Direct Assessment of Interlaminar Stresses in Composite Multilayered Plates Using a Layer-Wise Mixed Finite Element Model

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Abstract: An accurate determination of interlaminar transversal stresses in composite multilayered plates, especially near free-edge, is of great importance in the study of inter-ply damage modes, mainly in the initiation and growth of delamination. In this paper, interlaminar stresses are determined by layer-wise mixed finite element model. Each layer is analyzed as an isolated one where the displacement continuity is ensured by means of Lagrange multipliers (which represent the statics variables). This procedure allows the authors to work with any single plate model, obtaining the interlaminar stresses directly without loss of precision. The FSDT (first shear deformation theory) with transverse normal strain effects included is assumed in each layer, but Lagrange polynomials are used to describe the kinematic instead of Taylor's polynomial functions of the thickness coordinates, as is common. This expansion allows the authors to pose the interlaminar displacements compatibility simpler than the second one. The in-plane domain of the plate is discretized by four-node quadrilateral elements, both to the field of displacement and to the Lagrange multipliers. The mixed interpolation of tensorial components technique is applied to avoid the shear-locking in the finite element model. Several examples were carried out and the results have been satisfactorily compared with those available in the literature.

Key words: Interlaminar stresses, multilayered plates, mixed finite element, Lagrange multipliers.

1. Introduction

Advanced composite materials are widely used in many branches of the engineering, such as civil, naval, aeronautic and aerospace industries, because of their better properties compared to conventional materials. Among the different types of composite materials, laminates are the most popular due to variety of structural applications. Composite laminates are formed by stacking of different layers. Since the laminas are linked through their faces, transverse shear stresses are developed on the interface between two layers. Although these stresses are negligible in the regions away from the laminate boundaries, generally transverse stress (σ_{xz} , σ_{yz} , σ_{zz}) concentrations are exhibited near material and geometric discontinuities (the so-called free-edge effect) that can lead to damage

in the form of delamination, matrix cracking and adhesive joint separation, causing loss of stiffness and a complex load redistribution [1]. The study of this problem arises around 1970 when many laminate test coupons loaded in axial tension exhibited unusual failure modes, precipitated near the free edges. By this time, Pipes and Pagano [2, 3] developed an elastic solution, based on finite difference technique, for finite width composite laminates under uniform axial strain. Their observations have led to the understanding of the interlaminar stresses in the region near the free edge of a laminate. They showed strong evidence of a singularity in the interlaminar σ_{xz} and σ_{zz} at the intersection of the interface and free-edge for angle-ply laminates ($\pm 45^\circ$)_s. These high stresses in the neighborhood of the free edge may be expected to cause delamination of the laminate, in particular under fatigue loadings. At present, a three-dimensional elasticity exact solution to evaluate the nature of stress

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concentration near edge boundary in composite laminates is unavailable yet. A complete review of literature is given by Kant and Swaminathan [4].

Thus, a challenging task in composite plates/shells modelling is obtained from an accurate estimation of interlaminar transverse stresses a priori (i.e., without requiring any post-processing procedure, such as integration of the three-dimensional equilibrium equations, that leads to loss of precision). This paper presents a numerical solution based on a layered model for the direct determination of interlaminar stresses via finite elements with mixed variables (kinematics and statics).

The paper has been organized as follows: Section 2 provides the mathematical formulation of the problem, including the kinematics and the strong form of the equilibrium equations. Section 3 develops the finite element model from the weak form of the governing equations. The methodology to solve just the static variables is shown, based on a strategy to eliminate the kinematic unknowns. Several examples are solved and exposed in Section 4. Finally, Section 6 contains the conclusions.

2. Problem Formulation

Throughout, Greek and Capital Latin indexes take

on the values 1, 2, 3 and 1, 2, respectively. Also, the repeated indexes are summed over their ranges, i.e., Einstein notation convention is adopted.

Infinitesimal strain and small displacement are assumed. Due to this, no distinction is made between the material coordinates and spatial coordinates, between the Green-Lagrange strain tensor and infinitesimal strain tensor, and between the second Piola-Kirchhoff stress tensor and the Cauchy stress tensor.

2.1 Kinematics

Considering a plate of total thickness h composed of n orthotropic layers which are perfectly bonded together, a Cartesian reference system is defined in the undeformed midplane Ω of the plate, where (x, y) are the in-plane coordinates (Fig. 1). The z axis is taken positive downward from the midplane. The generic i^{th} layer is located between the points $(z_{\text{top}} = z_i)$ $(z_{\text{bot}} = z_{i+1})$ in the thickness direction. For the purpose of modelling, all layers are assumed here to have the same in-plane dimensions (i.e., it is assumed that $\Omega_i = \Omega$) but the thickness h_i of each might differ. The middle surface Ω_i of each layer is bounded by Γ_i (Γ_i^u and Γ_i^σ refer to those parts of Γ_i on which the geometrical and mechanical boundary conditions are prescribed, respectively). The

Fig. 1 Multilayered plate: notations and definitions.

integer i , which is extensively used as both subscripts or superscripts, denotes the layer number that starts from the plate top. Symbols that are not affected by the index i refer to the whole plate.

Using CUF (Carrera's unified formulation) [5], the displacement field $\{u_\alpha, \alpha = 1, 2, 3\}$ for any point $\{x, y, z\}$ of the i^{th} layer can be expressed in a compact notation as shown in Eq. (1):

$$u_\alpha^{(i)}(x, y, z) = F_k(z) u_{\alpha k}^{(i)}(x, y) \quad k=1, \dots, N^* \quad (1)$$

where, $F_k(z)$ is the set of thickness functions and $N^* \geq 2$ is the order of the expansion, which can both be arbitrarily chosen. In the present work, a linear thickness expansion ($N^* = 2$) based on Lagrange polynomial is used. Furthermore, the same functions are used for all components of the unknown field variables, thus, in explicit way:

$$u_1^{(i)} = F_t u_{1t}^{(i)} + F_b u_{1b}^{(i)} \quad (2a)$$

$$u_2^{(i)} = F_t u_{2t}^{(i)} + F_b u_{2b}^{(i)} \quad (2b)$$

$$u_3^{(i)} = F_t u_{3t}^{(i)} + F_b u_{3b}^{(i)} \quad (2c)$$

in which the thickness function $F_k(z)$ are:

$$F_t(z) = \frac{1 + \zeta_i(z)}{2} \quad (3a)$$

$$F_b(z) = \frac{1 - \zeta_i(z)}{2} \quad (3b)$$

The dimensionless thickness coordinate $\zeta_i(z) \in [-1, +1]$ is introduced at layer level by the following relation:

$$\zeta_i(z) = -\frac{2}{h_i} z + \frac{z_i + z_{i+1}}{h_i} \quad (4)$$

The chosen functions have the following interesting properties:

$$\zeta_i = \begin{cases} +1: & F_t=1; \quad F_b=0 \\ -1: & F_t=0; \quad F_b=1 \end{cases} \quad (5)$$

It can be noticed that the top and bottom values (denoted by t and b , respectively) are used as unknowns, instead of variables referred to the midplane as usual in Taylor expansion. This representation is more convenient for the purpose of implementing the

interlaminar continuity between two adjacent layers (i.e., local displacements $\{u_\alpha, \alpha = 1, 2, 3\}$ at the bottom of the layer i and the top of the layer $i + 1$, have to be continuous). In a Taylor polynomial thickness expansion for a displacement variable described in a layer-wise sense is not convenient since many degrees of freedom are involved in the writing of the interlaminar compatibility condition. Based on Eq. (1), the strong form of such a compatibility condition is:

$$u_{\alpha t}^{(i+1)} - u_{\alpha b}^{(i)} = 0 \quad (6)$$

The Cartesian components of the infinitesimal strain tensor at level layer are:

$$\varepsilon_{AB}^{(i)} = \frac{1}{2} F_k (u_{A k, B}^{(i)} + u_{B k, A}^{(i)}) \quad (7a)$$

$$\varepsilon_{A3}^{(i)} = \frac{1}{2} (F_k u_{3 k, A}^{(i)} + F_{k, 3} u_{A k}^{(i)}) \quad (7b)$$

$$\varepsilon_{33}^{(i)} = F_{k, 3} u_{3 k}^{(i)} \quad k = t, b \quad (7c)$$

where, the comma indicates the spatial partial differentiation. It can be noted that the transverse normal strain ε_{33} is included in the kinematics. Models without transverse normal strain are not capable of accurately determining interlaminar stresses near geometric and material discontinuities such as holes or cut-outs, traction free edges, delamination fronts and matrix crack fronts, or in localized regions of intense loading. In modelling these localized effects, inclusion of the transverse normal strain is important for two reasons: First of all, the transverse normal stress is usually a significant, if not dominant, stress in these regions perhaps is even singular [3]. Secondly, layer-wise models that neglect transverse normal strain do not satisfy traction-free boundary conditions for transverse shear stresses at the laminate edges [1].

2.2 Governing Equations

Multilayered system is obtained as a superposition of n ply, where the interlaminar continuity requirements (Eq. (6)) are mathematically ensured by means of Lagrange multipliers $\{s_\alpha^{(j)} \alpha = 1, 2, 3\}$ which

represent surfaces forces distributed along of interfaces between two layers. Thus, the potential energy functional Π_{mP1} is expressed as:

$$\Pi_{mP1} = \sum_{i=1}^n \left\{ \int_{V_i} \frac{1}{2} \sigma_{\alpha\beta}^{(i)} \varepsilon_{\alpha\beta}^{(i)} dV \right\} + \sum_{j=2}^n \left\{ \int_{S_{j+1}} (u_{\alpha \text{ top}}^{(j)} - u_{\alpha \text{ bot}}^{(j-1)}) S_{\alpha}^j dS \right\} - W^{\text{ext}} \quad k = t, b \quad (8)$$

where, $\sigma_{\alpha\beta}$ are the Cartesian components of the Cauchy stress tensor and W^{ext} is the work done by the external layer-forces $\{p_{\alpha}^{(ik)}, \alpha = 1, 2, 3, k = t, b\}$. The i -layer volume is denoted as V_i , while S_{j+1} refers to the interface between the layers j and $j+1$. The other quantities have already been defined previously. It can be noted that the first term in the right-hand side of Eq. (8) represents the internal work in each layer, and the second one variationally enforces the subsidiary conditions (Eq. (6)). The Lagrange multipliers introduced have the following physical interpretation [6] (Fig. 2):

$$S_A^{(j)} = \sigma_{A3}^{(j)} \quad (9a)$$

$$S_3^{(j)} = \sigma_{33}^{(j)} \quad (9b)$$

Indeed, the interface conditions (Eq. (6)) could be imposed in a procedure similar to standard assembly techniques through the thickness involved in a finite element implementation, such as Refs. [7, 8]. However,

the use of Lagrange multipliers is preferred herein in order to directly assess the interlaminar transverse stresses.

The principle associated to the functional Eq. (8) is commonly called “the first modified principle of potential energy with relaxed continuity requirements” because the displacement functions in each element (or layer, in this case) may be chosen independently without any concern about the conformity requirement [9]. This approach can be interpreted as a hybrid formulation in the thickness.

In the analysis, each lamina is treated as a homogeneous, orthotropic, elastic body whose behaviour is described by the Hooke’s law:

$$\sigma_{AB}^{(i)} = C_{ABCD}^{(i)} \varepsilon_{CD}^{(i)} + C_{AB33}^{(i)} \varepsilon_{33}^{(i)} \quad (10a)$$

$$\sigma_{A3}^{(i)} = C_{A3C3}^{(i)} \varepsilon_{A3}^{(i)} \quad (10b)$$

$$\sigma_{33}^{(i)} = C_{33AB}^{(i)} \varepsilon_{AB}^{(i)} + C_{3333}^{(i)} \varepsilon_{33}^{(i)} \quad (10c)$$

where, $C_{\alpha\beta\gamma\eta}$ are the Cartesian components of the fourth-order constitutive tensor.

On substituting of Eqs. (1), (7) and (10) into Eq. (8), making some trivial manipulations, and integrating by parts at layer level (details are omitted for brevity), the stationary conditions are:

$$\begin{aligned} \delta \Pi_{mP1} = & \sum_{i=1}^n \left\{ \int_{\Omega_i} \left(\delta u_{Ak}^{(i)} E_{kl}^{(i)} C_{ABCD}^{(i)} u_{C l, D, B}^{(i)} + \delta u_{Ak}^{(i)} E_{kl, 3}^{(i)} C_{AB33}^{(i)} u_{3 l, B}^{(i)} + \delta u_{3 k}^{(i)} E_{kl}^{(i)} C_{A3C3}^{(i)} u_{3 l, C, A}^{(i)} + \delta u_{3 k}^{(i)} E_{kl, 3}^{(i)} C_{A3C3}^{(i)} u_{C l, A}^{(i)} \right. \right. \\ & \left. \left. + \delta u_{Ak}^{(i)} E_{k, 3 l}^{(i)} C_{A3C3}^{(i)} u_{3 l, C}^{(i)} + \delta u_{Ak}^{(i)} E_{k, 3 l, 3}^{(i)} C_{A3C3}^{(i)} u_{C l}^{(i)} + \delta u_{3 k}^{(i)} E_{k, 3 l}^{(i)} C_{33CD}^{(i)} u_{C l, D}^{(i)} + \delta u_{3 k}^{(i)} E_{k, 3 l, 3}^{(i)} C_{3333}^{(i)} u_{3 l}^{(i)} \right) d\Omega \right\} \\ & + \sum_{j=1}^{n-1} \left\{ \int_{S_{j+1}} \left(s_{\alpha}^{(j+1)} \delta u_{\alpha t}^{(j+1)} - s_{\alpha}^{(j+1)} \delta u_{\alpha b}^{(j)} \right) dS + \int_{S_{j+1}} \left(u_{\alpha t}^{(j+1)} - u_{\alpha b}^{(j)} \right) \delta s_{\alpha}^{(j+1)} dS \right\} - \delta W^{\text{ext}} = 0 \quad (11) \end{aligned}$$

Fig. 2 Physical meaning of Lagrange multipliers: compatibility condition between two adjacent layers.

$$k = t, b; l = t, b$$

where, δ means the variational operator, and the integration over domain is represented as:

$$\int_{V_i} (\cdot) dV = \int_{\Omega_i} \int_{h_i} (\cdot) dz d\Omega \quad (12)$$

Integration across the thickness is carried out explicitly due to the displacement field in the transverse coordinate, reducing the three-dimensional problem to a two-dimensional one, as usual in plate

theory. To the purpose, the following definitions have been introduced:

$$\left(E_{kl}^{(i)}, E_{kl,3}^{(i)}, E_{k,3l}^{(i)}, E_{k,3l,3}^{(i)} \right) =$$

$$\int_{h_i} (F_k F_l, F_k F_{l,3}, F_{k,3} F_l, F_{k,3} F_{l,3}) dz \quad (13)$$

Since the variations are arbitrary, the paper obtains the $(9n + 3)$ Euler-Lagrange equations that governing the problem:

$$\delta u_{A t}^{(i)} : E_{tl}^{(i)} C_{ABCD}^{(i)} u_{C l, D B}^{(i)} + E_{tl,3}^{(i)} C_{AB33}^{(i)} u_{3 l, B}^{(i)} + E_{t,3l}^{(i)} C_{A3C3}^{(i)} u_{3 l, C}^{(i)} + E_{t,3l,3}^{(i)} C_{A3C3}^{(i)} u_{C l}^{(i)} + s_A^{(i)} = p_{A t}^{(i)} \quad (14)$$

$$\delta u_{3 t}^{(i)} : E_{tl}^{(i)} C_{A3C3}^{(i)} u_{3 l, C A}^{(i)} + E_{tl,3}^{(i)} C_{A3C3}^{(i)} u_{C l, A}^{(i)} + E_{t,3l}^{(i)} C_{33CD}^{(i)} u_{C l, D}^{(i)} + E_{t,3l,3}^{(i)} C_{3333}^{(i)} u_{3 l}^{(i)} + s_3^{(i)} = p_{3 t}^{(i)} \quad (15)$$

$$\delta u_{A b}^{(i)} : E_{bl}^{(i)} C_{ABCD}^{(i)} u_{C l, D B}^{(i)} + E_{bl,3}^{(i)} C_{AB33}^{(i)} u_{3 l, B}^{(i)} + E_{b,3l}^{(i)} C_{A3C3}^{(i)} u_{3 l, C}^{(i)} + E_{b,3l,3}^{(i)} C_{A3C3}^{(i)} u_{C l}^{(i)} - s_A^{(i+1)} = p_{A b}^{(i)} \quad (16)$$

$$\delta u_{3 b}^{(i)} : E_{bl}^{(i)} C_{A3C3}^{(i)} u_{3 l, C A}^{(i)} + E_{bl,3}^{(i)} C_{A3C3}^{(i)} u_{C l, A}^{(i)} + E_{b,3l}^{(i)} C_{33CD}^{(i)} u_{C l, D}^{(i)} + E_{b,3l,3}^{(i)} C_{3333}^{(i)} u_{3 l}^{(i)} - s_3^{(i+1)} = p_{3 b}^{(i)} \quad (17)$$

$$\delta s_{\alpha}^{(j)} : u_{\alpha t}^{(j)} - u_{\alpha b}^{(j-1)} = 0 \quad (18)$$

$$l = t, b; i = 1, \dots, n; j = 2, \dots, n$$

Explicit form of the governing equations for each layer can be written by expanding the introduced subscripts and superscripts over their ranges. The boundary conditions are omitted for brevity. Set of Eqs. (14)-(18) describes $(6n)$ second-order PDE (partial differential equations) coupling to $(3n - 3)$

algebraic equations in terms of $(9n - 3)$ variables.

3. Finite Element Discretization

The weak form of the equations that governing the problem are obtained multiplying the Euler-Lagrange equations by their respective variations, integrating over the element domain, and integrating by parts to weaken the differentiability of $\delta u_{ak}^{(i)}$ ($\alpha = 1, 2, 3$; $k = t, b$):

$$\int_{\Omega_i} \left(\delta u_{A t, B}^{(i)} E_{tl}^{(i)} C_{ABCD}^{(i)} u_{C l, D B}^{(i)} + \delta u_{A t, B}^{(i)} E_{tl,3}^{(i)} C_{AB33}^{(i)} u_{3 l, B}^{(i)} + \delta \gamma_{A3}^{(i)} \Big|_{u_{A t}} C_{A3C3}^{(i)} \gamma_{C3}^{(i)} + \delta u_{A t}^{(i)} s_A^{(i)} - \delta u_{A t}^{(i)} p_{A t}^{(i)} \right) d\Omega = 0 \quad (19)$$

$$\int_{\Omega_i} \left(\delta \gamma_{A3}^{(i)} \Big|_{u_{3 t, A}} C_{A3C3}^{(i)} \gamma_{C3}^{(i)} + \delta u_{3 t}^{(i)} E_{t,3l}^{(i)} C_{33CD}^{(i)} u_{C l, D}^{(i)} + \delta u_{3 t}^{(i)} E_{t,3l,3}^{(i)} C_{3333}^{(i)} u_{3 l}^{(i)} + \delta u_{A t}^{(i)} s_3^{(i)} - \delta u_{A t}^{(i)} p_{3 t}^{(i)} \right) d\Omega = 0 \quad (20)$$

$$\int_{\Omega_i} \left(\delta u_{A b, B}^{(i)} E_{bl}^{(i)} C_{ABCD}^{(i)} u_{C l, D B}^{(i)} + \delta u_{A b, B}^{(i)} E_{bl,3}^{(i)} C_{AB33}^{(i)} u_{3 l, B}^{(i)} + \delta \gamma_{A3}^{(i)} \Big|_{u_{A b}} C_{A3C3}^{(i)} \gamma_{C3}^{(i)} + \delta u_{A b}^{(i)} s_A^{(i)} - \delta u_{A b}^{(i)} p_{A b}^{(i)} \right) d\Omega = 0 \quad (21)$$

$$\int_{\Omega_i} \left(\delta \gamma_{A3}^{(i)} \Big|_{u_{3 b, A}} C_{A3C3}^{(i)} \gamma_{C3}^{(i)} + \delta u_{3 b}^{(i)} E_{b,3l}^{(i)} C_{33CD}^{(i)} u_{C l, D}^{(i)} + \delta u_{3 b}^{(i)} E_{b,3l,3}^{(i)} C_{3333}^{(i)} u_{3 l}^{(i)} - \delta u_{A b}^{(i)} s_3^{(i)} - \delta u_{A b}^{(i)} p_{3 b}^{(i)} \right) d\Omega = 0 \quad (22)$$

$$\int_{\Omega_i} (u_{\alpha t}^{(j)} - u_{\alpha b}^{(j-1)}) d\Omega = 0 \quad (23)$$

$$l = t, b; i = 1, \dots, n; j = 2, \dots, n$$

where, the notation $\delta(*)|_{(\cdot)}$ has been introduced to denote the variation of $(*)$ with respect to $(\cdot)$.

The kinematic variables require C^0 continuity

in-plane due to just their first derivative are present in the Eqs. (19)-(23), while the statics variables can be discontinuous (C^{-1} continuity). However, by simplicity the same interpolation is used for every unknown.

Using isoparametric finite element [10], the Eq. (24) is shown as follows:

$$u_{\alpha k}^{(i)} = \sum_{m=1}^{NN} N_m \hat{u}_{\alpha k}^{m(i)} \quad k=t, b; i=1, \dots, n \quad (24a)$$

$$s_{\alpha}^{(j)} = \sum_{m=1}^{NN} N_m \hat{s}_{\alpha}^{m(j)} \quad j=2, \dots, n \quad (24b)$$

$$x_A = \sum_{m=1}^{NN} N_m \hat{x}_A^m \quad (24c)$$

where, $N_m(\xi, \eta)$ are the shape functions associated to m^{th} -node and NN is the number of nodes of the element (in this paper, a four-node quadrilateral element is used,

i.e., $NN = 4$). The circumflex tilde denotes nodal variable.

As is well known, the four-node quadrilateral element with transverse shear strain locks when it is thin (assuming full numerical integration) [10]. In order to avoid the shear-locking, the mixed interpolation of tensorial components [11] technique is applied to assess the transverse shear strain at layer level, resulting the following Eq. (25):

$$\begin{aligned} \gamma_{13}^{(i)} = & \frac{C_y + \xi B_y}{8 \det(\mathbf{J})} \left\{ (1 + \eta) \left[F_t \frac{\hat{u}_{3t}^{1(i)} - \hat{u}_{3t}^{2(i)}}{2} + F_b \frac{\hat{u}_{3b}^{1(i)} - \hat{u}_{3b}^{2(i)}}{2} - \frac{\hat{x}^1 - \hat{x}^2}{4h_i} (\hat{u}_{1t}^{1(i)} + \hat{u}_{1t}^{2(i)}) \right. \right. \\ & - \frac{\hat{y}^1 - \hat{y}^2}{4h_i} (\hat{u}_{2t}^{1(i)} + \hat{u}_{2t}^{2(i)}) + \frac{\hat{x}^1 - \hat{x}^2}{4h_i} (\hat{u}_{1b}^{1(i)} + \hat{u}_{1b}^{2(i)}) + \frac{\hat{y}^1 - \hat{y}^2}{4h_i} (\hat{u}_{2b}^{1(i)} + \hat{u}_{2b}^{2(i)}) \left. \right] \\ & + (1 - \eta) \left[F_t \frac{\hat{u}_{3t}^{4(i)} - \hat{u}_{3t}^{3(i)}}{2} + F_b \frac{\hat{u}_{3b}^{4(i)} - \hat{u}_{3b}^{3(i)}}{2} - \frac{\hat{x}^4 - \hat{x}^3}{4h_i} (\hat{u}_{1t}^{4(i)} + \hat{u}_{1t}^{3(i)}) \right. \\ & - \frac{\hat{y}^4 - \hat{y}^3}{4h_i} (\hat{u}_{2t}^{4(i)} + \hat{u}_{2t}^{3(i)}) + \frac{\hat{x}^4 - \hat{x}^3}{4h_i} (\hat{u}_{1b}^{4(i)} + \hat{u}_{1b}^{3(i)}) + \frac{\hat{y}^4 - \hat{y}^3}{4h_i} (\hat{u}_{2b}^{4(i)} + \hat{u}_{2b}^{3(i)}) \left. \right] \\ & - \frac{A_y + \eta B_y}{8 \det(\mathbf{J})} \left\{ (1 + \xi) \left[F_t \frac{\hat{u}_{3t}^{1(i)} - \hat{u}_{3t}^{2(i)}}{2} + F_b \frac{\hat{u}_{3b}^{1(i)} - \hat{u}_{3b}^{2(i)}}{2} - \frac{\hat{x}^1 - \hat{x}^4}{4h_i} (\hat{u}_{1t}^{1(i)} + \hat{u}_{1t}^{4(i)}) \right. \right. \\ & - \frac{\hat{y}^1 - \hat{y}^4}{4h_i} (\hat{u}_{2t}^{1(i)} + \hat{u}_{2t}^{4(i)}) + \frac{\hat{x}^1 - \hat{x}^4}{4h_i} (\hat{u}_{1b}^{1(i)} + \hat{u}_{1b}^{4(i)}) + \frac{\hat{y}^1 - \hat{y}^4}{4h_i} (\hat{u}_{2b}^{1(i)} + \hat{u}_{2b}^{4(i)}) \left. \right] \\ & + (1 - \xi) \left[F_t \frac{\hat{u}_{3t}^{2(i)} - \hat{u}_{3t}^{1(i)}}{2} + F_b \frac{\hat{u}_{3b}^{2(i)} - \hat{u}_{3b}^{1(i)}}{2} - \frac{\hat{x}^2 - \hat{x}^3}{4h_i} (\hat{u}_{1t}^{2(i)} + \hat{u}_{1t}^{1(i)}) \right. \\ & - \frac{\hat{y}^2 - \hat{y}^3}{4h_i} (\hat{u}_{2t}^{2(i)} + \hat{u}_{2t}^{1(i)}) + \frac{\hat{x}^2 - \hat{x}^3}{4h_i} (\hat{u}_{1b}^{2(i)} + \hat{u}_{1b}^{1(i)}) + \frac{\hat{y}^2 - \hat{y}^3}{4h_i} (\hat{u}_{2b}^{2(i)} + \hat{u}_{2b}^{1(i)}) \left. \right] \\ & - \frac{\hat{y}^2 - \hat{y}^3}{4h_i} (\hat{u}_{2t}^{2(i)} + \hat{u}_{2t}^{1(i)}) + \frac{\hat{x}^2 - \hat{x}^3}{4h_i} (\hat{u}_{1b}^{2(i)} + \hat{u}_{1b}^{1(i)}) + \frac{\hat{y}^2 - \hat{y}^3}{4h_i} (\hat{u}_{2b}^{2(i)} + \hat{u}_{2b}^{1(i)}) \left. \right] \end{aligned} \quad (25a)$$

$$\begin{aligned} \gamma_{23}^{(i)} = & \frac{C_x + \xi B_x}{8 \det(\mathbf{J})} \left\{ (1 + \eta) \left[F_t \frac{\hat{u}_{3t}^{1(i)} - \hat{u}_{3t}^{2(i)}}{2} + F_b \frac{\hat{u}_{3b}^{1(i)} - \hat{u}_{3b}^{2(i)}}{2} - \frac{\hat{x}^1 - \hat{x}^2}{4h_i} (\hat{u}_{1t}^{1(i)} + \hat{u}_{1t}^{2(i)}) \right. \right. \\ & - \frac{\hat{y}^1 - \hat{y}^2}{4h_i} (\hat{u}_{2t}^{1(i)} + \hat{u}_{2t}^{2(i)}) + \frac{\hat{x}^1 - \hat{x}^2}{4h_i} (\hat{u}_{1b}^{1(i)} + \hat{u}_{1b}^{2(i)}) + \frac{\hat{y}^1 - \hat{y}^2}{4h_i} (\hat{u}_{2b}^{1(i)} + \hat{u}_{2b}^{2(i)}) \left. \right] \end{aligned}$$

$$\begin{aligned}
& + (1-\eta) \left[F_t \frac{\hat{u}_{3t}^{(i)} - \hat{u}_{3t}^{(i)}}{2} + F_b \frac{\hat{u}_{3b}^{(i)} - \hat{u}_{3b}^{(i)}}{2} - \frac{\hat{x}^4 - \hat{x}^3}{4h_i} (\hat{u}_{1t}^{(i)} + \hat{u}_{1t}^{(i)}) \right. \\
& \left. - \frac{\hat{y}^4 - \hat{y}^3}{4h_i} (\hat{u}_{2t}^{(i)} + \hat{u}_{2t}^{(i)}) + \frac{\hat{x}^4 - \hat{x}^3}{4h_i} (\hat{u}_{1b}^{(i)} + \hat{u}_{1b}^{(i)}) + \frac{\hat{y}^4 - \hat{y}^3}{4h_i} (\hat{u}_{2b}^{(i)} + \hat{u}_{2b}^{(i)}) \right] \\
& - \frac{A_x + \eta B_x}{8 \det(\mathbf{J})} \left\{ (1+\xi) \left[F_t \frac{\hat{u}_{3t}^{(i)} - \hat{u}_{3t}^{(i)}}{2} + F_b \frac{\hat{u}_{3b}^{(i)} - \hat{u}_{3b}^{(i)}}{2} - \frac{\hat{x}^1 - \hat{x}^4}{4h_i} (\hat{u}_{1t}^{(i)} + \hat{u}_{1t}^{(i)}) \right. \right. \\
& \left. \left. - \frac{\hat{y}^1 - \hat{y}^4}{4h_i} (\hat{u}_{2t}^{(i)} + \hat{u}_{2t}^{(i)}) + \frac{\hat{x}^1 - \hat{x}^4}{4h_i} (\hat{u}_{1b}^{(i)} + \hat{u}_{1b}^{(i)}) + \frac{\hat{y}^1 - \hat{y}^4}{4h_i} (\hat{u}_{2b}^{(i)} + \hat{u}_{2b}^{(i)}) \right] \right. \\
& \left. + (1-\xi) \left[F_t \frac{\hat{u}_{3t}^{(i)} - \hat{u}_{3t}^{(i)}}{2} + F_b \frac{\hat{u}_{3b}^{(i)} - \hat{u}_{3b}^{(i)}}{2} - \frac{\hat{x}^2 - \hat{x}^3}{4h_i} (\hat{u}_{1t}^{(i)} + \hat{u}_{1t}^{(i)}) \right. \right. \\
& \left. \left. - \frac{\hat{y}^2 - \hat{y}^3}{4h_i} (\hat{u}_{2t}^{(i)} + \hat{u}_{2t}^{(i)}) + \frac{\hat{x}^2 - \hat{x}^3}{4h_i} (\hat{u}_{1b}^{(i)} + \hat{u}_{1b}^{(i)}) + \frac{\hat{y}^2 - \hat{y}^3}{4h_i} (\hat{u}_{2b}^{(i)} + \hat{u}_{2b}^{(i)}) \right] \right\} \quad (25b)
\end{aligned}$$

The determinant of the Jacobian matrix is:

$$\det(\mathbf{J}) = x_{,\xi} y_{,\eta} - y_{,\xi} x_{,\eta} \quad (26)$$

and

$$\begin{aligned}
A_x &= \hat{x}^1 - \hat{z}^2 - \hat{z}^3 + \hat{z}^4, & B_x &= \hat{x}^1 - \hat{x}^2 + \hat{x}^3 - \hat{x}^4, & C_x &= \hat{x}^1 + \hat{x}^2 - \hat{x}^3 - \hat{x}^4 \\
A_y &= \hat{y}^1 - \hat{y}^2 - \hat{y}^3 + \hat{y}^4, & B_y &= \hat{y}^1 - \hat{y}^2 + \hat{y}^3 - \hat{y}^4, & C_y &= \hat{y}^1 + \hat{y}^2 - \hat{y}^3 - \hat{y}^4
\end{aligned} \quad (27)$$

Introducing Eqs. (24) and (25) into Eqs. (19)-(23), after some manipulations, the finite element model is obtained, for a three-layered plate:

$$\begin{bmatrix}
\mathbf{K}_{tt}^{(1)} & \mathbf{K}_{tb}^{(1)} & \mathbf{0} & & & \\
\mathbf{K}_{tb}^{(1)\top} & \mathbf{K}_{bb}^{(1)} & -\mathbf{Q}_{tb} & & & \\
\mathbf{0} & -\mathbf{Q}_{tb} & \mathbf{0} & \mathbf{Q}_{tb} & \mathbf{0} & \mathbf{0} \\
& & \mathbf{Q}_{tb} & \mathbf{K}_{tt}^{(2)} & \mathbf{K}_{tb}^{(2)} & \mathbf{0} \\
& & \mathbf{0} & \mathbf{K}_{tb}^{(2)\top} & \mathbf{K}_{bb}^{(2)} & -\mathbf{Q}_{tb} \\
& & \mathbf{0} & \mathbf{0} & -\mathbf{Q}_{tb} & \mathbf{0} \\
& & & \mathbf{Q}_{tb} & \mathbf{0} & \mathbf{Q}_{tb} & \mathbf{0} \\
& & & \mathbf{0} & \mathbf{K}_{tt}^{(3)} & \mathbf{K}_{tb}^{(3)} \\
& & & & \mathbf{K}_{tb}^{(3)\top} & \mathbf{K}_{bb}^{(3)}
\end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{u}}_t^{(1)} \\ \hat{\mathbf{u}}_b^{(1)} \\ \hat{\mathbf{s}}^{(2)} \\ \hat{\mathbf{u}}_t^{(2)} \\ \hat{\mathbf{u}}_b^{(2)} \\ \hat{\mathbf{s}}^{(3)} \\ \hat{\mathbf{u}}_t^{(3)} \\ \hat{\mathbf{u}}_b^{(3)} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_t^{(1)} \\ \mathbf{F}_b^{(1)} \\ \mathbf{0} \\ \mathbf{F}_t^{(2)} \\ \mathbf{F}_b^{(2)} \\ \mathbf{0} \\ \mathbf{F}_t^{(3)} \\ \mathbf{F}_b^{(3)} \end{bmatrix} \quad (28)$$

where, $\mathbf{K}_{tt}^{(i)}$, $\mathbf{K}_{tb}^{(i)}$ and $\mathbf{K}_{bb}^{(i)}$ are the stiffness matrices related to top-top, top-bottom and bottom-bottom

degrees of freedom, respectively, and $\mathbf{Q}_{tb}$ is the matrix resulting from the Lagrangian constraint problem. The

arrays ($\mathbf{F}_t^{(i)}$, $\mathbf{F}_b^{(i)}$) represent the equivalent nodal loads applied at the i^{th} -layer. Explicit form of mentioned arrays could be obtained as is common in finite element technique; herein, these are omitted by

$$\begin{bmatrix} \mathbf{Q}_{tb} \cdot \bar{\mathbf{K}}_b^{(1)-1} \cdot \mathbf{Q}_{tb} + \mathbf{Q}_{tb} \cdot \bar{\mathbf{K}}_t^{(2)-1} \cdot \mathbf{Q}_{tb} & \mathbf{Q}_{tb} \cdot \bar{\mathbf{K}}_{tb}^{(2)} \cdot \mathbf{Q}_{tb} \\ \mathbf{Q}_{tb} \cdot \bar{\mathbf{K}}_{tb}^{(2)T} \cdot \mathbf{Q}_{tb} & \mathbf{Q}_{tb} \cdot \bar{\mathbf{K}}_b^{(2)-1} \cdot \mathbf{Q}_{tb} + \mathbf{Q}_{tb} \cdot \bar{\mathbf{K}}_t^{(3)-1} \cdot \mathbf{Q}_{tb} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{s}}^{(2)} \\ \hat{\mathbf{s}}^{(3)} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{F}}^{(2)} \\ \bar{\mathbf{F}}^{(3)} \end{bmatrix} \quad (29)$$

where:

$$\bar{\mathbf{K}}_t^{(i)} = \mathbf{K}_t^{(i)} - \mathbf{K}_{tb}^{(i)} \cdot \mathbf{K}_{bb}^{(i)-1} \cdot \mathbf{K}_{tb}^{(i)T} \quad (30)$$

$$\bar{\mathbf{K}}_b^{(i)} = \mathbf{K}_{bb}^{(i)} - \mathbf{K}_{tb}^{(i)T} \cdot \mathbf{K}_t^{(i)-1} \cdot \mathbf{K}_{tb}^{(i)} \quad (31)$$

$$\bar{\mathbf{K}}_{tb}^{(i)} = \bar{\mathbf{K}}_t^{(i)-1} \cdot \mathbf{K}_{tb}^{(i)} \cdot \mathbf{K}_{bb}^{(i)-1} \quad (32)$$

$$\bar{\mathbf{F}}^{(i)} = \mathbf{Q}_{tb}^{(i-1)} \cdot \bar{\mathbf{K}}_{tb}^{(i-1)T} \cdot \mathbf{F}_t^{(i-1)} - \mathbf{Q}_{tb}^{(i-1)} \cdot \bar{\mathbf{K}}_b^{(i-1)T} \cdot \mathbf{Q}_{tb}^{(i-1)} \cdot \mathbf{F}_b^{(i-1)} - \mathbf{Q}_{tb}^{(i)} \cdot \bar{\mathbf{K}}_t^{(i)-1} \cdot \mathbf{Q}_{tb}^{(i)} \mathbf{F}_t^{(i)} + \mathbf{Q}_{tb}^{(i)} \cdot \bar{\mathbf{K}}_{tb}^{(i)} \cdot \mathbf{F}_b^{(i)} \quad i = 2, \dots, n \quad (33)$$

The obtained matrix (Eq. (29)) is both symmetric and positive-definite. Necessary but not sufficient stability condition to solve it is [12]:

$$n_u \geq n_s \quad (34)$$

where, n_u and n_s are the total number of degrees of freedom associated with the kinematic and static variables, respectively.

Note that, if desired, the displacements could be subsequently determined from the static variables.

4. Numerical Results

The first example is a free-edge bending problem solved by Tahani and Nosier [13]. Considering a $(0^\circ/90^\circ)_s$ cross-ply laminate subjected to uniform transverse load q_o on its top surface, the load is applied downward. The geometric of the plate, simply supported on two opposite edge and free in the other remaining two, meets the following relationships: $a = 10h$ and $b = 4h$, where, (a, b) are the dimensions coincident with the (x, y) directions, respectively, and h is the total thickness (Fig. 1). Each lamina is assumed to be of the same thickness h_i and has the following material properties:

$$E_L = 20 \times 10^6 \text{ psi}; \quad E_T = 2.1 \times 10^6 \text{ psi};$$

$$G_{LT} = G_{TT} = 0.85 \times 10^6 \text{ psi}; \quad \nu_{LT} = \nu_{TT} = 0.21 \quad (35)$$

brevity.

Static variables (interlaminar transverse stresses) can be computed directly eliminating the displacement variables. After some algebra, one can write:

where, the subscripts L and T signify the direction parallel to the fibers and the transverse direction perpendicular to the fibers, respectively. Due to the symmetry, just one quarter of the plate is modelled. The in-plane mesh has 5×11 four-node quadrilateral elements. All elements have the same length (i.e., $a/2/5$). However, a fine finite element mesh near the free edges was used in the modelling (Fig. 3). Thus, the width of the elements decreases as the free edge at $(x, b/2, z)$ is approached. The widths of the 11 rows of elements, as one moves away from the refined free edge, are $h_i/32, h_i/32, h_i/16, h_i/8, h_i/4, h_i/2, h_i/2, h_i, 1.5h_i, 1.5h_i$ and $2.5h_i$. By the same criterion, the thickness of the numerical layers decreases as the interface between two physical layers is approached. From the top to the midplane, the layer thickness are $h_i/3, h_i/3, h_i/6, h_i/6, h_i/6, h_i/6, h_i/3, h_i/6$ and $h_i/6$ (the same distribution is from the bottom to the midplane). The boundary conditions applied are:

$$\begin{aligned} (0, y, z): u_1 = u_3 = 0; \quad (a/2, y, z): u_1 = 0; \\ (x, 0, z): u_2 = 0 \end{aligned} \quad (36)$$

All the numerical results are normalized as follows:

$$\bar{\sigma}_{A3} = \frac{h}{q_o a} \sigma_{A3} \quad (37a)$$

$$\bar{\sigma}_{33} = \frac{1}{q_o} \sigma_{33} \quad (37b)$$

Fig. 4 illustrates the variation of transverse normal stress through the thickness at $x = 5h$. The distribution obtained is in good accordance with those by Tahani and Nosier [13]. The plot shown suggests that a stress singularity may exist at the intersections of the $0^\circ/90^\circ$ interfaces and the free-edges, where the maximum bending stress $\sigma_{33} = q_0$ is approximately tripled. However, such a conjecture cannot be proved by the use of approximate solutions such as numerical methods. This interlaminar stress concentration near the edge of the laminate can initiate delamination damage resulting in a significant loss of stiffness.

Typical distributions of the interlaminar shear stress σ_{23} across the middle plane ($z = 0$) and the upper $0^\circ/90^\circ$ interface ($z = -h/4$) laminate at $x = 2h$ are shown in

Fig. 3 The modelling: (a) in-plane finite element mesh; (b) thickness discretization.

Fig. 4 Normalized transverse normal stress σ_{33} through the thickness at $x = 5h$ and the free-edge (laminate $(0^\circ/90^\circ)_s$) (comparison with solution of Ref. [14]).

Fig. 5. It is noted that the magnitude of σ_{23} is increased in the boundary layer region and exhibits singular behaviour near the free edge at $z = h/4$. It is seen that the numerical value of σ_{23} may not become zero at the free edge, contrary to what is expected. Whitcomb et al. [14] suggest that this behaviour is due to the non-symmetry of the stress tensor at a singular point.

Based on the previous problem, the effect of the lamination scheme is studied. Fig. 6 shows the variation of the transverse normal stress at $x = 0.50h$ as the free edge is approached for $(0^\circ/90^\circ)_2$ and $(90^\circ/0^\circ)_2$ laminates.

Because of the different position of the rigid 0° layer, a different bending behaviour is expected. It is noted that the maximum magnitude of σ_{33} in the $(0^\circ/90^\circ)_2$

Fig. 5 Distributions of transverse shear stress σ_{23} throughout the width at $z = 0$ and $z = h/4$ (laminates $(0^\circ/90^\circ)_s$).

Fig. 6 Normalized transverse normal stress σ_{33} through the thickness at $x = 0.50h$: (a) laminate $(0^\circ/90^\circ)_2$; (b) laminate $(90^\circ/0^\circ)_2$.

lay-up is greater than that of the $(90^\circ/0^\circ)_2$, as expected for the highest rigidity scheme. Also, while in the first case the σ_{33} peaks are compressive, in the second one are tensile. Further away from the free edge, peaks disappear and the transverse normal stress is completely compressive in both cases. It is seen that the transverse normal stress profile near the free edge is strongly influenced by the stacking sequence.

5. Conclusions

A layer-wise mixed finite element model was presented to evaluate the interlaminar transverse

stresses directly without any post-processing technique by means of Lagrange multipliers. Several examples have been carried out. It is shown that the obtained results are very close to those available in the literature. Further, the lay-up sensibility was studied, concluding that the stacking sequence has a high influence on the free-edge effect in bending.

References

- [1] Reddy, J. N. 2003. *Mechanics of Composite Plates and Shells*. Florida: CRC Press, (2003).
- [2] Pipes, R. B., and Pagano, N. J. 1970. "Interlaminar Stresses in Composite Laminates under Uniform Axial

- Extension.” *J. Compos. Mat.* 4: 538-48.
- [3] Pagano, N. J., and Pipes, R. B. 1971. “The Influence of Stacking Sequence of Laminate Strength.” *J. Compos. Mat.* 5 (1): 50-7.
- [4] Kant, T., and Swaminathan, K. 2000. “Estimation of Transverse/Interlaminar Stresses in Laminated Composites—A Selective Review and Survey of Current Developments.” *Compos. Struc.* 49: 65-75.
- [5] Carrera, E. 2003. “Theories and Finite Elements for Multilayered Plates and Shells: A Unified Compact Formulation with Numerical Assessment and Benchmarking.” *Arch. Comput. Meth. Engng.* 10 (3): 215-96.
- [6] Alliney, S., and Carnicer, R. S. 1992. “A Hybrid Finite Element Model for Multilayered Plates.” *Comp. Mech.* 10 (5): 319-33.
- [7] Reddy, J. N. 1987. “A Generalization of Two-Dimensional Theories of Laminated Composite Plates.” *Comm. Appl. Num. Meth.* 3: 173-80.
- [8] Reddy, J. N., Barbero, E. J., and Teply, J. L. 1989. “A Plate Bending Element Based on a Generalized Laminate Plate Theory.” *Inter. J. Num. Meth. Engn.* 28: 2275-92.
- [9] Washizu, K. 1975. *Variational Methods in Elasticity and Plasticity*. Oxford: Pergamo Press.
- [10] Bathe, K. J. 1996. *Finite Element Procedure*. New Jersey: Prentice Hall.
- [11] Bathe, K. J., and Dvorkin, E. N. 1985. “A Four-Node Plate Bending Element Based on Mindlin/Reissner Plate Theory and a Mixed Interpolation.” *Comp. J. Num. Meth Engn.* 21: 367-83.
- [12] Zienkiewicz, O. C., and Taylor, R. L. 2000. *The Finite Element Method: The Basis*. Oxford: Butterworth-Heinemann.
- [13] Tahani, M., and Nosier, A. 2003. “Edge Effects of Uniformly Loaded Cross-ply Composite Laminates.” *Materials and Design* 24: 647-58.
- [14] Whitcomb, J. D., Raju, I. S., and Goree, J. G. 1982. “Reliability of the Finite Element Method for Calculating Free Edge Stresses in Composite Laminates.” *Comp. & Struct.* 15: 23-37.