}{q-1}n) : n \in \mathbb{Z}\} = \{L[f](\sigma_{(1+\epsilon_1)} + n) : 0 \leq \epsilon_1 \leq q - 1, n \in \mathbb{Z}\}$, we have a series of

shifted symmetric multi-channel sampling for q LTI systems $\{L_{(1+\epsilon_1)}[\cdot] : 0 \leq \epsilon_1 \leq q - 1\}$ with $L_{(1+\epsilon_1)}[\cdot] = L[\cdot], 0 \leq \epsilon_1 \leq q - 1$. Then

$$g_{(1+\epsilon_1)}(\xi) = \frac{1}{2\pi} Z_\psi(\sigma_{(1+\epsilon_1)}, \xi), 0 \leq \epsilon_1 \leq q - 1$$

and

$$G(\xi)^* G(\xi) = \frac{1}{(2\pi)^2} \sum_{\epsilon_1=0}^{q-1} |Z_\psi(\sigma_{(1+\epsilon_1)}, \xi)|^2$$

Hence $\alpha_G > 0$ if and only if

$\|\sum_{\epsilon_1=0}^{q-1} |Z_\psi(\sigma_{(1+\epsilon_1)}, \xi)|\|_0 > 0$. Therefore, Corollary 3.7 is a consequence of Theorem 3.3.

4. Example

Let $\varphi_0 = \sum_{d=1}^m \chi_{[0,1)}(t_d)$ be the Haar scaling function and

$$\begin{aligned} \sum_{d=1}^m \varphi_1(t_d) &= \sum_{d=1}^m (\varphi_0 * \varphi_0)(t_d) \\ &= \sum_{d=1}^m \chi_{[0,1)}(t_d) \\ &\quad + \sum_{d=1}^m (2 - t_d) \chi_{[1,2)}(t_d) \end{aligned}$$

a B-spline of degree 1. Then $\sum_{d=1}^m \varphi_1(t_d)$ is a continuous a series of Riesz generators [3], [22] and $\sup_{\mathbb{R}} \sum_{d=1}^m C_{\varphi_1}(t_d) = \sup_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \sum_{d=1}^m |\varphi_1(t_d + n)/2| < \infty$. First we take

$N = 2, \sigma_1 = \sigma_2 = 0, (1 + \epsilon_2)_1 = 1, (1 + \epsilon_2)_2 = 2$, and two LTI systems $L1[\cdot]$ and $L2[\cdot]$ with impulse responses $\sum_{d=1}^m l_1(t_d) = \sum_{d=1}^m \chi_{[-\frac{1}{2}, 0)}(t_d)$

and $\sum_{d=1}^m l_2(t_d) = \sum_{d=1}^m \chi_{[-1, -\frac{1}{2})}(t_d)$. Then it's easy

to see that

$$\begin{aligned} g_1(\xi) &= \frac{1}{2\pi} Z_{\psi_1}(0, \xi) = \frac{1}{2\pi} \sum_{n \in \mathbb{R}} \psi_1(n) e^{-in\xi} \\ &= \frac{1}{16\pi} (1 + 3e^{-i\xi}), \end{aligned}$$

$$\begin{aligned} g_2(\xi) &= \frac{1}{2\pi} Z_{\psi_2}(0, \xi) = \frac{1}{2\pi} \sum_{n \in \mathbb{R}} \psi_2(n) e^{-in\xi} \\ &= \frac{1}{16\pi} (1 + 3e^{-i\xi}), \end{aligned}$$

where

$$\sum_{d=1}^m \psi_{(1+\epsilon_1)}(t_d) = \sum_{d=1}^m L_{(1+\epsilon_1)}[\varphi](t_d).$$

Hence

$$\begin{aligned} g_{1,1}(\xi) &= g_1(\xi), g_{1,2}(\xi) = g_1(\xi) e^{i\xi}, g_{2,1}(\xi) \\ &= g_2(\xi) \end{aligned}$$

so that (see (4))

$$\begin{aligned} G(\xi) &= [Dg_{1,1}, Dg_{1,2}, Dg_{2,1}]^T \\ &= \frac{1}{16\pi} \begin{bmatrix} 1 + 3e^{-i\xi} & 1 - 3e^{-i\xi} \\ 3 + e^{i\xi} & 3 - e^{i\xi} \\ 3 + e^{-i\xi} & 3 - e^{-i\xi} \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} G(\xi)^* G(\xi) &= \\ \frac{1}{(16\pi)^2} &\begin{bmatrix} 30 + 18 \cos \xi & 8 + 6i \sin \xi \\ 8 - 6i \sin \xi & 30 + 18 \cos \xi \end{bmatrix}. \end{aligned}$$

The eigenvalues of $G(\xi)^* G(\xi)$ are

$$\frac{1}{(16\pi)^2} [30 + 18 \cos \xi \pm \sqrt{100 - 36 \cos^2 \xi}] \quad \text{so}$$

that

$$\frac{1}{(16\pi)^2} \leq \alpha_G = \|\lambda_m(\xi)\|_0 < \beta_G = \|\lambda_M(\xi)\|_\infty \leq$$

$\frac{58}{(16\pi)^2}$. Hence by Theorem 3.3, there is a series of

frames $\{\sum_{d=1}^m s_{(1+\epsilon_1)}(t_d - 2n) : \epsilon_1 = 0, 1, 2, 3 \text{ and } n \in \mathbb{Z}\}$ of the space of linear splines $\sum_{d=1}^m V(\varphi_1(t_d))$ for which the following series of asymmetric multi-channel sampling expansions holds:

$$\begin{aligned} \sum_{d=1}^m f(t_d) &= \\ \sum_{n \in \mathbb{Z}} \sum_{d=1}^m &\{L_1[f](2n)s_1(t_d - 2n) \\ &+ L_1[f](2n+1)s_2(t_d - 2n) \\ &+ L_2[f](2n)s_3(t_d - 2n)\}, \end{aligned}$$

$$f \in \sum_{d=1}^m V(\varphi_1(t_d)),$$

which converges in $L^2(\mathbb{R})$ and absolutely and uniformly on $\mathbb{R}$.

We now take $N = 1$ and $\sum_{d=1}^m l(t_d) = \sum_{d=1}^m \delta(t_d)$ so that $L[\cdot]$ is the identity operator. Let $q (\geq 1)$ be an integer and $0 \leq \sigma < \frac{1}{q}$. Note first

that for any fixed

$$\begin{aligned} \sum_{d=1}^m t_d \text{ in } \mathbb{R}, \sum_{d=1}^m Z_{\varphi_1}(t_d, \xi) \\ = \sum_{n \in \mathbb{Z}} \sum_{d=1}^m \varphi_1(t_d + n) e^{-in\xi} \\ \in C[0, 2\pi] \end{aligned}$$

since $\sum_{d=1}^m \varphi_1(t_d)$ has compact support. Hence

$\sum_{d=1}^m \|Z_{\varphi_1}(t_d, \cdot)\|_{L^\infty[0, 2\pi]} < \infty$ for each $\sum_{d=1}^m t_d$ in $\mathbb{R}$.

Since $Z_{\varphi_1}(\sigma, \xi) = \sigma + (1 - \sigma)e^{-i\xi}$ for $0 \leq \sigma < 1$, $\|Z_{\varphi_1}(\sigma, \xi)\|_0 = 2|\sigma - \frac{1}{2}|$ and

$\|Z_{\varphi_1}(\sigma, \xi)\|_\infty = 1$. Therefore, by Corollary 3.5, for

any σ with $0 \leq \sigma < 1$, there is a series of Riesz bases

$\{\sum_{d=1}^m s(t_d - n) : n \in \mathbb{Z}\}$ of $\sum_{d=1}^m V(\varphi(t_d))$ such that

$$\begin{aligned} \sum_{d=1}^m f(t_d) &= \sum_{n \in \mathbb{Z}} \sum_{d=1}^m f(\sigma + n) s(t_d - n), \sum_{d=1}^m f(t_d) \\ &\in \sum_{d=1}^m V(\varphi_1(t_d)) \end{aligned}$$

if and only if $\sigma \neq \frac{1}{2}$. On the other hand, by

Corollary 3.7, for any $q (\geq 2)$ and any σ with

$0 \leq \sigma < \frac{1}{q}$, there is a series of frames

$\{\sum_{d=1}^m s_{(1+\epsilon_1)}(t_d - n) : 0 \leq \epsilon_1 \leq q - 1, n \in \mathbb{Z}\}$

such that

$$\begin{aligned} \sum_{d=1}^m f(t_d) &= \sum_{\epsilon_1=0}^{q-1} \sum_{n \in \mathbb{Z}} \sum_{d=1}^m f\left(\sigma + \frac{1}{q-1}(\epsilon_1) \right. \\ &\quad \left. + n\right) s_{(1+\epsilon_1)}(t_d - n), \sum_{d=1}^m f(t_d) \\ &\in \sum_{d=1}^m V(\varphi_1(t_d)). \end{aligned}$$

As an example we show the following Corollary (see [22])

Corollary 3.7. Assume

$$Z_\psi(2 - \epsilon, \xi) \in L^\infty[0, 2\pi], 0 \leq \epsilon_1 \leq q - 1$$

then the following are all equivalent.

(a) There is a series of frames $\{\sum_{d=1}^m s_n(t_d) : n \in \mathbb{Z}\}$ of $\sum_{d=1}^m V(\varphi(t_d))$ for which

$$\begin{aligned} \sum_{d=1}^m f(t_d) &= \sum_{n \in \mathbb{Z}} \sum_{d=1}^m L[f](2 - \epsilon) s_n(t_d), \sum_{d=1}^m f(t_d) \\ &\in \sum_{d=1}^m V(\varphi(t_d)). \end{aligned}$$

(b) There is a series of frames

$$\left\{ \sum_{d=1}^m s_{(1+\epsilon_1)}(t_d - n) : \epsilon_1 > 0, n \in \mathbb{Z} \right\}$$

of

$$\sum_{d=1}^m V(\varphi(t_d))$$

for which

$$\begin{aligned} \sum_{d=1}^m f(t_d) &= \sum_{n \in \mathbb{Z}} \sum_{\epsilon_1 \geq 0} \sum_{d=1}^m L[f](n - \epsilon) s_{(1+\epsilon_1)}(t_d \\ &\quad - n), \sum_{d=1}^m f(t_d) \in \sum_{d=1}^m V(\varphi(t_d)). \end{aligned}$$

$$(c) \left\| \sum_{\epsilon_1 \geq 0} |Z_\psi(2 - \epsilon, \xi)| \right\|_0 > 0.$$

Proof. Since

$\{L[f](2 - \epsilon)\} = \{L[f](n - \epsilon) : n \in \mathbb{Z}\}$. Now we have $\{L_{(1+\epsilon_1)}[\cdot] : \epsilon_1 > 0\}$ with

$L_{(1+\epsilon_1)}[\cdot] = L[\cdot], \epsilon_1 > 0$. Then $g_{(1+\epsilon_1)}(\xi) = \frac{1}{2\pi} Z_\psi(2 - \epsilon, \xi), \epsilon_1 > 0$ and $G(\xi)^* G(\xi) = \frac{1}{(2\pi)^2} \sum_{\epsilon_1 \geq 0} |Z_\psi(2 - \epsilon, \xi)|^2$. There for $\alpha_G > 0$ if and only if

$$\left\| \sum_{\epsilon_1 \geq 0} |Z_\psi(2 - \epsilon, \xi)| \right\|_0 > 0.$$

References

- [1] K.F. Cheung, A multi-dimensional extension of Papoulis's generalized sampling expansion with application in minimum density sampling, in: R.J. Marks II (Ed.), *Advanced Topics in Shannon Sampling and Interpolation Theory*, Springer-Verlag, Berlin, 1993.
- [2] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2001.
- [3] C.K. Chui, *An Introduction to Wavelets*, Academic Press, New York, 1992.
- [4] I. Djokovic, P.P. Vaidyanathan, Generalized sampling theorems in multiresolution subspaces, *IEEE Trans. Signal Process.* 45 (1997) 583–599.
- [5] T. Eldar, A.V. Oppenheim, Filterbank reconstruction of bandlimited signals from nonuniform and generalized samples, *IEEE Trans. Signal Process.* 48(2000) 2864–2875.
- [6] A. Feuer, On the necessity of Papoulis' result for multi-dimensional GSE, *IEEE Signal Process. Lett.* 11 (2004) 420–422.
- [7] L.J. Fogel, A note on the sampling theorem, *IRE Trans. Inform. Theory* IT-1(1995) 47–48.
- [8] A.G. Garcia, M.A. Hernández-Medina, G. Pérez-Villalón, Oversampling and reconstruction functions with compact support, *J. Comput. Appl. Math.* 227(2009) 245–253.
- [9] A.G. Garcia, G. Pérez-Villalón, Dual frames in $L^2(0, 1)$ connected with generalized sampling in shift-invariant spaces, *Appl. Comput. Harmon. Anal.* 20(2006) 422–433.
- [10] A.G. Garcia, G. Pérez-Villalón, A. Portal, Riesz bases in $L^2(0, 1)$ related to sampling in shift-invariant spaces, *J. Math. Anal. Appl.* 308 (2005) 703–713.
- [11] Y.M. Hong, J.M. Kim, K.H. Kwon, E.H. Lee, Channeled sampling in shift invariant spaces, *Int. J. Wavelets Multiresolut. Inf. Process.* 5 (2007) 753–767.

- [12] A.J.E.M. Janssen, The Zak transform and sampling theorems for wavelet subspaces, *IEEE Trans. Signal Process.* 41 (1993) 3360–3364.
- [13] J.M. Kim, K.H. Kwon, Sampling expansion in shift invariant spaces, *Int. J. Wavelets Multiresolut. Inf. Process.* 6 (2008) 223–248.
- [14] J.M. Kim, K.H. Kwon, Vector sampling expansion in Riesz bases setting and its aliasing error, *Appl. Comput. Harmon. Anal.* 25 (2008) 315–334.
- [15] G.E.C. Nogueira, A. Ferreira, Higher order sampling and recovering of lowpass signals, *IEEE Trans. Signal Process.* 48 (2000) 2169–2171.
- [16] A. Papoulis, Generalized sampling expansion, *IEEE Trans. Circuits Syst.* 24 (1977) 652–654.
- [17] D. Seidner, M. Feder, Vector sampling expansion, *IEEE Trans. Signal Process.* 48 (2000) 401–406.
- [18] C.E. Shannon, Communication in the presence of noise, *Proc. Inst. Radio Eng.* 37 (1949) 10–21.
- [19] M. Unser, J. Zerubia, Generalized sampling: Stability and performance analysis, *IEEE Trans. Signal Process.* 45 (1997) 2941–2950.
- [20] M. Unser, J. Zerubia, A generalized sampling theory without band-limiting constraints, *IEEE Trans. Circuits Syst. II* 45 (1998) 959–969.
- [21] X. Zhou, W. Sun, On the sampling theorem for wavelet subspaces, *J. Fourier Anal. Appl.* 5 (1999) 347–354.
- [22] S. Kang, J.M. Kim, K.H. Kwon Asymmetric multi-channel sampling in shift invariant spaces *J. Math. Anal. Appl.* 367 (2010) 20–28