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$GL(n, R)$-Equivalence of a Pair of Curves in Terms of Invariants

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Abstract: In this paper, the generator set of $R<x_1,x_2>^G$ is obtained in according to the group $G=Gl(n,R)$. The conditions of $G=Gl(n,R)$-equivalence of a pair of curves are found in terms of $G=Gl(n,R)$-invariants. And the independence of $GL(n,R)$-invariants is shown.

Keywords: $GL(n,R)$-invariants, differential invariants of curves, equivalence of curves.

1. Introduction

The theory of differential invariants consists of three fundamental theorems. The first of these is finding the generators for invariant functions. The second is finding the conditions of equivalence for curves and the third one is finding the relations (if it exists) between of these generators. We give the generator set of differential invariants for two curves and investigate the relations among them.

Let R be the field of real numbers and R^n be n-dimensional Euclidean space. The set

$$GL(n,R)=\left\{\begin{matrix} A=\left\|a_{ij}\right\|:i,j=1,\ldots,n \\ and\ \ a_{ij}\in R,\ \ which\ \ detA\neq 0\end{matrix}\right\}$$

is a group in according to multiplication of matrix. The action of the group $GL(n,R)$ on R^n is given by

$$g.x=\begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \cdots & \cdots & \cdots \\ g_{n1} & \cdots & g_{nn}\end{pmatrix}.\begin{pmatrix} x_1 \\ \vdots \\ x_n\end{pmatrix}=\begin{pmatrix} g_{11}x_1+\ldots+g_{1n}x_n \\ \vdots \\ g_{n1}x_1+\ldots+g_{nn}x_n\end{pmatrix}$$

for $g\in GL(n,R)$ and $x\in R^n$.

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Invariant theory is studied since earlier times [5, 6, 13, 14]. There are a lot of paper and books about the invariant theory of curves and surfaces [2, 3, 7, 8, 9, 10, 11, 12]. The generator set of differential invariants and the relations of them is obtained in [1] for special groups. For two curves, it is investigated the differential invariants and its applications to ruled surfaces for the group $SL(n,R)$ in [4].

In this paper, we investigate the differential invariants of a pair of curves for the group $GL(n,R)$. In section 1, we give some introductory definitions. In section 2, the generator system of differential invariants is found for the rational functions of a pair of curves. Then the conditions of equivalence for two pairs of curves is given by the differential invariants. Also it is shown that the set of generator invariants is minimal.

Definition 1.1. *A* C^∞*-function* $x:I\to R^n$ *will be called a parametric curve or briefly a curve in* R^n.

Definition 1.2. *Let* $\{x_1,x_2\}$ *and* $\{y_1,y_2\}$ *be two pairs of curves. If* $y_i=gx_i, i=1,2$ *for some* $g\in GL(n,R)$*, then these curve families will be called* $GL(n,R)$*-equivalent and denoted by* $\{x_1,x_2\}\overset{G}{\approx}\{y_1,y_2\}$ *for the group* $G=GL(n,R)$.

Definition 1.3. *Let* x_1 *and* x_2 *be two curve in* R^n*. The polynomial*

$$P\{x_1,x_2\}=P(x_1,x_2,x_1',x_2',\ldots,x_1^{(m)},x_2^{(m)})$$

for some natural number m will be called the differential polynomial of x_1 *and* x_2.

The derivation of $P\{x_1,x_2\}$ will be denoted by P' and this derivation is obtained as follows:

$$x_i^{(0)}=x_i,\ \left(x_i^{(m-1)}\right)'=x_i^{(m)}, i=1,2$$

Definition 1.4. *Let* P_1 *and* P_2 *be two differential polynomials. Then the function*

$$f<x_1,x_2>=\frac{P_1\{x_1,x_2\}}{P_2\{x_1,x_2\}},\ P_2\{x_1,x_2\}\neq 0$$

will be called a differential rational function.

If $f<gx,gy>=f<x,y>$ for all $g\in GL(n,R)$, the differential rational function f all called centro-affine invariant differential rational function.Centro-affine differential polynomial is defined by the same way.There no exists centro-affine invariant differential polynomial except constant. But there exists the centro-affine invariant differential rational function different from constant.

The set of all differential rational functions will be denoted by $R\langle x_1,x_2\rangle$. It is a field and R-algebra. Let G be the group $GL(n,R)$. The set of all centro-affine invariant differential rational functions will be denoted by $R\langle x_1,x_2\rangle^G$. $R\langle x_1,x_2\rangle^G$ is a differential subfield and subalgebra of $R\langle x_1,x_2\rangle$.

Definition 1.5. *Let* $f_1,f_2,\ldots,f_k\in R\langle x_1,x_2\rangle^G$. *If the differential field and algebra generated by these functions is equal to* $R\langle x_1,x_2\rangle^G$, *then these functions will be called the generator set of* $R\langle x_1,x_2\rangle^G$.

2. Centro-Affine Invariants of a Pair of Curves

Let $x_1,x_2,\ldots,x_n\in R^n$. We will be denoted the determinant $\begin{vmatrix} x_{11} & \ldots & x_{n1}\\ \ldots & \ldots & \ldots \\ x_{1n} & \ldots & x_{nn}\end{vmatrix}$ by $[x_1\ldots x_n]$ In here, k. column of this determinant is consist of the components of x_k, which are $x_{k1},x_{k2},\ldots,x_{kn}$.

Lemma 2.1. *Let* $x_0,x_1,\ldots,x_n,y_2,\ldots,y_n$ *be vectors in* R^n. *Then the following equality holds*:

$$\begin{aligned}&[x_1x_2\ldots x_n][x_0y_2\ldots y_n]-[x_0x_2\ldots x_n][x_1y_2\ldots y_n]\\&-\ldots-[x_1x_2\ldots x_0][x_ny_2\ldots y_n]=0.\end{aligned}\quad(2.1)$$

Proof. Page 53 in [1].

Definition 2.1. *A curve* x *in* R^n *will be called* $GL(n,R)$ *-regular (briefly regular) if* $[x_1x'_1\ldots x_1^{(n-1)}]\neq 0$. *Hence for all* t, $[x_1(t)x'_1(t)\ldots x_1^{(n-1)}(t)]\neq 0$.

Let G be the group $GL(n,R)$.

Theorem 2.1. *Let* x_1 *and* x_2 *be two curve in* R^n *such that* x_1 *is regular. Then the generator set of* $R\langle x_1,x_2\rangle^G$ *is*

$$\frac{\left[x_1\ldots x_1^{(i-1)}x_1^{(n)}x_1^{(i+1)}\ldots x_1^{(n-1)}\right]}{\left[x_1x_1'\ldots x_1^{(n-1)}\right]},\quad \frac{\left[x_1\ldots x_1^{(i-1)}x_2x_1^{(i+1)}\ldots x_1^{(n-1)}\right]}{\left[x_1x_1'\ldots x_1^{(n-1)}\right]}\quad(2.2)$$

for $i=0,\ldots,n-1$.

Proof. For the group $G=GL(n,R)$, the generator set of $R(x_\tau,\tau\in\Delta)^G$ is

$$\frac{[x_1\ldots x_{i-1}x_\tau x_{i+1}\ldots x_n]}{[x_1\ldots x_n]}, i=1,\ldots,n,\tau\in\Delta/\{1,\ldots,n\}$$

[1]. Let us take $x_1,x_2,x'_1,x'_2,\ldots,x_1^{(K)},x_2^{(K)},\ldots$ instead of the vectors x_τ. Then the generator set of $R(x_1,x_2,x'_1,x'_2,\ldots,x_1^{(K)},x_2^{(K)},\ldots)^G$ is

$$\frac{\left[x_1\ldots x_1^{(i-1)}x_1^{(\tau)}x_1^{(i+1)}\ldots x_1^{(n-1)}\right]}{\left[x_1x_1'\ldots x_1^{(n-1)}\right]},$$

$$i=0,\ldots,n-1,\ \tau\in\Delta/\{0,\ldots,n-1\}$$

$$\frac{\left[x_1\ldots x_1^{(i-1)}x_2^{(\tau)}x_1^{(i+1)}\ldots x_1^{(n-1)}\right]}{\left[x_1x_1'\ldots x_1^{(n-1)}\right]},\tau\geq 0$$

Firstly, we want to show that $\frac{\left[x_1..x_1^{(i-1)}x_1^{(\tau)}x_1^{(i+1)}..x_1^{(n-1)}\right]}{\left[x_1x_1'..x_1^{(n-1)}\right]}$, $\tau \geq n$ is generated by $\frac{\left[x_1..x_1^{(i-1)}x_1^{(n)}x_1^{(i+1)}..x_1^{(n-1)}\right]}{\left[x_1x_1'..x_1^{(n-1)}\right]}$, $i=0,\ldots,n-1$. Let $\tau = n$. Then the generator set of $R(x_1,x_2,x'_1,x'_2,\ldots,x_1^{(K)},x_2^{(K)},\ldots)^G$ is

$$\frac{\left[x_1^{(n)}x'_1\ldots x_1^{(n-1)}\right]}{\left[x_1\ldots x_1^{(n-1)}\right]},\frac{\left[x_1x_1^{(n)}\ldots x_1^{(n-1)}\right]}{\left[x_1\ldots x_1^{(n-1)}\right]},\ldots,\frac{\left[x_1x'_1\ldots x_1^{(n)}\right]}{\left[x_1\ldots x_1^{(n-1)}\right]}$$

So these are generated by the set (2.2).

Let $\tau > n$. By induction, for $\tau-1$ let the set (2.2) be the generator set. Therefore $\frac{\left[x_1..x_1^{(i-1)}x_1^{(\tau-1)}x_1^{(i+1)}..x_1^{(n-1)}\right]}{\left[x_1x_1'..x_1^{(n-1)}\right]}$ is generated by (2.2). We get

$$\left[x_1\ldots x_1^{(i-1)}x_1^{(\tau)}x_1^{(i+1)}\ldots x_1^{(n-1)}\right]=$$
$$\left[x_1\ldots x_1^{(i-1)}x_1^{(\tau-1)}x_1^{(i+1)}\ldots x_1^{(n-1)}\right]'$$
$$-\left[x_1\ldots x_1^{(i-2)}x_1^{(i)}x_1^{(\tau-1)}x_1^{(i+1)}\ldots x_1^{(n-1)}\right]$$
$$-\left[x_1\ldots x_1^{(i-1)}x_1^{(\tau-1)}x_1^{(i+1)}\ldots x_1^{(n-2)}x_1^{(n)}\right]$$

If we divide this equation by $\left[x_1x_1'\ldots x_1^{(n-1)}\right]$, it is obtained that

$$\frac{\left[x_1\ldots x_1^{(i-1)}x_1^{(\tau)}x_1^{(i+1)}\ldots x_1^{(n-1)}\right]}{\left[x_1x_1'\ldots x_1^{(n-1)}\right]}=$$
$$\frac{\left[x_1\ldots x_1^{(i-1)}x_1^{(\tau-1)}x_1^{(i+1)}\ldots x_1^{(n-1)}\right]'}{\left[x_1x_1'\ldots x_1^{(n-1)}\right]}$$
$$-\frac{\left[x_1\ldots x_1^{(i-2)}x_1^{(i)}x_1^{(\tau-1)}x_1^{(i+1)}\ldots x_1^{(n-1)}\right]}{\left[x_1x_1'\ldots x_1^{(n-1)}\right]}$$
$$-\frac{\left[x_1\ldots x_1^{(i-1)}x_1^{(\tau-1)}x_1^{(i+1)}\ldots x_1^{(n-2)}x_1^{(n)}\right]}{\left[x_1x_1'\ldots x_1^{(n-1)}\right]} \quad (2.3)$$

The first term in the right of equality (2.3) is obtained by the derivation of $\frac{\left[x_1..x_1^{(i-1)}x_2^{(\tau-1)}x_1^{(i+1)}..x_1^{(n-1)}\right]}{\left[x_1x_1'..x_1^{(n-1)}\right]}$.

The second term in the equality (2.3) is generated by the set (2.2) in according to induction hypothesis. In Lemma 1, if we take

$$x_1=x_1,\ x_2=x_1',\ldots,\ x_n=x_1^{(n-1)},$$
$$x_0=x_1^{(n)},\ y_2=x_1,\ldots,\ y_{i+1}=x_1^{(i-1)},$$
$$y_{i+2}=x_1^{(\tau-1)},\ y_{i+3}=x_1^{(i+1)},\ldots,\ y_n=x_1^{(n-2)}$$

eliminate the zero terms and divide $\left[x_1x_1'\ldots x_1^{(n-1)}\right]$ it is obtained that

$$\frac{\left[x_1^{(n)}x_1\ldots x_1^{(\tau-1)}\ldots x_1^{(n-1)}\right]}{\left[x_1x_1'\ldots x_1^{(n-1)}\right]}-$$
$$\frac{\left[x_1\ldots x_1^{(i-1)}x_1^{(n)}x_1^{(i+1)}\ldots x_1^{(n-1)}\right]}{\left[x_1x_1'\ldots x_1^{(n-1)}\right]}\cdot\frac{\left[x_1^{(i)}x_1\ldots x_1^{(\tau-1)}\ldots x_1^{(n-1)}\right]}{\left[x_1x_1'\ldots x_1^{(n-1)}\right]}$$
$$-\frac{\left[x_1\ldots x_1^{(n-2)}x_1^{(n)}\right]}{\left[x_1x_1'\ldots x_1^{(n-1)}\right]}\cdot\frac{\left[x_1^{(n-1)}x_1\ldots x_1^{(\tau-1)}\ldots x_1^{(n-2)}\right]}{\left[x_1x_1'\ldots x_1^{(n-1)}\right]}=0$$

So the term $\frac{\left[x_1^{(n)}x_1..x_1^{(\tau-1)}..x_1^{(n-1)}\right]}{\left[x_1x_1'..x_1^{(n-1)}\right]}$ generated by the set (2.2). Therefore the third term in the equality (2.3) is generated by the set (2.2).

Similarly, $\frac{\left[x_1..x_1^{(i-1)}x_2^{(\tau)}x_1^{(i+1)}..x_1^{(n-1)}\right]}{\left[x_1x_1'..x_1^{(n-1)}\right]}, \tau \geq 0$ is obtained by induction on τ. For $\tau=0$, $\frac{\left[x_1..x_1^{(i-1)}x_2x_1^{(i+1)}..x_1^{(n-1)}\right]}{\left[x_1x_1'..x_1^{(n-1)}\right]}$ is the generator. Let for $\tau=n$, $\frac{\left[x_1..x_1^{(i-1)}x_2^{(n)}x_1^{(i+1)}..x_1^{(n-1)}\right]}{\left[x_1x_1'..x_1^{(n-1)}\right]}$ generated by the set (2.2). Let us show that this is true for $\tau=n+1$.

$$\left(\frac{\left[x_1 \ldots x_1^{(i-1)} x_2^{(n)} x_1^{(i+1)} \ldots x_1^{(n-1)}\right]}{\left[x_1 x_1' \ldots x_1^{(n-1)}\right]}\right)' = \frac{\left[x_1 \ldots x_1^{(i-1)} x_2^{(n)} x_1^{(i+1)} \ldots x_1^{(n-1)}\right]'}{\left[x_1 x_1' \ldots x_1^{(n-1)}\right]} - \frac{\left[x_1 x_1' \ldots x_1^{(n-1)}\right]'}{\left[x_1 x_1' \ldots x_1^{(n-1)}\right]} \cdot \frac{\left[x_1 \ldots x_1^{(i-1)} x_2^{(n)} x_1^{(i+1)} \ldots x_1^{(n-1)}\right]}{\left[x_1 x_1' \ldots x_1^{(n-1)}\right]} \tag{2.4}$$

The second terms in the right of equality (2.4) are the generators. The first term is generated as follows;

$$\left[x_1 \ldots x_1^{(i-1)} x_2^{(n)} x_1^{(i+1)} \ldots x_1^{(n-1)}\right]' = \left[x_1 \ldots x_1^{(i-2)} x_1^{(i)} x_2^{(n)} x_1^{(i+1)} \ldots x_1^{(n-1)}\right] + \left[x_1 \ldots x_1^{(i-1)} x_2^{(n+1)} x_1^{(i+1)} \ldots x_1^{(n-1)}\right] + \left[x_1 \ldots x_1^{(i-1)} x_2^{(n)} x_1^{(i+1)} \ldots x_1^{(n-2)} x_1^{(n)}\right]$$

If we divide by $\left[x_1 x_1' \ldots x_1^{(n-1)}\right]$ this equality, we get

$$\frac{\left[x_1 \ldots x_1^{(i-1)} x_2^{(n+1)} x_1^{(i+1)} \ldots x_1^{(n-1)}\right]}{\left[x_1 x'_1 \ldots x_1^{(n-1)}\right]} = \frac{\left[x_1 \ldots x_1^{(i-1)} x_2^{(n)} x_1^{(i+1)} \ldots x_1^{(n-1)}\right]'}{\left[x_1 x'_1 \ldots x_1^{(n-1)}\right]} - \frac{\left[x_1 \ldots x_1^{(i-2)} x_1^{(i)} x_2^{(n)} x_1^{(i+1)} \ldots x_1^{(n-1)}\right]}{\left[x_1 x'_1 \ldots x_1^{(n-1)}\right]} - \frac{\left[x_1 \ldots x_1^{(i-1)} x_2^{(n)} x_1^{(i+1)} \ldots x_1^{(n-2)} x_1^{(n)}\right]}{\left[x_1 x'_1 \ldots x_1^{(n-1)}\right]}$$

The first term in the right side of the above equality is shown that can be generated. The second term is the generator. And it is shown that the third term can be generated using the Lemma 1. Therefore $\frac{\left[x_1 \ldots x_1^{(i-1)} x_2^{(n+1)} x_1^{(i+1)} \ldots x_1^{(n-1)}\right]}{\left[x_1 x'_1 \ldots x_1^{(n-1)}\right]}$ can be generated by the set (2.2).

By the induction hypothesis, The set (2.2) is generator set. □

Theorem 2.2. *Let* $G = GL(n,R)$ *and* $\{x_1, x_2\}$, $\{y_1, y_2\}$ *be two curve families such that* x_1 *and* y_1 *are regular. If for* $i = 0,\ldots,n-1$

$$\frac{\left[x_1 \ldots x_1^{(i-1)} x_1^{(n)} x_1^{(i+1)} \ldots x_1^{(n-1)}\right]}{\left[x_1 x'_1 \ldots x_1^{(n-1)}\right]} = \frac{\left[y_1 \ldots y_1^{(i-1)} y_1^{(n)} y_1^{(i+1)} \ldots y_1^{(n-1)}\right]}{\left[y_1 y'_1 \ldots y_1^{(n-1)}\right]}$$
$$\frac{\left[x_1 \ldots x_1^{(i-1)} x_2 x_1^{(i+1)} \ldots x_1^{(n-1)}\right]}{\left[x_1 x'_1 \ldots x_1^{(n-1)}\right]} = \frac{\left[y_1 \ldots y_1^{(i-1)} y_2 y_1^{(i+1)} \ldots y_1^{(n-1)}\right]}{\left[y_1 y'_1 \ldots y_1^{(n-1)}\right]} \tag{2.5}$$

then $\{x_1, x_2\} \overset{G}{\approx} \{y_1, y_2\}$.

Proof. Since x_1 and y_1 are regular, we get $\left[x_1 x'_1 \ldots x_1^{(n-1)}\right] \neq 0$ and $\left[y_1 y'_1 \ldots y_1^{(n-1)}\right] \neq 0$. Let us take the matrixes

$$A_{x_1} = \begin{pmatrix} x_{11}(t) & \ldots & x_{11}^{(n-1)}(t) \\ \ldots & \ldots & \ldots \\ x_{1n}(t) & \ldots & x_{1n}^{(n-1)}(t) \end{pmatrix} \text{ and}$$

$$A'_{x_1} = \begin{pmatrix} x'_{11}(t) & \ldots & x_{11}^{(n)}(t) \\ \ldots & \ldots & \ldots \\ x'_{1n}(t) & \ldots & x_{1n}^{(n)}(t) \end{pmatrix}$$

Since $\left[x_1 x_1' \ldots x_1^{(n-1)}\right] \neq 0$, there exists the inverse of A_{x_1} . Take the matrix $A_{x_1}^{-1}.A'_{x_1} = C.$ Then $A'_{x_1} = A_{x_1}.C$. So the matrix C has the form

$$C = \begin{pmatrix} 0 & \ldots & 0 & c_{1n} \\ 1 & \ldots & 0 & c_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & c_{nn} \end{pmatrix}$$

where

$$c_{1n}=\frac{\left[x_1^{(n)}x'_1\ldots x_1^{(n-1)}\right]}{\left[x_1x'_1\ldots x_1^{(n-1)}\right]},c_{2n}=\frac{\left[x_1x_1^{(n)}x''_1\ldots x_1^{(n-1)}\right]}{\left[x_1x'_1\ldots x_1^{(n-1)}\right]},\ldots,c_{nn}=\frac{\left[x_1x'_1\ldots x_1^{(n-2)}x_1^{(n)}\right]}{\left[x_1x'_1\ldots x_1^{(n-1)}\right]}$$

From the equalities (2.5), it is obtained that $A_{x_1}^{-1}.A'_{x_1}=A_{y_1}^{-1}.A'_{y_1}$. So we have that

$$\left(A_{y_1}.A_{x_1}^{-1}\right)'=A'_{y_1}.A_{x_1}^{-1}+A_{y_1}.\left(A_{x_1}^{-1}\right)'$$
$$=A'_{y_1}.A_{x_1}^{-1}+A_{y_1}.(-A_{x_1}^{-1}.A'_{x_1}.A_{x_1}^{-1})$$
$$=A_{y_1}.\left(A_{y_1}^{-1}.A'_{y_1}-A_{x_1}^{-1}.A'_{x_1}\right).A_{x_1}^{-1}=0$$

Therefore $A_{y_1}.A_{x_1}^{-1}=g,$ g is constant and $det\left(A_{y_1}.A_{x_1}^{-1}\right)=detA_{y_1}.detA_{x_1}^{-1}=detg\neq 0$. So $g\in GL(n,R)$. And we get $A_{y_1}=gA_{x_1}$. If we write this equality obviously, we have that

$$\begin{pmatrix} y_{11} & \cdots & y_{11}^{(n-1)} \\ \cdots & \cdots & \cdots \\ y_{1n} & \cdots & y_{1n}^{(n-1)} \end{pmatrix}=\begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \cdots & \cdots & \cdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}.\begin{pmatrix} x_{11} & \cdots & x_{11}^{(n-1)} \\ \cdots & \cdots & \cdots \\ x_{1n} & \cdots & x_{1n}^{(n-1)} \end{pmatrix}$$

and then $y_1(t)=gx_1(t),\ \forall t\in I$.

Let us take the matrix

$$D_{x_2}=\begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{pmatrix}$$

and take

$$\begin{pmatrix} x_{11} & x'_{11} & \cdots & x_{11}^{(n-1)} \\ x_{12} & x'_{12} & \cdots & x_{12}^{(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ x_{1n} & x'_{1n} & \cdots & x_{1n}^{(n-1)} \end{pmatrix}.\begin{pmatrix} h_{1n} \\ h_{2n} \\ \vdots \\ h_{nn} \end{pmatrix}=\begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{pmatrix}$$

Therefore $A_{x_1}^{-1}.D_{x_2}=H=\|h_{in}\|,i=1,\ldots,n$. Let us find the element of this matrix. We have that $D_{x_2}=A_{x_1}.H$. Then we get

$$x_{11}h_{1n}+x'_{11}h_{2n}\ldots+x_{11}^{(n-1)}h_{nn}=x_{21}$$
$$x_{12}h_{1n}+x'_{12}h_{2n}\ldots+x_{12}^{(n-1)}h_{nn}=x_{22}$$
$$\cdots$$
$$x_{1n}h_{1n}+x'_{1n}h_{2n}\ldots+x_{1n}^{(n-1)}h_{nn}=x_{2n}$$

The solution of this equation system in according to Crammer's rule;

$$h_{1n}=\frac{\left[x_2x'_1\ldots x_1^{(n-1)}\right]}{\left[x_1x'_1\ldots x_1^{(n-1)}\right]},h_{2n}=\frac{\left[x_1x_2\ldots x_1^{(n-1)}\right]}{\left[x_1x'_1\ldots x_1^{(n-1)}\right]},\ldots,h_{nn}=\frac{\left[x_2x'_1\ldots x_1^{(n-2)}x_2\right]}{\left[x_1x'_1\ldots x_1^{(n-1)}\right]}$$

Similarly, we can find the matrix $A_{y_1}^{-1}.D_{y_2}$ and from the equations (2.5), we will get $A_{x_1}^{-1}.D_{x_2}=A_{y_1}^{-1}.D_{y_2}$. Also, we know that, since $A_{y_1}=gA_{x_1}$ then

$$A_{x_1}^{-1}.D_{x_2}=(gA_{x_1})^{-1}.D_{y_2}=A_{x_1}^{-1}.g^{-1}.D_{y_2}$$

so, we will get $D_{x_2}=g^{-1}.D_{y_2}$ and $D_{y_2}=g.D_{x_2}$.

If we write this equality as matrixes

$$\begin{pmatrix} y_{21} \\ y_{22} \\ \vdots \\ y_{2n} \end{pmatrix}=\begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \cdots & \cdots & \cdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}.\begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{pmatrix}$$

Then we get $y_2(t)=gx_2(t),\ \forall t\in I$. So for the same $g\in GL(n,R)$, it is obtained that $y_1(t)=gx_1(t)$ and $y_2(t)=gx_2(t)$. Hence $\{x_1,x_2\}\overset{G}{\approx}\{y_1,y_2\}$. □

Theorem 2.3. *Let* $G=GL(n,R)$ *and*

$f_1(t), f_2(t), \ldots, f_n(t), f_{2i}(t), i = 0, \ldots, n-1$, $(t \in I)$ *be* C^∞ *-functions. Then there exists curves* x_1, x_2 *which* x_1 *is regular such that*

$$\frac{\left[x_1 \ldots x_1^{(i-1)} x_1^{(n)} x_1^{(i+1)} \ldots x_1^{(n-1)}\right]}{\left[x_1 x'_1 \ldots x_1^{(n-1)}\right]} = f_{i+1}(t), i = 0, \ldots, n-1$$

$$\frac{\left[x_1 \ldots x_1^{(i-1)} x_2 x_1^{(i+1)} \ldots x_1^{(n-1)}\right]}{\left[x_1 x'_1 \ldots x_1^{(n-1)}\right]} = f_{2i}(t), i = 0, \ldots, n-1$$

Proof. From the previous proof, we take the matrix multiplication $A_{x_1}^{-1}.A'_{x_1} = B$ such that $A'_{x_1} = A_{x_1}.B$. In here, matrix B has the form

$$B = \begin{pmatrix} 0 & \ldots & 0 & f_1(t) \\ 1 & \ldots & 0 & f_2(t) \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & f_n(t) \end{pmatrix}$$

Then we have the following differential equation system from this multiplication;

$$x_{11} f_1(t) + x'_{11} f_2(t) + \ldots + x_{11}^{(n-1)} f_n(t) = x_{11}^{(n)}$$

$$x_{12} f_1(t) + x'_{12} f_2(t) + \ldots + x_{12}^{(n-1)} f_n(t) = x_{12}^{(n)}$$

$$\ldots$$

$$x_{1n} f_1(t) + x'_{1n} f_2(t) + \ldots + x_{1n}^{(n-1)} f_n(t) = x_{1n}^{(n)}$$

Let us take $x_{1i} = y,\ i = 1, \ldots, n$. So we can write the above differential equation system as

$$f_1(t) y + f_2(t) y' + \ldots + f_n(t) y^{(n-1)} - y^{(n)} = 0$$

It is known that the theory of differential equations, there exist one solution of this differential equation. Let $x_1(t) = (y_1, y_2, \ldots, y_n)$ be the solution. Then the curve $x_1(t)$ satisfies the conditions of the theorem.

Take the matrixes

$$A_2 = \begin{pmatrix} x_{11} & \ldots & x_{11}^{(n-2)} & x_{21} \\ x_{12} & \ldots & x_{12}^{(n-2)} & x_{22} \\ \ldots & \ldots & \ldots & \ldots \\ x_{1n} & \ldots & x_{1n}^{(n-2)} & x_{2n} \end{pmatrix} \text{ and}$$

$$A_{x_1} = \begin{pmatrix} x_{11}(t) & \ldots & x_{11}^{(n-1)}(t) \\ \ldots & \ldots & \ldots \\ x_{1n}(t) & \ldots & x_{1n}^{(n-1)}(t) \end{pmatrix}$$

and let $A_{x_1}^{-1}.A_2 = H$. So $A_2 = A_{x_1}.H$. Then we get the matrix H as;

$$H = \begin{pmatrix} 1 & 0 & \ldots & 0 & f_{20}(t) \\ 0 & 1 & \ldots & 0 & f_{21}(t) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & f_{2n-2}(t) \\ 0 & 0 & \ldots & 0 & f_{2n-1}(t) \end{pmatrix}$$

Since $A_2 = A_{x_1}.H$, we have the following differential equation system:

$$x_{21} = x_{11} f_{20}(t) + x'_{11} f_{21}(t) + \ldots + x_{11}^{(n-1)} f_{2n-1}(t)$$

$$x_{22} = x_{12} f_{20}(t) + x'_{12} f_{21}(t) + \ldots + x_{12}^{(n-1)} f_{2n-1}(t)$$

$$\ldots$$

$$x_{2n} = x_{1n} f_{20}(t) + x'_{1n} f_{21}(t) + \ldots + x_{1n}^{(n-1)} f_{2n-1}(t)$$

So we get the curve $x_2 = \begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{pmatrix}$, hence curves x_1 and x_2 satisfies the theorems rules.

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