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# An Application of Bessel Functions: Study of Transient Flow in a Cylindrical Pipe

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**Abstract:** The Navier-Stokes equations for a fluid flow are applied to a pipe. Under conditions of symmetry these are reduced to the well-known equation of zeroth modified Bessel, which is solved to find the velocity profile and shape of the axial velocity.

## 1. Introduction

The fluids transport such as water, oil, etc. is done through pipes, so it is important to study the flow characteristics under different conditions such as flow rate, pipe size and then obtain mathematical expressions for the velocity profile, the axial velocity, flow rate, etc. Here under symmetry conditions, the resulting differential equations can be resolved accurately in terms of Bessel functions. This work is divided in five parts: This introduction, section 2 illustrates the method for which we obtain the modified Bessel equation from the Navier Stokes equations; in section 3 we resolve the Bessel differential equation; section 4 is an appendix with the most important characteristics about the modified Bessel functions used in this article. Finally, section 5 presents conclusions and future projects.

## 2. Navier-Stokes Equations

Suppose a long circular cylindrical pipe of radius $R$. We consider an incompressible, isothermal Newtonian flow (density $\rho = \text{constant}$, viscosity $\mu = \text{constante}$), with a velocity field $\vec{u} = \left(u_r, u_\phi, u_z\right)$ in terms of cylindrical coordinates $\left(r, \phi, z\right)$. The continuity equation for incompressible flow is [1, 2]

$$\frac{1}{r}\frac{\partial\left(ru_r\right)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0 \tag{1}$$

The $z$-component of the Navier-Stokes equations is written as

$$\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_z}{\partial \phi} + u_z\frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{dp}{dz} + \frac{\mu}{\rho}\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \phi^2} + \frac{\partial^2 u_z}{\partial z^2}\right) \tag{2}$$

In our development of flow in the pipe, considering the velocity components with axial symmetry, that is, they only depend on the radius and time, we have

$$u_r = u_r\left(r,t\right),\ u_\phi = u_\phi\left(r,t\right),\ u_z = u_z\left(r,t\right) \tag{3}$$

Without changes in $z$, the continuity equation for incompressible flow, becomes

$$\frac{\partial}{\partial r}\left(ru_r\right) = 0 \tag{4}$$

where we must have, $ru_r = f\left(\phi, z, t\right)$. However, under consideration of axial symmetry $ru_r = f\left(t\right)$. Moreover, $u_r = \frac{f\left(t\right)}{r} = 0$ at $r = R$, for which

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$f(t)$ must be zero. It follows that there is no radial velocity, that is, $u_r(r,t)=0$.

By similar arguments to those above, we find that $u_\phi = 0$, and $u_z = u_z(r,t)$.

Therefore, Navier-Stokes equation is written as

$$\frac{\partial u_z}{\partial t} = -\frac{1}{\rho}\frac{dp}{dz} + \frac{\mu}{\rho}\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) \quad (5)$$

If the pressure $G(t) = -\frac{dp}{dz}$, is periodic with frequency $w$, we can write:

$$G(t) = \mathrm{Re}\left(\sum_{l=0}^{\infty} G_l e^{i(lwt)}\right) \quad (6)$$

where Re denotes the real part. If this gradient has an infinite period, we can take the Fourier transform.

Similarly, for the Fourier series expansion of *z*-component of the velocity, we have

$$u_z(r,t) = \mathrm{Re}\left(\sum_{l=0}^{\infty} u_{zl} e^{i(lwt)}\right) \quad (7)$$

Substituting equations 6 and 7 into equation 5 and matching harmonics, one has

$$\rho i l w u_{zl} = G_l + \mu\left(\frac{\partial^2 u_{zl}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{zl}}{\partial r}\right) \quad (8)$$

If it is assumed that, the fluid does not slip on the surface, $u_{zl}(r=R)=0$ and a regularity condition on the cylinder axis, i.e. $u_{zl}(r=0)=\max$. The steady state component $(l=0)$ is simply Poiseuille flow [1]

$$G_0 + \mu\left(\frac{\partial^2 u_{z0}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{z0}}{\partial r}\right) = 0 \quad (9)$$

and the equation 8 can be written as

$$\frac{d^2 u_{z0}}{dr^2} + \frac{1}{r}\frac{du_{z0}}{dr} + \frac{G_0}{\mu} = 0 \quad (10)$$

The solution of the equation 10 is

$$u_{z0} = \frac{G_0 R^2}{4\mu}\left(1 - \frac{r^2}{R^2}\right) \quad (11)$$

where the maximum speed is given for

$$u_{z0\max} = \frac{G_0 R^2}{4\mu} \quad (12)$$

The velocity distribution is in the form of a parabola, with the fastest velocity in the vertex and friction cause the velocity decrease outwards. The figure 1 shows the velocity profile in the steady state.

The flow rate is calculated as follows

$$Q = \int u_{z0}\, dA = \int_0^R \frac{G_0 R^2}{4\mu}\left(1 - \frac{r^2}{R^2}\right) 2\pi r dr = \frac{\pi G_0 R^4}{8\mu} \quad (13)$$

where the integral extends over the entire cross sectional area $A$, of the pipe.

The specific flow rate, $q = \frac{Q}{A}$, is

$$q = \frac{Q}{\pi R^2} = \frac{G_0 R^2}{8\mu} \quad (14)$$

In Darcy's equation, the flow rate has the form

$$Q = -kA\frac{dp}{dz} \quad (15)$$

where the constant $k$ is the hydraulic conductivity, $A$ represents the cross-sectional area of the pipe, and $z$ is the axial direction.

Comparing equation 13 to equation 15, one has that

$$G_0 = -\frac{dp}{dz} \quad (16)$$

and

$$k = \frac{R^2}{8\mu} \quad (17)$$

for the hydraulic conductivity, that is a property of both the fluid and the pipe.

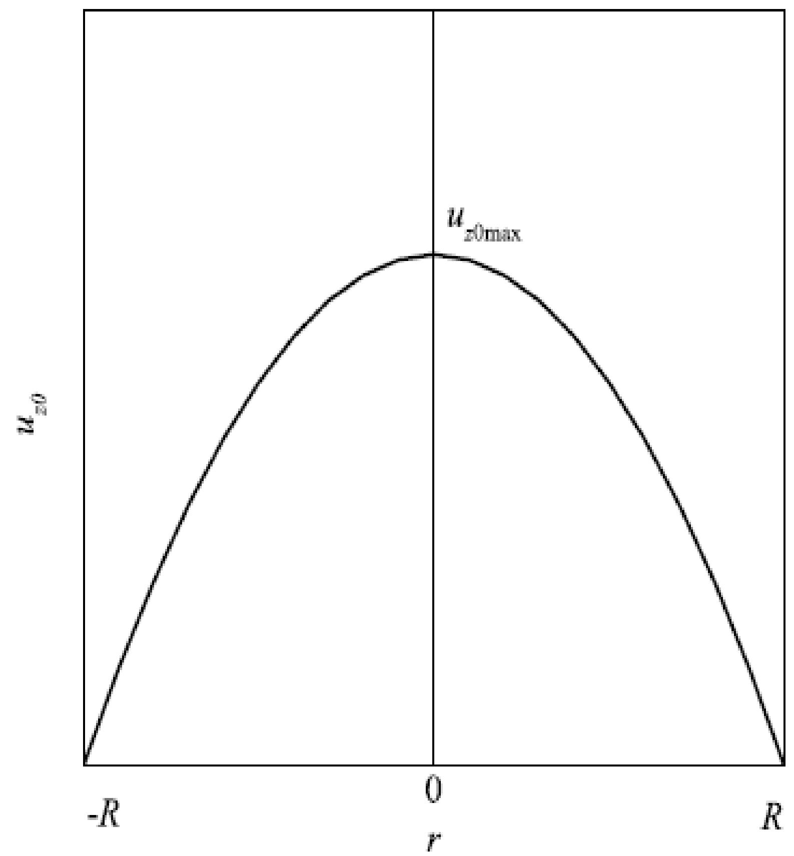


**Fig. 1 Velocity profile in a pipe or radius *R* for the steady state.**

On the other hand, the terms time independent $u_{zl}(r)$ with $l > 0$ obey the equation

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{du_{zl}}{dr}\right) - i(\rho lw)u_{zl} = -G_l \qquad (18)$$

Rearranging terms, equation 18 is written as

$$\frac{d^2u_{zl}}{dr^2} + \frac{1}{r}\frac{du_{zl}}{dr} - i\left(\frac{\rho lw}{\mu}\right)u_{zl} = -\frac{G_l}{\mu} \qquad (19)$$

Then, changing $x = r/R$ variables, finally we obtain

$$x^2\frac{d^2u_{zl}}{dx^2} + x\frac{du_{zl}}{dx} - ix^2\left(\frac{\rho lwR^2}{\mu}\right)u_{zl} = -x^2\frac{R^2G_l}{\mu} \qquad (20)$$

This equation 20 is the desired differential equation to describe the transient state of the fluid in the pipe, and in the next section, we will resolve it.

## 3. Modified Bessel Differential Equation

The homogeneous equation from equation 20 is

$$x^2\frac{d^2u_{zl}}{dx^2} + x\frac{du_{zl}}{dx} - ix^2\left(\frac{\rho lwR^2}{\mu}\right)u_{zl} = 0 \qquad (21)$$

which is equation of zeroth modified Bessel and its solution is written as

$$u_{zl}(x) = AI_0\left(i^{1/2}\sqrt{l}W_0x\right) + BK_0\left(i^{1/2}\sqrt{l}W_0x\right) \qquad (22)$$

where $A$, $B$ are arbitrary constants, and $I_0(y)$, $K_0(y)$ are the modified Bessel functions of the first and second kind of order zero, defined for the equations 47 and 50. In addition, the parameter $W_0$ is the Womersley dimensionless number [4], and it is a measure of the ration of the terms time dependent of the momentum equation to the viscous part, given for

$$W_0 = \sqrt{\frac{w}{\nu}} R = \sqrt{\frac{w\rho}{\mu}} R \quad (23)$$

The parameters $\nu$ , $4W_0^2$ are the dynamic viscosity and the kinetic Reynolds number [5] respectively.

On the other hand, the particular solution of the equation 20 is

$$u_{zl}(x) = -i\frac{G_l}{l\rho w} \quad (24)$$

The full solution is written as

$$u_{zl}(x) = AI_0\left(i^{1/2}\sqrt{l}W_0 x\right) + BK_0\left(i^{1/2}\sqrt{l}W_0 x\right) - i\frac{G_l}{l\rho w} \quad (25)$$

Since the velocity is finite at $r = 0$ and $K_0$ becomes infinite at this value, physically acceptable solution is $I_0$. Then, the solution is

$$u_{zl}(x) = AI_0\left(i^{1/2}\sqrt{l}W_0 x\right) - i\frac{G_l}{l\rho w} \quad (26)$$

Applying the boundary conditions, $u_{zl}(x=1) = 0$ in the equation 26, we obtain

$$A = \frac{i\dfrac{G_l}{l\rho w}}{I_0\left(i^{1/2}\sqrt{l}W_0\right)} \quad (27)$$

Substituting equation 27 in equation 26 and rearranging terms, we obtain

$$u_{zl}(x) = -i\frac{G_l}{l\rho w}\left(1 - \frac{I_0\left(i^{1/2}\sqrt{l}W_0 x\right)}{I_0\left(i^{1/2}\sqrt{l}W_0\right)}\right) \quad (28)$$

Using the relation of Bessel functions given by equation 43 in equation 28, one has

$$u_{zl}(x) = -i\frac{G_l}{l\rho w}\left(1 - \frac{J_0\left(i^{3/2}\sqrt{l}W_0 x\right)}{J_0\left(i^{3/2}\sqrt{l}W_0\right)}\right) \quad (29)$$

where $J_0(y)$ is the Bessel function of the first kind and order zero.

Then for a flow of axial symmetry of an isotropic, incompressible and Newtonian fluid without external forces, we have derived an analytical solution for laminar flow pulses in a pipe; this is often referred to as Womersley flow.

The axial velocity as a function of radial position r and time t is given by

$$u_z(r,t) = \frac{G_0 R^2}{4\mu}\left(1 - \frac{r^2}{R^2}\right) + \operatorname{Re}\left[\sum_{l=1}^{\infty}\left(-i\frac{G_l}{l\rho w}\right)\left(1 - \frac{J_0\left(i^{3/2}\sqrt{l}W_0 \frac{r}{R}\right)}{J_0\left(i^{3/2}\sqrt{l}W_0\right)}\right)e^{ilwt}\right] \quad (30)$$

where we used the equations 6, 7 and equation 29.

Note that in equation 30, $G_l$ is the time independent pressure gradient, and $G_l e^{ilwt}$ is the time dependent pressure gradient. The figure 2 show the velocity profiles for different components.

Using the equations 43 and 47, the equation 30 can be written as

$$u_z(r,t) = \frac{G_0 R^2}{4\mu}\left(1 - \frac{r^2}{R^2}\right) + \operatorname{Re}\left[\sum_{l=1}^{\infty}\left(-i\frac{G_l}{l\rho w}\right)\left(1 - \frac{ber\left(\sqrt{l}W_0 \frac{r}{R}\right) + ibei\left(\sqrt{l}W_0 \frac{r}{R}\right)}{ber\left(\sqrt{l}W_0\right) + ibei\left(\sqrt{l}W_0\right)}\right)e^{ilwt}\right] \quad (31)$$

where $ber$ and $bei$, are the real and imaginary parts of $I_0(y) = J_0(iy)$, and their values are given by the equations 48 and 49.

Finally, after doing some algebra, the equation for the velocity $u_z(r,t)$ is written as

$$u_z(r,t) = \frac{G_0 R^2}{4\mu}\left(1 - \frac{r^2}{R^2}\right) + \sum_{l=1}^{\infty} \frac{G_l}{l\rho w} \sin(lwt)$$
$$+ \sum_{l=1}^{\infty} \frac{G_l \cos(wlt)}{l\rho w} \frac{ber\left(\sqrt{l}W_0 \frac{r}{R}\right) bei\left(\sqrt{l}W_0\right) - bei\left(\sqrt{l}W_0 \frac{r}{R}\right) ber\left(\sqrt{l}W_0\right)}{\left(ber\left(\sqrt{l}W_0\right)\right)^2 + \left(bei\left(\sqrt{l}W_0\right)\right)^2}$$
$$- \sum_{l=1}^{\infty} \frac{G_l sin(wlt)}{l\rho w} \frac{bei\left(\sqrt{l}W_0 \frac{r}{R}\right) bei\left(\sqrt{l}W_0\right) - ber\left(\sqrt{l}W_0 \frac{r}{R}\right) ber\left(\sqrt{l}W_0\right)}{\left(ber\left(\sqrt{l}W_0\right)\right)^2 + \left(bei\left(\sqrt{l}W_0\right)\right)^2} \quad (32)$$

In particular, the axial velocity $u_z(r = 0, t)$ is given by

$$u_z(0,t) = \frac{G_0 R^2}{4\mu} + \sum_{l=1}^{\infty} \frac{G_l}{l\rho w} \sin(lwt) + \sum_{l=1}^{\infty} \frac{G_l}{l\rho w} \frac{bei\left(\sqrt{l}W_0\right) \cos(wlt) - ber\left(\sqrt{l}W_0\right) \sin(wlt)}{\left(ber\left(\sqrt{l}W_0\right)\right)^2 + \left(bei\left(\sqrt{l}W_0\right)\right)^2} \quad (33)$$

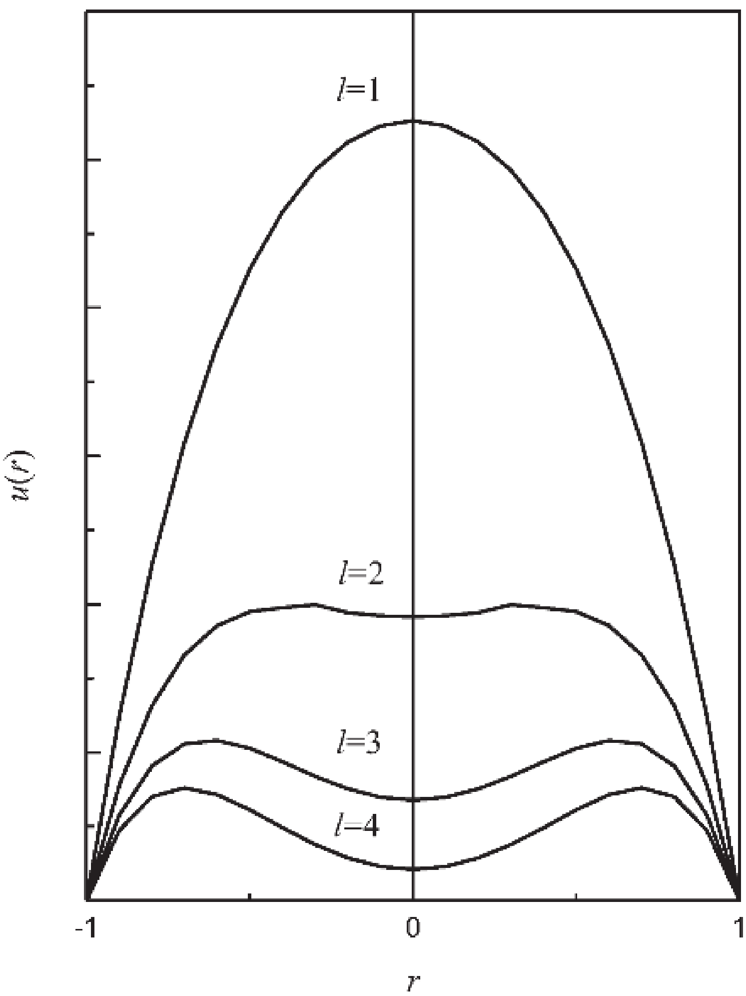


**Fig. 2 Velocity profiles in a water pipe of radius *R* for each component.**

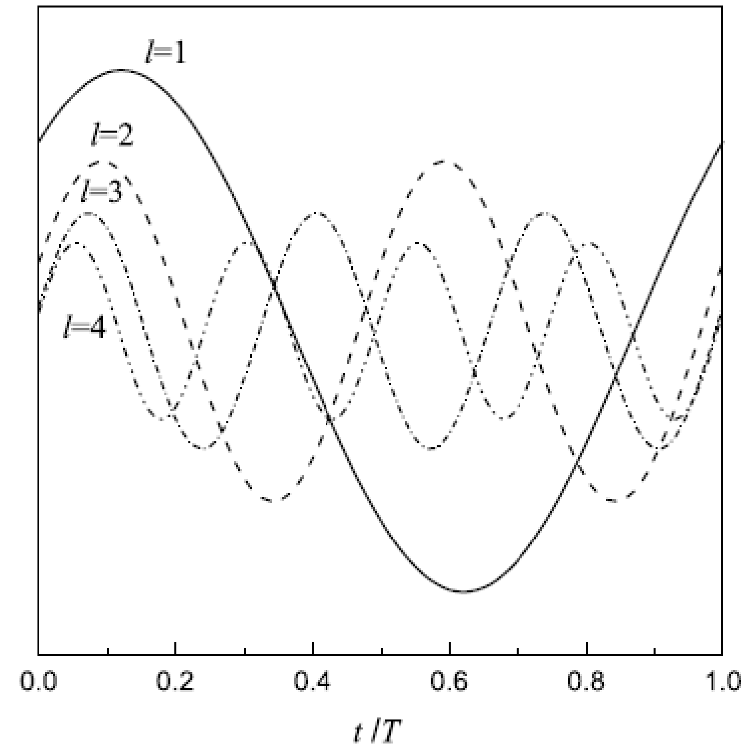


**Fig. 3 The sinusoidal axial velocity in a water pipe of radius *R* for each component.**

In equations 32 and 33 is omitted the subscript zero in *ber* and *bei* functions. In figure 3 it can be seen the axial velocity time dependent, i. e. the waveform for each component.

## 4. Appendix

Bessel functions are one of the most important functions in physics and mathematics. The Bessel´s differential equation [6] is

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \nu^2\right) y = 0 \qquad (34)$$

Thus, when ν is not an integer we may write the solution of the equation 34 in the form

$$u_{zl}\left(x\right) = AJ_{\nu}\left(x\right) + BJ_{-\nu}\left(x\right) \qquad (35)$$

where $A$, $B$ are arbitrary constants and $J_{\nu}\left(x\right)$ is known as the Bessel function of the first kind of order ν, and is defined by the equation [7]

$$J_{\nu}\left(x\right) = \frac{x^{\nu}}{2^{\nu}\Gamma\left(\nu+1\right)} {}_0F_1\left(\nu+1; -\frac{1}{4}x^2\right) = \frac{x^{\nu}}{2^{\nu}\Gamma\left(\nu+1\right)} \sum_{r=0}^{\infty} \frac{\left(-\frac{1}{4}x^2\right)^r}{r!} \qquad (36)$$

where ${}_2F_1\left(\alpha_1, \alpha_2; \beta; x\right)$ is the hypergeometric function of two parameters $\alpha$ and one parameter $\beta$.

However, when the number $\nu$ is an integer $n$, the complete solution is

$$y\left(x\right) = CJ_n\left(x\right) + DY_n\left(x\right) \qquad (37)$$

where $J_n\left(x\right)$ is defined by the equation 36 and $Y_n\left(x\right)$ is given by

$$Y_n\left(x\right) = \frac{2}{\pi}\left\{\gamma + \log\left(\frac{1}{2}x\right)\right\} J_n\left(x\right) - \frac{1}{\pi}\sum_{r=0}^{n-1} \frac{\left(n-r-1\right)!}{r!}\left(\frac{2}{x}\right)^{n-2r} - \frac{1}{\pi}\sum_{r=0}^{\infty} \frac{\left(-1\right)^r}{r!\left(n+r\right)!}\left(\frac{x}{2}\right)^{n+2r} \left\{\phi\left(r+n\right) + \phi\left(r\right)\right\} \qquad (38)$$

The function $Y_n(x)$ so defined is known as Bessel function of the second kind of order $n$ or Newmann function; the constant $\gamma$ is known as Euler´s constant, and $\phi(r)$ is given for the equation [6]

$$\phi(r)=\sum_{s=1}^{r}\frac{1}{s} \tag{39}$$

Other equation important is known as modified Bessel differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\left(x^2-\nu^2\right)y=0 \tag{40}$$

This equation 40 can be transform into the equation 34, when replacing $x$ by $ix$. However, this leads to a complex solution of the equation 34.

Similarly, to what happened with the solution of equation 34, there are two possible general solutions for the equation 40, that depend if $\nu$ an integer is or not.

When $\nu$ is not an integer the solution of this equation is

$$y(x)=AI_\nu(x)+BI_{-\nu}(x) \tag{41}$$

where the function $I_\nu(x)$ is defined by the equation

$$I_\nu(x)=\frac{x^\nu}{2^\nu\Gamma(\nu+1)}\,{}_0F_1\left(\nu+1;\frac{1}{4}x^2\right) \tag{42}$$

Comparing equation 42 with equation 36 we see that

$$I_\nu(x)=i^{-\nu}J_\nu(ix) \tag{43}$$

a result which might have been conjured from the differential equation 40.

If $\nu$ is an integer $n$, the general solution of the equation 40 is

$$y(x)=AI_n(x)+BK_n(x) \tag{44}$$

where the function $K_n(x)$ is defined by the equation

$$K_n(x)=(-1)^{n+1}\left\{\gamma+\log\left(\frac{1}{2}x\right)\right\}I_n(x)$$
$$+\frac{1}{2}\sum_{r=0}^{n-1}\frac{(-1)^r(n-r-1)!}{r!}\left(\frac{x}{2}\right)^{-n+2r}$$
$$+\frac{1}{2}(-1)^n\sum_{r=1}^{\infty}\frac{1}{r!(n+r)!}\left(\frac{x}{2}\right)^{n+2r}\left\{\phi(r+n)+\phi(r)\right\} \tag{45}$$

The functions $I_n(x)$, $K_n(x)$ defined by the equations 42 and 45 respectively are known as modified Bessel functions of the first and second kind of order n.

A particularly important case is when $n=0$. In this case the solution is

$$\begin{aligned} y(x)&=AI_0\left(\sqrt{i}x\right)+BK_0\left(\sqrt{i}x\right)\\ &=AJ_0\left(i^{3/2}x\right)+BK_0\left(\sqrt{i}x\right)\end{aligned} \tag{46}$$

where $I_0(y)$ and $K_0(y)$ are the modified Bessel functions of the first and second kind of order zero.

It is common to introduce two new functions $ber_n(x)$ and $bei_n(x)$ which are [7, 8] respectively the real and imaginary parts of $I_n\left(\sqrt{i}x\right)$, i. e.

$$I_0\left(\sqrt{i}x\right)=ber(x)+ibei(x) \tag{47}$$

In equation 47 is omitted the subscript zero in *ber* and *bei* functions.

From definition given in the equation 44

$$ber(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{[2s!]^2}\left(\frac{1}{4}x^2\right)^{2s} \tag{48}$$

and

$$bei(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{[(2s+1)!]^2}\left(\frac{1}{4}x^2\right)^{2s+1} \tag{49}$$

In similar way the functions $ker_n(x)$ and $kei_n(x)$ are defined to be respectively the real and imaginary parts of the complex function $K_n\left(\sqrt{i}x\right)$, i. e.

$$K_0\left(\sqrt{i}x\right)=ker\left(x\right)+ikei\left(x\right), \tag{50}$$

where is omitted the subscript zero in $ker\left(x\right)$ and $kei\left(x\right)$, and their expressions are given by

$$ker\left(x\right)=-\left\{\gamma+\log\left(\frac{1}{2}x\right)\right\}ber\left(x\right) + \frac{\pi}{4}bei\left(x\right)+\sum_{r=1}^{\infty}\frac{\left(-1\right)^r}{\left[2r!\right]^2}\left(\frac{x}{2}\right)^{4r}\phi\left(2r\right) \tag{51}$$

and

$$kei\left(x\right)=-\left\{\gamma+\log\left(\frac{1}{2}x\right)\right\}bei\left(x\right)- \frac{\pi}{4}ber\left(x\right)+\sum_{r=1}^{\infty}\frac{\left(-1\right)^r}{\left[\left(2r+1\right)!\right]^2}\left(\frac{x}{2}\right)^{4r+2}\phi\left(2r+1\right) \tag{52}$$

Finally, it is noteworthy that these four functions are very useful in applications to engineering problems.

## 5. Conclusions

Was successfully applied to the solution of the modified Bessel equation to find the velocity profile and the axial velocity for the transient flow of a fluid in a pipe. Furthermore, the Fourier transform is used in the analysis of the above solution to apply the superposition. Based on this work we hope to attack other flows where the density is not constant, but whose expression allows us according to the symmetry of the problem, expressing the Navier-Stokes equations in a Bessel equation, using the Fourier transform and obtaining information about the transient flow.

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