An Equation Related to Normal Distribution

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This paper gives an equation, $f(x) = 2f\left(\frac{x}{\sqrt{2}}\right)$, its resolutions are $f(x) = ax^2$, a is any number. This equation can be related to normal distribution. So it can give a good explanation of central limit theorem of probability theory. It is helpful to teach central limit theorem to student and is possible to proof some condition of central limit theorem easily.

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Introduction

Now we research an equation as follows,

$$f(x) = 2f\left(\frac{x}{\sqrt{2}}\right) \quad (1)$$

where $f(x)$ is an analytic function. see Shilov (1996). It is an algebra function but is not a differential equation.

Theorem 1. If an analytic function $f(x)$ satisfies equation (1), if and only if it has form

$$f(x) = ax^2 \quad (2)$$

where a can be any number.

Proof Because $f(x)$ is an analytic function, so it can be expressed as the form of the Taylor series as follows,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n \quad (3)$$

and we have

$$2f\left(\frac{x}{\sqrt{2}}\right) = 2 \sum_{n=0}^{\infty} a_n \left(\frac{x}{\sqrt{2}}\right)^n = \sum_{n=0}^{\infty} a_n 2^{1-\frac{n}{2}} x^n \quad (4)$$

Because $f(x)$ satisfies equation (1), formula (3) must be equal to formula (4), so every corresponded coefficient must be equal, that is

$$a_n = a_n 2^{1-\frac{n}{2}} (n = 0, 1, 2, \dots) \quad (5)$$

So we must have $a_n = 0 (n \neq 2)$ and a_2 keeps unknown, so $f(x)$ must have form formula (2).

Replace $f(x) = ax^2$ to right side of equation (1),

$$2f\left(\frac{x}{\sqrt{2}}\right) = 2a\left(\frac{x}{\sqrt{2}}\right)^2 = ax^2 = f(x)$$

Proof is complete.

From Theorem 1 we can get some interesting result which related to normal distribution of probability.

Get Normal Distribution

Theorem 2. If an analytic function $f(x)$ satisfies

$$f(x) = f^2\left(\frac{x}{\sqrt{2}}\right) \quad (6)$$

if and only if it has form

$$f(x) = e^{ax^2} \quad (7)$$

where a is any number.

Proof Let

$$g(x) = \log f(x) \quad (8)$$

so according equation (6),

$$g(x) = \log f^2\left(\frac{x}{\sqrt{2}}\right) = 2 \log f\left(\frac{x}{\sqrt{2}}\right) = 2g\left(\frac{x}{\sqrt{2}}\right) \quad (9)$$

so $g(x)$ satisfies equation (1), according Theorem 1,

$$g(x) = ax^2 \quad (10)$$

According formula (8) and formula (10),

$$f(x) = e^{g(x)} = e^{ax^2}$$

If $f(x) = e^{ax^2}$,

$$f^2\left(\frac{x}{\sqrt{2}}\right) = \left[e^{a\left(\frac{x}{\sqrt{2}}\right)^2} \right]^2 = e^{2a\frac{x^2}{2}} = e^{ax^2} = f(x)$$

it satisfies equation (6).

Proof is complete.

Now we talk characteristic function of a random variable. see Heathcote (1971). The definition of characteristic function $\phi(t)$ of a random variable X is

$$\phi(t) = E[e^{iXt}] \quad (i = \sqrt{-1}) \quad (11)$$

It is easy to proof

$$|\phi(t)| \leq \phi(0) = 1 \quad (12)$$

Now we talk normal distribution of probability.

A random variable X of normal distribution is a continuous random variable whose probability density function $f(x)$ has the form

$$f(x) = ce^{ax^2+bx} \quad (13)$$

Where a, b, c are constants. But in text book of probability, for convenient, $f(x)$ is often expressed by the form which is easily to see the mean μ and variance σ^2 :

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (14)$$

and X is noted as $X \sim N(\mu, \sigma^2)$. If $X \sim N(\mu, \sigma^2)$, its characteristic function $\phi(t)$ is

$$\phi(t) = e^{it\mu - \frac{1}{2}\sigma^2 t^2} \quad (15)$$

When students study probability theory and is introduced the normal distribution and is told formula (14) and formula (15), but they still are strange how the formulas are thought by human being. Here we can give a method to derive these formulas simply.

By the way, because characteristic function $\phi(t)$ has property $\phi(-t) = \overline{\phi(t)}$, if $\phi(t) = e^{at^2} \neq 1$, from $e^{a(-t)^2} = e^{at^2} = \overline{e^{at^2}}$, it must have $\bar{a} = a$, so a must be a real number, and because $|\phi(t)| \leq 1$, so $a < 0$. Let $\sigma^2 = -2a$, so $\phi(t) = e^{-\frac{1}{2}\sigma^2 t^2}$, it means normal distribution with $\mu = 0$.

Theorem 3. Assuming X, Y are two independent random variables and they do not obey one-point distribution, $Z = X + Y$. If $X, Y, \frac{Z}{\sqrt{2}}$ have moments of any order and have same distribution, their distribution must be normal distribution.

Proof Because $X, Y, \frac{Z}{\sqrt{2}}$ have same distribution, we assume their common characteristic function is $\phi(t)$, so

$$\phi(t) = E[e^{-itX}] = E[e^{-itY}] = E[e^{-it\frac{Z}{\sqrt{2}}}] = E[e^{-it\frac{X+Y}{\sqrt{2}}}] = \phi^2\left(\frac{t}{\sqrt{2}}\right)$$

So $\phi(t)$ satisfies equation (6), according to theorem 2, $\phi(t)$ must have form $\phi(t) = e^{at^2}$. Because $\phi(t)$ is a characteristic function, X do not obey one-point distribution so that $\phi(t) \neq 1, a \neq 0, a < 0$. So $X, Y, \frac{Z}{\sqrt{2}}$ all obey same normal distribution with mean equal to 0.

Proof is complete.

An Explanation to Central Limit Theorem

Definition 1. (Odd sequence, Even sequence) For a given sequence of random variable $\{X_1, X_2, X_3, \dots\}$, we can divide it to two sequences, one is $\{X_1, X_3, X_5, \dots\}$ which is called odd sequence, another is $\{X_2, X_4, X_6, \dots\}$ which is called even sequence.

So every sequence of random variable has its odd sequence and even sequence.

When a scientist need to use central limit theorem (LCT), he or she often thinks like such: a noise often is generating by a lot of random variables summing together, these random variables can have different distribution and different mean and variance, but if their number is big enough, then the sum of them will obey normal distribution approximately. Of cause this approximate must meet some condition, like Lyapunov CLT or Lindeberg CLT, but these condition is hard to understand and proof are difficult.

According theorem 3, we have chance to give concept of central limit theorem a good explanation.

Definition 2. (same shape) $f(x)$ and $g(x)$ are two functions. If there are three constant numbers a, b, c to let $af(bx+c) = g(x)$, we say $f(x)$ and $g(x)$ have same shape.

When people learn probability theory, often are introduced some fact that two independent random variables X, Y with some same distribution add together, the sum has same distribution. For example, the sum of two independent random variables with binominal distribution is also having binominal distribution, the sum of two independent random variables with passion distribution is still passion distribution, etc.

But these thinking may generate some misleading.

The theorem 3 tell us that if $Z = X + Y$ and the distribution functions of X, Y, Z have same shape, then their distribution must be normal distribution.

For any sequence of random variables $\{X_1, X_2, \dots\}$, the sequence and its odd sequence and even sequence often have same description. For example, giving a sequence $\{X_1, X_2, \dots\}$, they are all have same distribution and independent each other, so the corresponded odd sequence and even sequence still have same description.

$$\text{So let } W_n = \frac{X_1 + X_2 + \dots + X_{2n}}{\sqrt{2n}}, \quad U_n = \frac{X_1 + X_3 + \dots + X_{2n-1}}{\sqrt{n}}, \quad V_n = \frac{X_2 + X_4 + \dots + X_{2n}}{\sqrt{n}},$$

So

$$U_n + V_n = \frac{X_1 + X_2 + \dots + X_{2n}}{\sqrt{n}} = \sqrt{2}W_n$$

If the distribution function $F_n(x)$ of W_n week convergence to function $F(x)$, because these three sequences have same description, so distribution function of U_n and V_n also week convergence to function $F(x)$, so the distribution function of them must be normal distribution according theorem 3.

Conclusion

Because of theorem 3, we can use the concept of odd sequence and even sequence to prove some condition in which the central limit theorem can be true. So the equation 1 should become a tool to prove some other theorem. But author doubt that the word analytic can be cancel, but he is not enough clever to proof it, so gives a conjecture as follows:

Conjecture 1. if function $f(x)$ meets

$$f(x) = 2f\left(\frac{x}{\sqrt{2}}\right)$$

It must have form $f(x) = ax^2$.

References

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