Asset Pricing Under Ambiguity*

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The capital asset pricing model is a model which describes how financial market works under the hypothesis that each investor wants to buy a portfolio of assets which gives the highest return for any level of risk. The common sense suggests that risky investments will generally yield higher returns than investments free of risk. With the development of the capital asset pricing model, it is possible to quantify risk and the reward for bearing it. One of the hypotheses of the model is that the uncertainty of the market is modelled by an objective probability distribution on the set $\Omega$ of states of nature. In the present work, a subjective probability distribution on $\Omega$ is considered for each investor; to take into account this latter probability, the concept of ambiguity is introduced. Consequently an advancement in asset pricing is given with the demonstration of the capital asset pricing model with the additional hypothesis introduced.

Keywords: financial markets, objective probability, subjective probability, model uncertainty, ambiguity, mean-variance portfolio-selection theory, capital asset pricing model

Introduction

One of the main problems in decision making is to investigate how a decision maker makes a choice among a set of feasible acts, when uncertainty is about the realization of states of nature, for example, in portfolio selection, investors have to choose an action, from a non-empty finite set of possible actions. The consequence of each action depends on which one of a given non-empty set $\Omega$ of states of nature comes true. Suppose that all the investors share the same well-known probability distribution $p$ defined on $\Omega$. As introduced in the seminal work of Markowitz (1952), the basic portfolio selection problem is to minimize the risk of portfolio under a constraint for its expected return or, equivalently, to maximize that expected return under a risk constraint. Moreover, when investors are deciding whether or not to invest in a particular stock, they want to know how the stock will contribute to the risk and expected return of their portfolios. The capital asset pricing model (Eichberger & Harper, 1997) provides an expression which relates the expected return of an asset to its systematic risk.

Supposing now that each investor, in addition to the objective official probability measure $p$ on $\Omega$, wants to decide also referring to a subjective probability measure on that set, which reflects his personal ideas. The purpose of the present paper is to give an advancement in asset pricing by the demonstration of the capital asset pricing model with the additional hypothesis that each agent has also a subjective probability distribution on the

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set $\Omega$, which is not shared with other agents. The importance of this study is documented by the extensive literature and by the many generalizations that the capital asset pricing model has had.

The paper is articulated in only two sections: In the first one, the classical capital asset pricing model is recalled, while the second section is dedicated to the demonstration of the model under the hypothesis of ambiguity.

**Literature Review**

The importance of model uncertainty has long been identified in finance by the recognition of the fact that the investors do not have a perfect knowledge of the probability distribution on the set of the states of the world.

Different problems have been treated by different authors. Dow and Werlang (1992) used the uncertainty averse preference model developed by Schmeidler (1989) to study a single period portfolio selection problem. Kandel and Stambaugh (1996), Brennan (1998), Barberis (2000), and Xia (2001) showed that parameter uncertainty can affect investor’s portfolio choice. Frost and Savarino (1986), Gennette (1986), Balduzzi and Liu (1999), Pastor (2000), and Uppal and Wang (2003) considered the case of multiple risky asset and studied the implication of uncertainty in portfolio selection. Maccheroni, Marinacci, and Ruffino (2013) and Collier (2011) studied the effects of higher ambiguity aversion on optimal portfolio rebalancing. Izhakian and Benninga (2008) found similar results on the separation of risk and ambiguity. Garlappi, Uppal, and Wang (2007) and Chen, Ju, and Miao (2013) found similar results on the separation of risk and ambiguity. Garlappi, Uppal, and Wang (2007) and Chen, Ju, and Miao (2013) considered the optimal portfolio allocation from the Bayesian and the ambiguity models. In Sharpe’s work (1964), the capital asset pricing model is founded on the theory for optimal portfolio selection of Markowitz (1952), although different variants of the model, taking into account the uncertainty of investors, are present in the literature, for example, Merton (1973) proposed an intertemporal model for the capital market. Lucas (1978), Breeden (1979), and Grossman and Shiller (1981) derived consumption-based asset-pricing models. More recent development of the model have been treated in Chen and Epstein (2002), Collard, M usherji, Sheppard, and Tallon (2011), Ju and Miao (2012), and Izhakian (2012). In Chen and Epstein’s model (2002), excess returns also reflect a compensation for risk and a separate compensation for ambiguity. In models of Collard et al. (2011) and Ju and Miao (2012), the investor’s pessimistic behavior is connected to a variety of quantities (risk-free rate equity premium, etc.). In Izhakian (2012), the model is based on shadow probability theory, while in Ruffino (2014), which contains a rich bibliography on the ambiguity, the model refers to smooth preferences of Klibanoff, Marinacci, and Mukerji (2005).

**The Capital Asset Pricing Model**

Assume that there are $n - 1$ risky securities: $S_1$, $S_2$, ..., $S_{n-1}$, and a risk-free security $S_n$, authors indicated with $R_j$ the return of the security $S_j$ and with $x_j$ the portion of total investment funds devoted to this security. Thus,

$$\sum_{j=1}^{n} x_j = 1$$

The vector $x = (x_1, x_2, \ldots, x_n)^T$ is the portfolio of the considered consumer.

In the real setting, one can seldom obtain the return rate $R_j$ ($j = 1, \ldots, n - 1$) without any uncertainty, furthermore since returns vary from time to time, they are assumed to be random variables and will be denoted
by $R_j$ ($j = 1, \ldots, n - 1$).

Considering the following data set in which the returns of the $n$ securities $S_1$, $S_2$, \ldots, $S_n$, are given for $k$ different states of the world, thus the $j$-th return $R_j$ is a random variable represented as the $j$-th column of the following $k \times n$ matrix:

$$
R = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\
R_{21} & R_{22} & \cdots & R_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
R_{k1} & R_{k2} & \cdots & R_{kn} 
\end{pmatrix}
$$

$S_n$ is supposed to be risk-free, thus it is:

$$R_{1n} = R_{2n} = \cdots = R_{kn} = r_f$$

The value of the portfolio $x$ in the $i$-th state of the world is:

$$W_i(x) = \sum_{j=1}^{n} R_{ij} x_j$$

Supposing that all agents assume the same discrete probability distribution $p = (p_1, p_2, \ldots, p_k)$ on the states of the world, the expected return of security $S_j$ is:

$$E(R_j) = \sum_{i=1}^{k} p_i R_{ij}$$

The expected return and the variance of $x$ can be written as:

$$E(x) = \sum_{i=1}^{k} p_i W_i(x) = \sum_{j=1}^{n} x_j E(R_j)$$

(1)

$$\sigma^2(x) = \sum_{j=1}^{n} x_j \sum_{j=1}^{n} x_j \sigma(R_i, R_j)$$

(2)

where $\sigma(R_i, R_j)$ is the covariance between $R_i$ and $R_j$. Thus the portfolio’s expected return is simply the weighted average of the expected returns of its component securities, a portfolio variance is a more complicated concept, and it depends on more than just the variances of the component securities.

Investors are risk averse, so they will choose to hold a portfolio of securities to take advantage of the benefits of diversification. Therefore, when they are deciding whether or not to invest in a particular stock, they want to know how the stock will contribute to the risk and expected return of their portfolios. The capital asset pricing model (Sharpe, 1964; Eichberger & Harper, 1997) provides an expression which relates the expected return on an asset to its systematic risk.

The market portfolio is a vector $A = (A_1, A_2, \ldots, A_n)$ in which $A_j$ is the aggregate quantity of the $j$-th risky asset available in the economy.

If $q_j$ denotes the price by which consumers may trade freely the $j$-th asset, the sum is as follows:

$$W_0 = \sum_{j=1}^{n} q_j A_j$$

(3)

It represents the market value of the market portfolio and it is the value of the aggregate endowments available in the economy.

The wealth of the market portfolio at the $i$-th state is:
The expected return on the market portfolio is defined as:

\[ E(A) = \sum_{j=1}^{k} p_j W_j(A_j) \]  

(4)

It follows directly from the definition of expected return of a random variable:

\[ E(A) = \sum_{i=1}^{k} p_i W_i(A) = \sum_{i=1}^{k} p_i (\sum_{j=1}^{n} R_{ij} A_j) = \sum_{i=1}^{k} p_i R_{ij} A_j = \sum_{j=1}^{n} E(R_j) A_j \]

Using the finance theorist’s preferred mode of operation and measuring asset returns as pay-offs per unit invested and asset quantities in units of expenditure, the capital asset pricing model equation states that, in equilibrium, the difference between the expected rate of return on each risky asset and the riskless rate of return is proportional to the difference between the expected rate of return on the market portfolio and the riskless rate of return.

If each investor wants to optimize its preference in the mean/variance space, the relationship, known as security market line equation, is expressed as follow:

\[ E(R_j) = r_f + [E(A) - r_f] \beta_j \]  

(5)

where \( r_f \) is the risk-free rate; the factor of proportionality \( \beta_j \) has the following expression:

\[ \beta_j = \frac{\sigma(A, R_j)}{\sigma^2(A)} \]

where \( \sigma(A, R_j) \) is the covariance between the return of the market portfolio and the return of the \( j \)-th asset; \( \sigma^2(A) \) is the variance of the market portfolio.

If the covariance between the return of the \( j \)-th asset and the market portfolio is greater than the variance of the market portfolio, the risk premium required by the market in equilibrium will exceed that required on the entire portfolio of risky assets.

**Capital Asset Pricing Model Under Ambiguity**

In the previous section, it is supposed that all agents assume the same prior probability distribution \( p = (p_1, p_2, \ldots, p_k) \) on the set of the states of nature \( \Omega \). Suppose now that the \( i \)-th agent not only assumes \( p \) on \( \Omega \), but also has a subjective probability distribution on that set, then \( p^i = (p'_1, p'_2, \ldots, p'_k) \). The aim of the present section is to derive the capital asset pricing model depending on \( p \) and on each \( p^i \) \( (i = 1, \ldots, I) \). The expected return of the risky asset \( S_j \) depending also on \( p^i \) varies from agent to agent and it has the following expression:

\[ E^i(R_j) = E(R_j, p^i) = E(R_j) + e^i_j \]  

(6)

where \( |e^i_j| \) is the error that the \( i \)-th agent does computing the expected return of asset \( S_j \) assuming the prior probability distribution \( p \) on \( \Omega \) rather than his subjective probability distribution \( p^i \). Let \( E'(R_j) \) be the ambiguous expected return of asset \( S_j \). Analogously, the ambiguous expected return of a portfolio held by the \( i \)-th consumer is:
This paper observes that,

\[ e^i_j = E^i(R_j) - E(R_j) = \sum_{k=1}^{k} (p^i_k - p_k) R_j \]  

\[ \tilde{e}^i = E^i(x) - E(x) = \sum_{j=1}^{n} [E^i(R_j) - E(R_j)] x_j = \sum_{j=1}^{n} e^i_j x_j \]  

Thus substituting equation (9) in equation (7) is:

\[ E^i(x) = E(x) + \sum_{j=1}^{n} e^i_j x_j = \sum_{j=1}^{n} [E(R_j) + e^i_j] x_j \]  

The ambiguous variance of the portfolio held by the \( i \)-th consumer is,

\[ \sigma^2(x) = \sum_{j=1}^{n} x_j \sum_{h=1}^{n} \sigma^i(R_h, R_j) \]  

where

\[ \sigma^i(R_h, R_j) = E^i(R_h, R_j) - [E^i(R_h)E^i(R_j)] = E(R_h, R_j) + e^i_{hj} - [(E(R_h) + e^i_{h})[E(R_j) + e^i_{j}]] \]  

This paper defines,

\[ E^i_j(x) = \frac{\partial E^i(x)}{\partial x_j} = E(R_j) + e^i_j(R_j, p) = E^i(R_j) \]  

\[ \sigma^2_j(x) = \frac{\partial \sigma^j(x)}{\partial x_j} = 2 \sigma^i(x, R_j) = 2 \sum_{l=1}^{n} x_l \sigma^i(R_j, R_l) \]  

The \( i \)-th consumer wants to solve the following optimization problem:

\[ \max_{x} V^i[E^i(x), \sigma^2_j(x)] \]  

So that

\[ \sum_{j=1}^{n} q_j x_j = \sum_{j=1}^{n} q_j x_j \]
where \( V(E'(x), \sigma^2(x)) \) is a representation of the preferences of the \( i \)-th consumer in the mean/variance space. Thus, all the hypotheses for which this representation holds true are supposed to be satisfied. \( q_j \) is the price for unit of asset \( S_j \) and \( \bar{x}_j \) is the initial endowment of \( S_j \) held by the \( i \)-th consumer, indicating with the followings:

\[
V_i'[E'(x), \sigma^2'(x)] := \frac{\partial V'[E'(x), \sigma^2'(x)]}{\partial E'(x)}
\]

\[
V_2'[E'(x), \sigma^2'(x)] := \frac{\partial V'[E'(x), \sigma^2'(x)]}{\partial \sigma^2'(x)}
\]

The first order conditions, for \( j = 1, \ldots, n \), are

\[
V_i'[E'(x), \sigma^2'(x)]E_j'(x) + V_2'[E'(x), \sigma^2'(x)]\sigma_j^2'(x) - \lambda q_j = 0
\]

(18)

\[
\sum_{j=1}^{n} q_j \bar{x}_j = \sum_{j=1}^{n} q_j \bar{x}_j
\]

(19)

where \( \lambda \) is the lagrangian multiplier of the problem. In this exchange economy, a general equilibrium is a vector of prices \( q^* = (q_1^*, q_2^*, \ldots, q_n^*) \) for the considered securities and a vector of demands for each consumer

\[
x^i* = (x_1^i*, x_2^i*, \ldots, x_n^i*), \quad i = 1, \ldots, I
\]

so that for all \( j \) in \( (1, \ldots, n) \),

\[
\sum_{i=1}^{I} x_j^i = \sum_{i=1}^{I} x_j^i := A_j
\]

(20)

\( A_j \), as previously defined, is the aggregate quantity of \( S_j \) in the economy. Since the asset \( S_n \) is risk-free, thus:

\[
E_n'(x) := \frac{\partial E'(x)}{\partial x_n} = r_f
\]

(21)

\[
\sigma_n^2(x) := \frac{\partial \sigma^2'(x)}{\partial x_n} = 0
\]

(22)

Substituting these in equation (18) for \( j = n \), and choosing \( q_n = 1 \), it is \( V_i'[E'(x^i*), \sigma^2'(x^i*)]r_f = \lambda \), substituting in equation (18), the founded expressions of \( \lambda \), \( E_j'(x) \) and \( \sigma_j^2'(x) \), it follows:

\[
V_i'[E'(x^i*), \sigma^2'(x^i*)][E'(R_j) - q_j^* r_f ] + 2V_2'[E'(x^i*), \sigma^2'(x^i*)]\sigma_j^i'(x^i*) + \sum_{i=1}^{I} x_j^i* \sigma^i_j(R_j, R_j) = 0
\]

(23)

setting \( \theta'(x^i*) = \frac{V_i'(x^i*)}{2V_2'(x^i*)} \) it is,

\[
\theta'(x^i*)[E'(R_j) - q_j^* r_f ] = \sum_{i=1}^{I} x_j^i* \sigma_j^i(R_j, R_j)
\]

(24)
This paper sets \( \sum_{i=1}^{l} \theta^i (x^*) := \theta(x^*) \), summing on the set of consumers and taking into account condition (20), it is:

\[
\sum_{i=1}^{l} \theta^i (x^*) [E^i(R_j) - q_j^* r_j] = \sum_{i=1}^{l} x_j^* \sigma_{ji}
\]  \hfill (25)

the first member of equation (25) can be written as:

\[
\sum_{i=1}^{l} \theta^i (x^*) [E(R_j) + e_j - q_j^* r_j] = \sum_{i=1}^{l} \theta^i (x^*) [E(R_j) - q_j^* r_j] + \sum_{i=1}^{l} \theta^i (x^*) e_j^i
\]  \hfill (26)

Let the authors arrange the second member taking into account equation (20):

\[
\sum_{i=1}^{l} \sum_{j=1}^{n} x_j^* \sigma_{ji} = \sum_{i=1}^{l} \sum_{j=1}^{n} (\sigma_j + \varepsilon_{ij}) = \sigma(A, R_j) + \sum_{j=1}^{n} x_j^* \varepsilon_{ij}
\]  \hfill (27)

Based on equations (26) and (27), it is:

\[
\theta(x^*) [E(R_j) - q_j^* r_j] = \sigma(A, R_j) + \sum_{j=1}^{n} x_j^* e_{ij} - \sum_{i=1}^{l} \theta^i (x^*) e_j^i
\]  \hfill (28)

Let

\[
\delta_j := \sum_{i=1}^{l} \sum_{j=1}^{n} x_j^* e_{ij} - \sum_{i=1}^{l} \theta^i (x^*) e_j^i
\]

then, equation (28) becomes:

\[
\theta(x^*) [E(R_j) - q_j^* r_j] = \sigma(A, R_j) + \delta_j
\]  \hfill (29)

Multiplying equation (29) for \( A_j \) and summing on the set of securities, it is:

\[
\theta(x^*) \left[ \sum_{j=1}^{n} E(R_j) A_j - r_j \sum_{j=1}^{n} q_j^* A_j \right] = \sum_{j=1}^{n} A_j \sigma(A, R_j) + \sum_{j=1}^{n} \delta_j A_j
\]  \hfill (30)

Thus taking into account equations (3) and (4), it follows:

\[
\theta(x^*) [E(A) - r_j W_0(A)] = \sigma^2(A) + \sum_{j=1}^{n} \delta_j A_j
\]  \hfill (31)

Computing \( \theta(x^*) \) from equation (29) and substituting it in equation (31), it is:

\[
E(R_j) - q_j^* r_j = \frac{\sigma(A, R_j) + \delta_j}{\sigma^2(A) + \sum_{j=1}^{n} \delta_j A_j} [E(A) - r_j W_0(A)]
\]

dividing by \( q_j^* \) and highlighting \( W_0(A) \), equation (31) becomes:

\[
\frac{E(R_j)}{q_j^*} - r_j = \frac{\sigma(A, R_j) + \delta_j}{\sigma^2(A) + \sum_{j=1}^{n} \delta_j A_j} W_0(A) \left[ \frac{E(A)}{W_0(A)} - r_j \right]
\]

This paper gives an interpretation of the following quantities:
\[
\hat{E}(R_j) := \frac{E(R_j)}{q_j^*}
\]

Return of asset \( S_j \) is the expected, with respect to the prior probability, for unit of money invested in it:

\[
\hat{E}(A) := \frac{E(A)}{W_0(A)}
\]

Return of the market portfolio is the expected, with respect to the prior probability, for unit of money invested in it:

\[
\hat{\sigma}^2(A) = \frac{\sigma^2(A)}{W_0(A)} + \frac{\sum_{j=1}^n \delta_j A_j}{W_0(A)} = \hat{\sigma}^2(A) + \frac{\sum_{j=1}^n \delta_j A_j}{W_0(A)}
\]

where \( \hat{\sigma}^2(A) \) and \( \sigma^2(A) \) are the ambiguous variance and the variance of the market portfolio respectively, for unit of money invested in it:

\[
\hat{\sigma}^2(A, R_j) := \frac{\sigma(A, R_j)}{q_j^*} + \frac{\delta_j}{q_j^*} = \hat{\sigma}(A, R_j) + \frac{\delta_j}{q_j^*}
\]

where \( \hat{\sigma}^2(A, R_j) \) and \( \hat{\sigma}(A, R_j) \) are the ambiguous covariance and the covariance between the market portfolio and asset \( S_j \) respectively, for unit of money invested in it. By these positions, the ambiguous capital asset pricing model is:

\[
\hat{E}(R_j) - r_f = \hat{\beta}^i_j [\hat{E}(A) - r_f]
\]

(32)

where

\[
\hat{\beta}^i_j = \frac{\hat{\sigma}(A, R_j) + \delta_j}{\hat{\sigma}^2(A) + \frac{\sum_{j=1}^n \delta_j A_j}{W_0(A)}}
\]

(33)

where \( \hat{\beta}^i_j \) is called the ambiguous beta and it is a measure of a stock’s (or portfolio’s) volatility in relation to the rest of the market, under the hypothesis that each investor also has a subjective probability distribution on the set of the states of the world.

**Conclusions**

In this work, an advancement in asset pricing is given with the demonstration of the capital asset pricing model with the additional hypothesis that the uncertainty of the market is modelled, not only by an objective probability distribution on the set \( \Omega \) of states of nature, but also by a subjective probability for each investor. The ambiguity may be defined in different ways and different models are derived, as a future prospective of research, it could be interesting to apply those models to a real case in order to compare the different results obtained.

**References**


