Discriminant Analysis of the Linear Separable Data

- Japanese 44 Cars -

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There are four serious problems in the discriminant analysis. We developed an optimal linear discriminant function (optimal LDF) based on the minimum number of misclassification (minimum NM) using integer programming (IP). We call this LDF as Revised IP-OLDF. Only this LDF can discriminate the cases on the discriminant hyperplane (Problem1). This LDF and a hard-margin SVM (H-SVM) can discriminate the lineary separable data (LSD) exactly. Another LDFs may not discriminate the LSD theoretically (Problem2). When Revised IP-OLDF discriminate the Swiss banknote data with six variables, we find MNM of two-variables model such as (X4, X6) is zero. Because MNM_k decreases monotounusly ($MNM_k \geq MNM(k+1)$), sixteen MNMs including (X4, X6) are zero. Until now, because there is no research of the LSD, we surveyed another three linear separable data sets such as: 18 exam scores data sets, the Japanese 44 cars data and six microarray datasets. When we discriminate the exam scores with $MNM=0$, we find the generalized inverse matrix technique causes the serious Problem3 and confirmed this fact by the cars data. At last, we claim the discriminant analysis is not the inferential statistics because there is no standard errors (SEs) of error rates and discriminant coefficients (Problem4). Therefore, we porposed the “100-fold cross validation for the small sample” method (the method). By this break-through, we can choose the best model having minimum mean of error rate (M2) in the validation sample and obtaine two 95% confidence intervals (CIs) of error rate and discriminant coefficients. When we discriminate the exam scores by this new method, we obtaine the surprising results seven LDFs except for Fisher’s LDF are almost the same as the trivial LDFs. In this research, we discriminate the Japanese 44 cars data because we can discuss four problems. There are six independent variables to discriminate 29 regular cars and 15 small cars. This data is linear separable by the emission rate (X1) and the number of seats (X3). We examine the validity of the new model selection procedure of the discriminant analysis. We proposed the model with minimum mean of error rates (M2) in the validation samples is the best model. We had examined this procedure by the exam scores, and we obtain good results. Moreover, the 95% CI of eight LDFs offers us real perception of the discriminant theory. However, the exam scores are different from the ordinal data. Therefore, we apply our theory and procedure to the Japanese 44 cars data and confirmed the same conclusion.

Keywords: Model Selection Procedure, Means of Error Rates, Fisher’s LDF, Logistic Regression, Support Vector Machine (SVM), Minimum Number of Misclassifications (minimum NM, MNM), Revised IP-OLDF based on MNM criterion, Revised IPLP-OLDF, Revised LP-OLDF, Linear Separable Data and Model, K-fold Cross validation.

Introduction

In this research, we examine the new model selection procedure of the discriminant analysis by the “k-fold cross validation for small sample” method (the method). Although Lachenbruch et al. [6] had proposed a leave-one-out (LOO) procedure for model selection of the discriminant analysis, they could not achieve the new method because of lack of computer power. If we fix “k=100”, we can obtain 100 LDFs and 100 error rates in the training and validation samples. From the 100 error rates, we calculate two means of error rates such M1 and M2 in the training and validation samples. We consider the model with minimum M2 among all possible combination models [5] is the best model. We apply this new procedure for three data sets of exam scores [18] and obtain good results. We had better distinguished these computer-intensive approaches from the traditional inferential statistics with the SE. Genuine statisticians without computer power had established the inferential statistics by their intellectual brain. Now, we can utilize the computer power with statistical and MP solvers such as JMP [7] and LINGO [8] [9]. The researchers who wish to analyze their research data can obtain the 95% confidence interval (CI) of the error rate and discriminant coefficients [14] [16]. These statistics give us precise and deterministic judgment about the model selection procedure of the discriminant analysis.

Method

In this research, we discriminate the Japanese 44 cars data (the data) by eight LDFs. We focus on two means of error rates such as “M1 and M2” in the training and validation samples and propose the model with minimum M2s is the best model (the model selection procedure, the procedure). We compare eight M2s of the best model of eight LDFs and the 95% CI for discriminant coefficients.

Eight LDFs

In this research, we compare two statistical LDFs and six MP-based LDFs [19] [20]. Two statistical LDFs are Fisher’s LDF [2] and a logistic regression [1]. Fisher proposed Fisher’s LDF based on the variance-covariance matrices and found the discriminant analysis. After Fisher’s LDF, we can use a quadratic discriminant function (QDF). From 1971 to 1974, we analyzed electrocardiogram (ECG) data in order to develop the diagnostic logic between normal and abnormal symptoms by Fisher’s LDF and QDF. Our research was inferior to the decision tree logic developed by the medical doctor. After this experience, we found four problems of the discriminant analysis [21][25] and concluded these discriminant functions are fragile for the discrimination of the normal and abnormal diseases. Therefore, most medical researchers used the logistic regression in the equation (1). If some independent variable increases or decreases, the probability p belongs to class1 (abnormal diseases) increases from 0 (class2) to 1 (class1).

$$\text{Log}(p/(1-p)) = f(x) \quad (1)$$

p: the probability belongs to class1; x: the independent variables.

We can obtain the maximum/minimum value of the function by MP, regardless of the presence or absence of constraints. Therefore, Schrage [8] introduced several definitions of regression models. Quadratic Programming (QP) defines the ordinal least square method. Linear Programming (LP) defines the “Least Absolute Values (LAV) Regression”. Nonlinear Programming (NLP) represents several Lp norm regression. However, there were few researches about the regression analysis. On the other hands, there were many researches about MP-based discriminant models [46]. However, statistical users rarely used these discriminant functions because

these functions were not evaluated by the real data. Vapnik [47] proposed three different SVM models examined by the actual data. H-SVM in equation (2) indicates the discrimination of LSD clearly.

$$\text{MIN} = \|b\|^2/2;$$

$$y_i * (x_i^T b + b_0) \geq 1; \quad (2)$$

$y_i = 1 / -1$ for $x_i \in \text{class1/class2}$; x_i : p-independent variables (p-variables);

b : p-discriminant coefficients; b_0 : the intercept and free variable.

Real data are rarely linearly separable. For this reason, S-SVM has been defined in equation (3) with two objects. These two objects are combined by defining some “penalty c .” However, S-SVM does not have a rule to determine ‘ c ’ correctly. In this research, two S-SVMs such as SVM4 ($c=10000$) and SVM1 ($c=1$) are examined. We know the “M1 & M2” of SVM4 are almost better than SVM1.

$$\text{MIN} = \|b\|^2/2 + c * \sum e_i ; y_i * (x_i^T b + b_0) \geq 1 - e_i; \quad (3)$$

c : penalty c ; e_i : non-negative decision variable.

On the other hand, Shinmura [10-13] [15] developed an revised optimal LDF by Integer Programming (Revised IP-OLDF) based on the minimum NM (MNM) criterion in equation (4) and found several new facts about the discriminant theory. Only Revised IP-OLDF can avoid the cases on the discriminant hyperplane (Problem1). Moreover, only H-SVM and Revised IP-OLDF can recognize linear separable model theoretically. Another LDFs cannot recognize LSD and cannot judge the data is overlap or not theoretically (Problem2).

$$\text{MIN} = \sum e_i ; y_i * (x_i^T b + b_0) \geq 1 - M * e_i ; \quad (4)$$

e_i : 0/1 integer decision variable; M : big M constant ($M=10000$); b_0 : free decision variables.

If e_i is a non-negative real variable, we utilize Revised LP-OLDF, which is an L1-norm LDF. Revised IPLP-OLDF is a combined model of Revised LP-OLDF and Revised IP-OLDF [17]. In the first step, Revised LP-OLDF is applied for all cases, and e_i fixed to 0 for cases that are discriminated correctly by Revised LP-OLDF. In the second step, Revised IP-OLDF is used for cases, e_i of which are not zero in the first step. It is important that e_i changes from real decision variable to binary integer variable. For this reason, Revised IPLP-OLDF can be expected to obtain an estimate of MNM faster than Revised IP-OLDF for large samples [22]. On the other hand, Revised IP-OLDF can do feature selection for six microarray data (the datasets) very easy, and find the dataset consists of several small subspaces with $\text{MNM}=0$ and other high-dimension subspaces that is not linear separable [27-43].

In this research, we compare Revised IP-OLDF with seven LDFs by the “k-fold cross validation for small sample” method. We evaluated Revised IP-OLDF by several small samples. It was difficult for us to compare Revised IP-OLDF with seven LDFs because we could not validate the effectiveness of Revised IP-OLDF. Therefore, we proposed the method and can evaluate eight LDFs by two means of error rates such as M1 and M2. Although Fisher developed Fisher’s LDF, he never formulated the equation of SEs of error rates and discriminant coefficients. Therefore, there were no good model selection procedures instead of the LOO method.

In this research, we propose the new model selection procedure as follows:

We discriminate an original data by eight LDFs and two discriminant functions such as QDF and a Regularized Discriminant Analysis (RDA) [4]. In principal, we analyzed all possible combination models made

by independent variables. Goodnight established this procedure in the regression analysis by the sweep operator [5]. By this procedure, we can overlook the whole picture of the study.

We discriminate re-sampling samples by the method. In this research, we fix $k=100$ in order to obtain two means of error rates and 95% CI of error rates and discriminant coefficients.

We consider the best model with the minimum values of M2 (minimum M2 standard) and compare eight M2s of eight best models. Although Vapnik defined the generalization ability, we claim the model with minimum M2 has good generalization ability. Moreover, we discuss the 95% CI of discriminant coefficients.

Forty-four Japanese Car Data

Let us consider the discrimination of 29 regular cars and 15 small cars having six variables in **Table 1**. Small cars have a unique Japanese specification. Women buy them as second cars because they cost efficient. The emission rate and capacity (number of seats) of small cars are smaller than the regular cars. The emission rate of small and regular cars ranges from [0.657, 0.658] and [0.996, 3.456], respectively. The seats of small and regular cars are 4 and [5, 8], respectively. 'p' is the number of variables selected by the forward stepwise procedure. QDF and Revised IP-OLDF can find the one variable model (X1: Emission) is linearly separable. Last two columns are NMs of RDA [2]. Before 2012, JMP [5] switched QDF by RDA when QDF found the problems of data. However, both QDF and RDA misclassified all regular cars to the small cars in the three variables because the seats of the small cars are four. If we add little random noise to the constant values, NMs become zero. This problem may be the defect of the generalized inverse matrix technique incremented in QDF of JMP. After this fact, modified RDA was released. We must choose two parameters of λ and γ . We choose the best combination such $\lambda=\gamma=0.1$ by 11*11 trials. Although the best combination is valid for this data, we use these values for another data because we think the survey of best combination is not meaningful for our research.

Table 1

Comparison of MNM and NMs

p	Var.	t	LDF	QDF	MNM	$\lambda=\gamma=0.8$	0.1
1	Emission (X1)	11.37	2	0	0	2	0
2	Price (X2)	5.42	1	0	0	4	0
3	Capacity (X3)	8.93	1	29	0	3	0
4	CO2 (X4)	4.27	1	29	0	4	0
5	Fuel (X5)	-4	0	29	0	5	0
6	Sales (X6)	-0.82	0	29	0	5	0

Results

The Outlook of Data

Fig.1 shows the Box-Whisker-Plots of the emission (X1) and capacity (X3). This graph tells us that we can obtain two 1-variable models are linear separable. Therefore, 48 models including (X1) or (X3) are linear separable among 63 models. Other 15 models are not linear separable.

Figure 1. Box-Whisker-Plots of Emission and Capacity (-1: Small Car, 1: Regular Car).

Table2. is the NMs of 48 linear separable models. We categorize these models to three groups. First 16 models from SN=1 to SN=16 include X1 and X3. Next 16 models from SN=17 to SN=32 include X1. Last 16 models from SN=33 to SN=48 include X3. The NMs of six MP-based LDFs, logistic regression and RDA are zero. Therefore, these results are omitted from the table. On the other hand, Fisher's LDF cannot recognize 41 models those are linear separable. Moreover, QDF misclassified 29 regular cars to the small cars for the first and third groups because these models include X3. If we add little random noise to X3 belonging to the small car class, all NMs become zero. Therefore, we conclude only Fisher's LDF can discriminate seven linear separable models among 48 linear separable models correctly.

Table 2

NMs of eight LDFs by 48 linear separable models

SN	Emission	Price	Capacity	CO2	Fuel	Sales	LDF	QDF
1	1	0	1	0	0	0	2	29
2	1	1	1	0	0	0	1	29
3	1	0	1	1	0	0	1	29
4	1	0	1	0	1	0	1	29
5	1	0	1	0	0	1	2	29
6	1	1	1	1	0	0	1	29
7	1	1	1	0	1	0	1	29
8	1	1	1	0	0	1	1	29
9	1	0	1	1	1	0	1	29
10	1	0	1	1	0	1	1	29
11	1	0	1	0	1	1	1	29
12	1	1	1	1	1	0	0	29
13	1	1	1	1	0	1	0	29
14	1	1	1	0	1	1	1	29
15	1	0	1	1	1	1	1	29
16	1	1	1	1	1	1	0	29
17	1	0	0	0	0	0	2	0
18	1	1	0	0	0	0	1	0
19	1	0	0	1	0	0	1	0
20	1	0	0	0	0	1	2	0
21	1	0	0	0	1	0	2	0

Table 2 continued

SN	Emission	Price	Capacity	CO2	Fuel	Sales	LDF	QDF
22	1	1	0	1	0	0	1	0
23	1	1	0	0	1	0	1	0
24	1	1	0	0	0	1	1	0
25	1	0	0	1	1	0	1	0
26	1	0	0	1	0	1	1	0
27	1	0	0	0	1	1	4	0
28	1	1	0	1	1	0	0	0
29	1	1	0	1	0	1	1	0
30	1	1	0	0	1	1	1	0
31	1	0	0	1	1	1	2	0
32	1	1	0	1	1	1	0	0
33	0	0	1	0	0	0	0	29
34	0	1	1	0	0	0	5	29
35	0	0	1	0	1	0	3	29
36	0	0	1	0	0	1	1	29
37	0	0	1	1	0	0	0	29
38	0	1	1	1	0	0	5	29
39	0	1	1	0	1	0	6	29
40	0	1	1	0	0	1	6	29
41	0	0	1	1	1	0	3	29
42	0	0	1	0	1	1	3	29
43	0	0	1	1	0	1	1	29
44	0	1	1	1	1	0	4	29
45	0	1	1	1	0	1	5	29
46	0	1	1	0	1	1	6	29
47	0	0	1	1	1	1	4	29
48	0	1	1	1	1	1	5	29

Comparison of fifteen Discriminant Function

Table 3 shows the NMs, and the number of cases of the discriminant-hyperplane $f(x_i)=0$ by 15 non-linear separable models. Four variables correspond to four columns. Because these models do not include X1 and X3, these two variables are omitted from the table. Sixth columns is MNM. Next eight columns are Diffs of eight discriminant functions. ‘Diff’ is the difference of (eight NM – MNM). We omit the NM of H-SVM because it cannot discriminate these models. Last column is the number of cases on $f(x_i)=0$ of Revised LP-OLDF. Other LDFs are free from **Problem1** for this data.

Table 3

Diffs and the number on $f(x)=0$ by 15 non-linear separable models

SN	Variable				MNM		Diff					f(x _i)=0		
	X2	X4	X5	X6	RIP	SVM4	SVM1	LP	IPLP	Logistic	LDF	QDF	RDA	LP
49	1	0	0	0	5	1	1	1	0	0	8	2	7	0
50	0	0	1	0	10	1	1	1	0	1	1	2	2	1
51	0	1	0	0	10	1	1	1	0	1	3	2	2	1
52	0	0	0	1	13	2	2	2	2	1	2	1	2	0

Table 3 continued

Variable					MNM			Diff					f(xi)=0	
SN	X2	X4	X5	X6	RIP	SVM4	SVM1	LP	IPLP	Logistic	LDF	QDF	RDA	LP
53	1	0	1	0	4	2	2	2	0	1	4	3	6	0
54	1	1	0	0	4	2	2	2	0	2	7	4	8	0
55	0	0	1	1	8	6	6	6	1	3	3	1	1	0
56	1	0	0	1	4	2	2	2	0	1	8	3	8	0
57	0	1	1	0	10	2	2	1	0	2	1	4	1	1
58	0	1	0	1	8	3	3	3	0	4	5	4	4	0
59	1	0	1	1	4	2	2	2	0	1	4	3	6	0
60	1	1	1	0	4	0	0	0	0	0	5	5	9	0
61	1	1	0	1	4	2	2	2	0	1	7	4	7	0
62	0	1	1	1	8	7	7	7	0	3	2	2	3	0
63	1	1	1	1	3	1	1	1	0	1	6	5	9	0

The 100-fold Cross validation Method

In this chapter, we compare six MP-based LDFs and two statistical LDFs by the method. In this research, we examine seven models made by three variables such as X1, X2, and X3 because other three variables are not important for the discrimination explained by **Table1**. Therefore, there are six linear separable models including X1 or X3, and one 1-variable model (X2) that is not linear separable. **Table 4** shows the results by the method. We omit QDF and RDA because those are not LDFs.

We examine seven discriminant models of seven LDFs and six models of H-SVM. First seven rows are seven discriminant models of Revised IP-OLDF (RIP). ‘Model’ column shows the independent variable. M1s and M2s are the mean of error rates in the training and validation samples. Six M1s and M2s of Revised LP-OLDF are zero. Six M1s and five M2s of Revised IP-OLDF and Revised IPLP-OLDF are zero. Six M1s and three M2s of HSVM, SVM4 and logistic regression are zero. Only two M1s and M2s of SVM1 are zero. All M1s and M2s of Fisher’s LDF are not zero. We can summarize these results about six linear separable models as follows:

We can roughly evaluate the ranking of eight LDFs as follows. Revised LP-OLDF is the first rank. Revised IP-OLDF and Revised IPLP-OLDF are the second rank. HSVM, SVM4 and logistic regression are the third rank. Although SVM1 and Fisher’s LDF are the fourth and fifth rank, these cannot recognize the linear separable models.

We can choose the fifth and sixth models are the best model because of two reasons. Although there are several models having “minimum M2 = 0”, we choose these models by the principal parsimony because those are 1-variable model. Revised IP-OLDF, SVM4, Revised LP-OLDF, Revised IPLP-OLDF and logistic regression choose these two models as the best models.

Seventh model is not linear separable. Six values of ‘M2Diff.’ are 27.89, 0.23, 0.23, 0.02, 0.26 and 14.28%, respectively. SVM4 and Fisher’s LDF are 27.89% and 14.28% worse than Revised IP-OLDF. Although the discrimination of one variable is not important, we must investigate the reason why these two results are so bad. On the other hand, absolute values of other LDFs are within 0.26%. In the pass/fail determinations [18][19], Revised IP-OLDF is superior to other seven LDFs for the linear separable data. We confirm these facts in this data.

Table 4

M1s and M2s of eight LDFs.

MNM		M1	M2	Diff2.	Model	
50s	1	0	0	0	X1, X2, X3	
	2	0	0	0	X1, X2	
	3	0	0.07	0.07	X1, X3	
	4	0	0	0	X2, X3	
	5	0	0	0	X1	
	6	0	0	0	X3	
	7	9.55	12.75	3.2	X2	
SVM4		M1	M2	Diff2.	M1Diff	M2Diff
40s 1		0	0.11	0.11	0	0.11
2		0	0.2	0.2	0	0.2
3		0	0	0	0	-0.07
4		0	0.11	0.11	0	0.11
5		0	0	0	0	0
6		0	0	0	0	0
7		40.45	40.64	0.18	30.91	27.89
SVM1		M1	M2	Diff2.	M1Diff	M2Diff
44S 1		1.14	1.2	0.07	1.14	1.2
2		0.98	1.7	0.73	0.98	1.7
3		0	0	0	0	-0.07
4		0.34	0.5	0.16	0.34	0.5
5		0.73	0.84	0.11	0.73	0.84
6		0	0	0	0	0
7		12.39	12.98	0.59	2.84	0.23
LP		M1	M2	Diff2.	M1Diff	M2Diff
1		0	0	0	0	0
2		0	0	0	0	0
3		0	0	0	0	-0.07
4		0	0	0	0	0
5		0	0	0	0	0
6		0	0	0	0	0
7		12.39	12.98	0.59	2.84	0.23
IPLP		M1	M2	Diff2.	M1Diff	M2Diff
1m40s1		0	0.27	0.27	0	0.27
2		0	0	0	0	0
3		0	0	0	0	-0.07
4		0	0	0	0	0
5		0	0	0	0	0
6		0	0	0	0	0
7		9.55	12.77	3.23	0	0.02
HSVM		M1	M2	Diff2.	M1Diff	M2Diff
1		0	0.11	0.11	0	0.11
2		0	0.2	0.2	0	0.2
3		0	0	0	0	-0.07
4		0	0.11	0.11	0	0.11
5		0	0	0	0	0
6		0	0	0	0	0

Table 4 to be continued

Logi	M1	M2	Diff2.	M1Diff	M2Diff
1	0	0.36	0.36	0	0.36
2	0	0.05	0.05	0	0.05
3	0	0.02	0.02	0	-0.05
4	0	0	0	0	0
5	0	0	0	0	0
6	0	0	0	0	0
7	12.3	13.01	0.72	2.75	0.26
LDF	M1	M2	Diff2.	M1Diff	M2Diff
1	1.5	2.35	0.85	1.5	2.35
2	1.89	2.91	1.03	1.89	2.91
3	4.52	4.75	0.23	4.52	4.68
4	10.43	12.83	2.4	10.43	12.83
5	5.36	5.74	0.38	5.36	5.74
6	4.7	6.09	1.38	4.7	6.09
7	26.89	27.03	0.14	17.34	14.28

The 95% CI of Discriminant Coefficients by six MP-based LDFs

Table 5 shows the 95% CI of eight LDFs. Because all intercepts fix to one, those are omitted from the table.

(1) Revised IP-OLDF

The equation (5) is the full model of Revised IP-OLDF showed by the median. Although the 95% CI of X3 is [-0.2222, -0.2222] and the constant, we guess this result caused by the “X3=4” of the small car. If X3 of the small car vary, the 95% CI is not the constant. Because both coefficients of X1 and X2 are zero, Revised IP-OLDF tells us that X1 and X2 are not important for the discrimination of this data. Although we choose two best models by the minimum M2 standard, we can judge the best model is the fifth or sixth model. Moreover, the forward stepwise procedure and t-test chose the X1 at first in the Table1. We claim the statistical suggestions are different from the 95% CI of coefficients. The equation (5) means the discriminant hyperplane is $X_3 = 4.504505$. If $X_3 < 4.5$, we can judge the car belongs to the small car. Otherwise, if $X_3 > 4.5$, we can judge the car belongs to the regular car. This discriminant rule is as same as the fact that the small cars have four seats and the regular cars have over five seats.

$$\text{RIP123: } 0 \times X_1 + 0 \times X_2 - 0.2222 \times X_3 + 1 = -0.2222 \times X_3 + 1 = 0 \quad (5)$$

The equations (6) is 2-variables models such as (X1, X2). Because the emission rate of small and regular cars ranges from [0.657, 0.658] and [0.996, 3.456], we can judge this model is linear separable by X1 without X2. The discriminant hyperplane is $X_1 = 0.82713$. The 95% CI of X1 in 1-variable model (X1) has almost the same 95% CI of fifth model. Both equations (5) and (6) tell us X3 is more important than X1 for the discrimination this data is linear separable.

$$\text{RIP12: } -1.209 \times X_1 + 0 \times X_2 + 1 = -1.209 \times X_1 + 1 = 0 \quad (6)$$

The equations (7) is 2-variables models such as (X1, X3). Although each variable recognize the linear separable model, both coefficients of (X1, X3) are zero at the 5% level. We guess each variable disturbs other variable and investigate this fact in near future.

$$\text{RIP13: } -1.078 \times X_1 + 0 \times X_3 = 0 \quad (7)$$

The equations (8) are 2-variables models such as (X2, X3). The 95% CI of X3 has the same 95% CI of X3 in the full model, and 1-variable model (X3).

Discriminant Analysis of the Linear Separable Data
- Japanese 44 Cars -

$$\text{RIP23: } 0 \times X_2 - 0.2222 \times X_3 + 1 = -0.2222 \times X_3 + 1 = 0 \quad (8)$$

The equation (9) is 1-variable model such as (X2). The discriminant hyperplane is $X_2=1333333$. The prices of four regular cars are less than 1,333 thousand yen and those of two small cars are higher than this price. Therefore, the error rate is 0.090909.

$$\text{RIP2: } -0.00000075 \times X_2 + 1 = 0 \quad (9)$$

(2) Comparison of H-SVM and Revised IP-OLDF

Fourth, sixth and fifth models of H-SVM and Revised IP-OLDF are the same. However, first, second and third models of both LDFs are quite different. This fact tells us as follows: Because H-SVM fixes some cases on two support vectors (SVs), three coefficients of X_1 , X_2 , and X_3 of H-SVM have ranges such as $[-0.11, -0.071]$, $[-7.2\text{E-}8, 4.5\text{E-}8]$, $[-0.218, -0.177]$, respectively. On the other hand, although Revised IP-OLDF has two SVs, it does not take effect of SVs and 100 LDFs become the same equation (5).

Table 5

The 95% confidence interval (CI) of eight LDFs.

	RIP	X1	X2	X3	H-SVM	X1	X2	X3
1	97.50%	0	0	-0.222	97.5%	-0.071	4.50E-08	-1.77E-01
	Median	0	0	-0.222	Median	-0.073	0	-0.209
	2.50%	0	0	-0.222	2.5%	-0.116	-7.20E-08	-0.218
2	97.50%	-1.053	0		97.5%	-0.5115	9.62E-07	
	Median	-1.209	0		Median	-1.2092	0	
	2.50%	-2.184	0		2.5%	-2.4382	-3.50E-07	
3	97.50%	379174		0	97.5%	-0.0707		-0.1978
	Median	-1.0776		0	Median	-0.0707		-0.2092
	2.50%	-1.2092		-74249	2.5%	-0.1155		-0.2092
4	97.50%		0	-0.222	97.5%		0.00E+00	-0.222
	Median		0	-0.222	Median		0.00E+00	-0.222
	2.50%		0	-0.222	2.5%		0.00E+00	-0.222
5	97.50%	-1.053			97.5%	-1.053		
	Median	-1.209			Median	-1.209		
	2.50%	-1.209			2.5%	-1.209		
6	97.50%			-0.222	97.5%			-0.222
	Median			-0.222	Median			-0.222
	2.50%			-0.222	2.5%			-0.222
7	97.50%		-6.70E-07					
	Median		-7.50E-07					
	2.50%		-8.20E-07					

The 95% CI of Discriminant Coefficients by Fisher's LDF and logistic regression

Table 6 shows the 95% CI of Fisher's LDF and logistic regression. We suppose the y_i as the object variable and analyze the data by the regression analysis because the obtained regression coefficients are propotional to the Fisher's LDF by the plug-in rule. First row is the coefficients. Second row is the standard error (SE). Third row is the p-value by t-test. We know only coefficient of X2 in the fourth model (X2, X3) is zero. Another coefficients are rejected at 5% or 1% levels. Therefore, it is difficult for us to choose good model among six models. On the other hand, we know all coefficient of six linear separable logistic regression models are zero [3]. Because seventh model is not linear separable, this 1-variable model (X2) is rejected at 1% level. We had better not trusted the SE and p-value of Fisher's LDF by the plug-in rule. In addition to this recommendation, we cannot use the SE and p-value of logistic regression.

Table 6

The 95% CI of Fisher's LDF and logistic regression.

	LDF	X1	X2	X3	c	logistic	X1	X2	X3	c
1	Coeff.	1.84	-7.90E-07	0.179	-1.709	Coeff.	-19.488	6.93E-06	-5.37E+01	251.697
	SE	0.25	1.56E-07	0.08	0.32	SE	0	2.317	2979878	1283091
	P	0.0001	0.0001	0.031	0.0001	P	.	1	1	1
2	Coeff.	2.119	-8.41E-07		-1.085	Coeff.	-179.039	1.75E-03		87.858
	SE	0.227	1.62E-07		0.164	SE	0	1.774		2497413
	P	0.0001	0.0001		0.0001	P	.	1		1
3	Coeff.	0.78		0.235	-2.003	Coeff.	-3.777		-34.414	158.844
	SE	0.175		0.101	0.399	SE	11909		8185.144	27565
	P	0.0001		0.024	0.0001	P	0.9997		0.9966	0.9954
4	Coeff.		1.62E-07	0.474	-2.443	Coeff.		-2.77E-07	-62.889	283.495
	SE		1.31E-07	0.105	0.46	SE		2.12E+00	2721988	11568553
	P		0.222	0.0001	0.0001	P		1	1	1
5	Coeff.	1.068			-1.191	Coeff.	-94.471			79.66
	SE	0.131			0.207	SE	6289			5492.898
	P	0.0001			0.0001	P	0.988			0.988
6	Coeff.			0.549	-2.528	Coeff.			-37.05	166.806
	SE			0.086	0.458	SE			3702.95	17078
	P			0.0001	0.0001	P			0.992	0.992
7	Coeff.		5.04E+00		-0.636	Coeff.		-6.97E-06		9.233
	SE		1.29E-07		0.273	SE		2.60E-06		3.421
	P		3.00E-04		0.025	P		7.30E-03		0.007

Consideration of the Statistical Analysis

Fig. 2 is score plots by the PCA by six independent variables. X-axis is first principal component and Y-axis are second and third principal components. The left small 99% probability ellipses are the small cars plotted by the symbol ‘.’. The right large 99% probability ellipses are the regular cars plotted by the symbol ‘+’. Although this data is linear separable, the regular car ellipses include the small car.

Figure 2. Score Plots by the principal components analysis (PCA).

If we use the indicator y_i as the dependent variable and analyze the car data by the regression analysis, the obtained regression coefficients are proportional to the discriminant coefficients by the plug-in rule. Therefore, we can use the model selection procedures and statistics of the regression analysis. Table 7 is a summary of all possible combinations of six variables. We sort six 1-variable models in descending order by R-squares. After 2-variables model, the forward stepwise procedure selects these five models. AIC and BIC recommend fifth model, and C_p statistics suggests full mode. Because our research suggests 1-variable model (X3), we must examine whether we can use these statistics for the discriminant analysis in near future.

Table 7

A summary of all possible combinations of six variables

p	Model	R-square	RMSE	AICc	BIC	C_p	RIP	LDF
1	X1	0.61	0.3	24.2	28.9	47.8	0	2
1	X3	0.49	0.35	35.9	40.7	74.8	0	0
1	X2	0.27	0.42	52	56.8	125.5	6	13
1	X5	0.25	0.42	53.1	57.8	129.4	11	11
1	X4	0.23	0.42	54	58.8	133.2	12	13
1	X6	0.02	0.48	65	69.8	182.4	15	15
2	X1, X2	0.77	0.24	4.3	10.4	14.9	0	1
3	X1-X3	0.79	0.23	1.7	9	11	0	1
4	X1-X4	0.82	0.22	-1.4	7	7.3	0	1
5	X1-X5	0.84	0.21	<u>-3.4</u>	<u>6</u>	5	0	0
6	X1-X6	0.84	0.21	-0.4	9.8	<u>7</u>	0	0

Conclusion

In this research, we discuss the new model selection procedure of the discriminant analysis. We discriminate the Japanese 44 cars data. Although AIC and BIC suggest 5-variables model and Cp suggests full model by the regression analysis, the “M2 minimum standard” procedure recommends 1-variable model (X3). Revised IP-OLDF, H-SVM, SVM4, SVM1, Revised IPLP-OLDF and logistic regression support this model. Moreover, the 95% CI of the discriminant coefficients suggest the best LDF as $f(x_3) = -0.2222 \cdot X_3 + 1$. The discriminant scores of the fifteen small cars are -0.8888 and the scores of the twenty-nine regular cars are greater than 1.111. This fact tells us that we need not discriminate the car data by the discriminant analysis. We can obtain clear judgment by looking at the distribution of six variables.

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