Rational Portfolio Investment Based on Consumer's Preferences: Black-Scholes Model and Stochastic Control

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Abstract: In the paper, it demonstrated a system engineering value driven approach within determination of portfolio investment policy. The principles of rationality and market efficiency lead to modern portfolio theory and to the Black-Scholes theory for option valuation. In the paper, the financial market is modeled by Black-Scholes stochastic differential equation and the rational portfolio solutions in risk are specified based on the individual consumers' preferences represented as utility objective function.

Key words: Preferences, utility, optimal portfolio, Hamilton-Jacobi-Bellman equation, Black-Scholes model, financial market.

1. Introduction

The principles of rationality and market efficiency lead to modern portfolio theory, and to the Black-Scholes theory for option valuation. Financial economic formally considers investment under certainty and uncertainty (risk) and hence contributes to determine rational (even optimal) financial business strategy. Financial econometrics is the branch of financial economics that uses econometric techniques to parameterize the relationships suggested. Arrive to econometrics we inevitably run into the notion of utility.

Utility theory is one of the popular methodologies in Multi-Attribute Decision Making. The Utility theory permits development of complex models in which human participation is reflected analytically starting from marginal information as human preferences. In fact the Utility theory permits mathematical inclusion of the decision maker in the mathematical modeling and mathematical descriptions of processes with human participations [1-3].

In the paper is demonstrated a system engineering value driven approach within determination of the investment policy in a continuous-time financial market modeled by a stochastic differential equation [4]. The problem consists in determination of investment policy which represents the amount invested in a risky asset at any moment of the financial process. The investment policy solution represents the optimal portfolio allocation determined as optimal stochastic control at any moment.

2. Mathematical Formulations

According to social-cognitive theories, people's strategies are guided both by internal expectations about their own capabilities of getting results, and by external feedback [1, 3]. Probability theory, stochastic programming and stochastic optimization and utility theory address decision making under these conditions [1, 3-5]. The mathematical description or modeling on such a fundamental level requires basic mathematical terms and their gradual elaboration to more complex and specific terms like value and utility functions, operators on mathematically structured sets as well, and equivalency of these descriptions. In the financial market is modeled by Black-Scholes stochastic

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differential equation.

2.1 Black-Scholes Model and Objective Function

Consider a non-risky asset S^0 and risky one S . Following the sources [4] the Black-Scholes stochastic differential equation is given by:

$$dS_t^0 = S_t^0 r dt \quad \text{and} \quad dS_t = S_t \mu dt + \sigma S_t dW_t.$$

In this paper r , μ and σ are constants (in the paper: $r = 0.03$, $\mu = 0.05$ and $\sigma = 0.3$) and W is a one dimensional Brownian motion. By X_t we denote the state space vector of the controlled time continuous financial market. The investment policy is defined by a progressively adapted process $\pi = \{\pi_t, t \in [0, T]\}$ where π_t represents (defines) the amount ($X_t \pi_t$) ($\pi_t \in [0, 1]$) invested in the risky process at time t . The remaining wealth ($X_t - \pi_t X_t$) at the same time moment t is invested in the non-risky process. The time period T is 50 weeks. The dynamic of the liquidation value X_T of a self-financing strategy satisfies is given by the SDE:

$$\begin{aligned} dX_t^\pi &= \pi X_t^\pi \frac{dS_t}{S_t} + (X_t^\pi - \pi X_t^\pi) \frac{dS_t^0}{S_t^0} = \\ &= (rX_t^\pi + (\mu - r)\pi X_t^\pi)dt + \sigma \pi X_t^\pi dW_t \end{aligned} \quad (1)$$

It is obvious that in these conditions and parameters is true $E \int_0^T (\pi X_t^\pi)^2 dt < \infty$. Here E denotes mathematical expectation defined in the initial filtered probability space $(\Omega, \mathcal{F}, F, P)$ with canonical filtration $F = \{\mathcal{F}_t, t \geq 0\}$ of the Brownian motion defined over the probability space $(\Omega, \mathcal{F}, P)$. More precisely, E denotes the mathematical expectation over the probability space $(\Omega, \mathcal{F}, P)$. The objective of the investor (decision maker-DM) is to choose the control (the amount π_t invested in the risky process) so as to maximize the expected utility of his terminal wealth at moment T , i.e.: $V(t, x) := \sup_{\pi \in [0, 1]} E[U(X_T^{t, x, \pi})]$, where $X^{t, x, \pi}$ is the solution of the controlled stochastic differential equation with initial condition

(initial wealth) x at time t [4]. For the liquidation value it is supposed that if the state space vector is zero in a moment t then it remains zero until the end T ($X_T^{t, x, \pi} = 0$).

We assume that the outcome set X is a two-attribute product set $V \times W$, with generic element $x = (v, w)$. The sets V and W are attribute sets where V designates the first attribute—the amount π_t , ($X_t \pi_t$, $\pi_t \in [0, 1]$) invested in the risky process and W designates the second attribute, the quantity of money in BGN's.

When the utility function over the consequences is known, it is very easy to exploit it using a computer. However, in practice, the effective construction of function raises numerous problems. Indeed, although the construction of single-attribute utility functions is generally quite easy, that of multiattribute utility functions is usually very hard to perform due to the cognitive limitations of decision makers [3, 6-8]. Hence, the usual requirement is that they be decomposable as a simple combination of single-attribute more easily constructed utility functions [3]. The aggregation of the two attributes in a multiattribute utility function needs investigation of the *Utility independence* in between the risky investment and the quantity of money [3]. We mark as $(v, w_1) \mathcal{L} (v, w_2)$ the lottery $\langle (v, w_1), (v, w_2), \alpha \rangle$: α is the probability of the appearance of the alternative (v, w_1) , and $(1 - \alpha)$ — the probability of the appearance of the alternative (v, w_2) . The probability α (subjective or objective) describes the uncertainty with one investment event $\mathcal{L}$ [8, 9]. It is said that the second attribute w is utility independent if:

$$\begin{aligned} (v, w_1) \mathcal{L} (v, w_2) \mathcal{I} (v, w_3) \mathcal{L} (v, w_4) &\Leftrightarrow \\ (v', w_1) \mathcal{L} (v', w_2) \mathcal{I} (v', w_3) \mathcal{L} (v', w_4), \end{aligned} \quad (2)$$

for all $v, v' \in V$ and for all $w_1, w_2, w_3, w_4 \in W$.

That is, preferences on W do not depend on the particular deterministic level at which $v \in V$ is fixed. A convenient implication of preferential independence is that changing v does not affect rank-ordering in W . The following theorem describes a well known

result [3].

Theorem 1: Assume that the image of the function $f(w) \rightarrow U(v, w)$ is an interval for all v , where $U(v, w)$ is the two attribute utility function. Then w is utility independent if and only if

$U(v, w) = f(v)p(w) + g(v)$ for some functions f, p, g with f positive.

Let W be relevant over a range w^0 to w^* and V over a range v^0 to v^* and assume that $U(v, w^*) > U(v, w^0)$ for all v and $U(v^*, w) > U(v^0, w)$ for all w . We may write this independency condition as: $U(v, w) = U(v, w^0) + [U(v, w^*) - U(v, w^0)] U(v^0, w)$.

We suppose that the preferences over the set W (the amount X_t at any moment t — quantity of money) do not depend on the particular deterministic level at which the risky investment $\pi_t (v \in V)$ is fixed on at any moment t . In other words, we suppose that the preferences associated with the quantity of BGN's (set W) are utility independent from the level of risky investment π (set V). For description of the objective function is needed evaluation of the single-attribute utility functions $U(v, w^*)$, $U(v, w^0)$ and $U(v^0, w)$ following the conclusions of the theorem. This multiattribute utility is relevant with the decision maker's preferences and permits optimal control design of the investment process in agreement with the intuition and the empirical knowledge of the of the decision maker.

2.2 Value, Utility and Polynomial Utility Approximation

The value based decision making in this paper is in the framework of the axiomatic decision making, but realize the prescriptive decision making approach [2, 5, 10]. The preference relation in the ordering scale (x is preferable to y) is denoted by $(x \succ y)$. A “value” is a function $u(\cdot)$ for which it is fulfilled [2, 3]:

$$((x, y) \in X^2, x \succ y) \Leftrightarrow (u(x) > u(y)). \quad (3)$$

We cannot talk about distance between the different alternatives. Here only ordinal evaluations within different mathematical processing of the information may be used. If with the ordering of the alternatives

we can evaluate the distance between them we can talk about interval scale [2, 10]. For these scales the distances between the alternatives have the meaning of real numbers. The transition from one interval scale to another is achieved with affine transformation:

$$x = ay + b, (x, y) \in X^2, a > 0, b \in R. \quad (4)$$

Among these types of scales is also the measurement of the utility function through the so called “lottery approach” [3, 9, 11]. Let X be the set of alternatives ($X \subseteq R^m$). The DM's preferences over X are expressed by $(\succ)$. The “indifference” relation $(\approx)$ is based on $(\succ)$ and is defined by $((x \approx y) \Leftrightarrow \neg((x \succ y) \vee (y \succ x)))$. Let P is a convex set of probability distributions over X . A utility function $u(\cdot)$ will be any function for which the following is fulfilled [2], [3], [10]:

$$(p \succ q, (p, q) \in P^2) \Leftrightarrow (\int u(\cdot) dp > \int u(\cdot) dq). \quad (5)$$

The interpretation of the above formula is that the integral of the utility function $u(\cdot)$ is a measure with respect to the comparison of the probability distributions p and q defined over X . The notation $(\succ)$ expresses the preferences of DM over P including those over X (single point distribution, $X \subseteq P$). The presumption of existence of a utility function $u(\cdot)$ leads to the existence of:

$$\text{Asymmetry } (x \succ y) \Rightarrow \neg(x \succ y),$$

$$\text{Transitivity } (x \succ y) \wedge (y \succ z) \Rightarrow (x \succ z) \quad (6)$$

and Transitivity of the “indifference” relation $(\approx)$.

There are quite different utility evaluation methods that are based prevalingly on the “lottery” approach (gambling approach). A “lottery” is called every discrete probability distribution over X . We mark as $\langle x, y, \alpha \rangle$ the lottery: α is the probability of the appearance of the alternative x and $(1-\alpha)$ — the probability of the alternative y . The most used evaluation approach is the following assessment: $z \approx \langle x, y, \alpha \rangle$, where $(x, y, z) \in X^3$, $(x \succ z \succ y)$ and $\alpha \in [0, 1]$ [3, 11]. Starting from the gambling approach for the definitions and the presentation of the expert's preferences we use the following sets:

$$A_{u^*} = \{(\alpha, x, y, z) / (\alpha u^*(x) + (1-\alpha)u^*(y)) > u^*(z)\},$$

$$B_{u^*} = \{(\alpha, x, y, z) / (\alpha u^*(x) + (1-\alpha)u^*(y)) < u^*(z)\} \quad (7)$$

In the formula above the notation $u^*(.)$ is the DM's empirical utility assessment. Through stochastic recurrent algorithms we approximate functions recognizing the above two sets [5]. The proposed assessment process is machine-learning based on the DM's preferences. The machine learning is a probabilistic pattern recognition because $(A_{u^*} \cap B_{u^*} \neq \emptyset)$ and the utility evaluation is a stochastic approximation with noise (uncertainty) elimination. The following presents the evaluation procedure. The DM compares the "lottery" $\langle x, y, \alpha \rangle$ with the simple alternative $z, z \in Z$ ("better-", $f(x, y, z, \alpha) = 1$ ", "worse-", $f(x, y, z, \alpha) = (-1)$ " or "can't answer or equivalent- $\sim$, $f(x, y, z, \alpha) = 0$ ", $f(.)$ denotes the qualitative DM's answer). This determines a learning point $((x, y, z, \alpha), f(x, y, z, \alpha))$. The following recurrent stochastic algorithm constructs the utility polynomial approximation $u(x) = \sum_i c_i \Phi_i(x)$:

$$c_i^{n+1} = c_i^n + \gamma_n \left[f(t^{n+1}) - \overline{(c^n, \Psi(t^{n+1}))} \right] \Psi_i(t^{n+1})$$

$$\sum_n \gamma_n = +\infty, \sum_n \gamma_n^2 < +\infty, \forall n, \gamma_n > 0. \quad (8)$$

In the formula are used the following notations (based on A_u): $t = (x, y, z, \alpha)$, $\Psi_i(t) = \Psi_i(x, y, z, \alpha) = \alpha \Phi_i(x) + (1 - \alpha) \Phi_i(y) - \Phi_i(z)$, where $(\Phi_i(x))$ is a family of polynomials. The line above the scalar product $\overline{v} = \overline{(c^n, \Psi(t))}$ means: $(\overline{v} = 1)$, if $(v > 1)$, $(\overline{v} = -1)$ if $(v < -1)$ and $(\overline{v} = v)$ if $(-1 < v < 1)$. The coefficients c_i^n take part in the polynomial presentation $g^n(x) = \sum_{i=1}^n c_i^n \Phi_i(x)$ and $(c^n, \Psi(t)) = \alpha g^n(x) + (1 - \alpha) g^n(y) - g^n(z) = G^n(x, y, z, \alpha)$ is a scalar product. The learning points are set with a pseudo random sequence.

The mathematical procedure describes the following assessment process: The expert relates intuitively the "learning point" (x, y, z, α) to the set A_{u^*} with probability $D_1(x, y, z, \alpha)$ or to the set B_{u^*} with probability $D_2(x, y, z, \alpha)$. The probabilities $D_1(x, y, z, \alpha)$ and $D_2(x, y, z, \alpha)$ are mathematical expectation of

$f(.)$ over A_{u^*} and B_{u^*} respectively, $(D_1(x, y, z, \alpha) = M(f/x, y, z, \alpha))$ if $(M(f/x, y, z, \alpha) > 0)$, $(D_2(x, y, z, \alpha) = (-)M(f/x, y, z, \alpha))$ if $(M(f/x, y, z, \alpha) < 0)$. Let $D'(x, y, z, \alpha)$ is the random value: $D'(x, y, z, \alpha) = D_1(x, y, z, \alpha)$ if $(M(f/x, y, z, \alpha) > 0)$; $D'(x, y, z, \alpha) = (-D_2(x, y, z, \alpha))$ if $(M(f/x, y, z, \alpha) < 0)$; $D'(x, y, z, \alpha) = 0$ if $(M(f/x, y, z, \alpha) = 0)$. We approximate $D'(x, y, z, \alpha)$ by a function of the type $G(x, y, z, \alpha) = (\alpha g(x) + (1 - \alpha) g(y) - g(z))$, where $g(x) = \sum_i c_i \Phi_i(x)$. The coefficients c_i^n take part in the approximation of the function $G(x, y, z, \alpha)$:

$$G^n(x, y, z, \alpha) = (c^n, \Psi(t)) = \alpha g^n(x) + (1 - \alpha) g^n(y) - g^n(z),$$

$$g^n(x) = \sum_{i=1}^N c_i^n \Phi_i(x). \quad (9)$$

The function $G^n(x, y, z, \alpha)$ is positive over A_{u^*} and negative over B_{u^*} depending on the degree of approximation of $D'(x, y, z, \alpha)$. The function $g^n(x)$ is the approximation of the utility function $u(.)$.

The single-attribute utility functions $U(v, w^*)$, $U(v, w^0)$ and $U(v^0, w)$ are evaluated by the proposed stochastic recurrent algorithm and the utility function is shown in Fig. 1. The black ciseaux line in the second utility surface is a stochastic optimal control process flow.

We underline that it is supposed that the preferences over the amount X_t (quantity of money) at any moment t do not depend on the particular deterministic level at which the risky investment π_t is fixed on at the same moment t . In other words, we suppose that the preferences associated with the quantity of BGN's (set W) are utility independent from the level of risky investment π (set V).

3. Black-Scholes Model and Optimal Portfolio Allocation

We recall the stochastic Black-Scholes model (1):

$$dX_t^\pi = (rX_t^\pi + (\mu - r)\pi X_t^\pi)dt + \sigma \pi X_t^\pi dW_t.$$

Here r, μ and σ are constants (in the paper: $r = 0.03$, $\mu = 0.05$ and $\sigma = 0.3$) and W is a one dimensional Brownian motion [4]. By X_t we denote the state space

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Fig. 1

vector of the controlled dynamic process. The investment policy is defined by a progressively adapted process $\pi = \{\pi_t, t \in [0, T]\}$ where π_t represents (defines) the amount (X_t, π_t) ($\pi_t \in [0, 1]$) invested in the risky asset at time t . The remaining wealth $(X_t - \pi_t X_t)$ at the same moment t is invested in the non-risky asset. The time period T is 50 weeks. The liquidation value of a self-financing strategy satisfies the following stochastic differential equation:

$$\begin{aligned} dX_t^\pi &= \pi X_t^\pi \frac{dS_t}{S_t} + (X_t^\pi - \pi X_t^\pi) \frac{dS_t^0}{S_t^0} = \\ &= (rX_t^\pi + (\mu - r)\pi X_t^\pi)dt + \sigma\pi X_t^\pi dW_t \end{aligned} \quad (10)$$

The numeric solution could be obtained by the Kushner-Dupuis scheme method for example [12]. In this paper is choosing a different approach exposed in the monograph [13]. Following the presentations in [4, 13] and passing through generalized solution of the Black-Scholes stochastic differential equation we found a polynomial approximations of the HJB

(Hamilton-Jacobi-Bellman) function $w(t, X)$ and of the control manifold $\pi(t, X)$. We show them in Figs. 2 and 3.

The stochastic process is started in 30 different initial points; from 1,000 BGN's to 30,000 BGN's. The next Fig. 4 shows the processes with optimal stochastic control.

Fig. 5 shows one sample of optimal control solutions. This figure shows not only the quantitative optimal control investment but also permits deduction of an investment business strategy in agreement with the decision maker's preferences. From the exposed Fig. 5, it follows that if the initial amount is between 1,000 BGN's and 10,000 BGN's is recommended to invest the entire initial wealth in the risky investment. If the initial wealth is between 25,000 BGN's and 30,000 BGN's it is advisable to start with the non-risky investment and after the 30 or 40 weeks to invest a part in the risky-investment. We emphasize

Fig. 2

Fig. 3

Fig. 4

Fig. 5

that this is true only for the concrete process parameters and for the concrete utility function.

4. Discussions

The two attribute utility function $U(v, w) = U(v, w^0) + [U(v, w^*) - U(v, w^0)] U(v^0, w) = U^*(v, w^0) + U_1(v, w^*) U_2(v^0, w)$ permits a more complex point of view. We determine the SDE (stochastic differential equation), which describes dynamically the utility $U(v, w)$ by the Ito formula:

$$\begin{aligned} dU_t^\pi = & (rX_t^\pi + (\mu - r)\pi_t X_t^\pi) U_1(\pi_t) \frac{\delta U_2(X_t^\pi)}{\delta X_t^\pi} + \\ & 0.5(\sigma\pi_t X_t^\pi)^2 U_1(\pi_t) \frac{\delta^2 U_2(X_t^\pi)}{\delta X_t^{\pi^2}} + \\ & (\sigma\pi_t X_t^\pi) U_1(\pi_t) \frac{\delta U_2(X_t^\pi)}{\delta X_t^\pi} dW_t \end{aligned} \quad (11)$$

The HJB equation for determination of the Belman's function $w(t, U)$ is:

$$\frac{\delta w(t, U_t^\pi)}{\delta t} + \sup_{\pi} \Lambda(w(t, U_t^\pi)) \quad (12)$$

where

$$\begin{aligned} \Lambda(w(t, U_t^\pi)) = & \{ (rX_t^\pi + (\mu - r)\pi_t X_t^\pi) U_1(\pi_t) \frac{\delta U_2(X_t^\pi)}{\delta X_t^\pi} + \\ & 0.5(\sigma\pi_t X_t^\pi)^2 U_1(\pi_t) \frac{\delta^2 U_2(X_t^\pi)}{\delta X_t^{\pi^2}} \} \frac{\delta w(t, U_t^\pi)}{\delta U_t^\pi} + \\ & 0.5(\sigma\pi_t X_t^\pi U_1(\pi_t) \frac{\delta U_2(X_t^\pi)}{\delta X_t^\pi})^2 \frac{\delta^2 w(t, U_t^\pi)}{\delta U_t^{\pi^2}} \end{aligned} \quad (13)$$

The solution is in agreement with the decision maker's preferences. However the presence of the first and second derivative makes difficult the utilization of a polynomial approximation of the HJB function. The numeric solution could be obtained by the Kushner-Dupuis scheme for example [4, 12].

The previous formulations suggest a modification of the Black-Scholes model. The new SDE has the form:

$$dX_t^\pi = (rX_t^\pi + (\mu - r)\pi_t X_t^\pi) [U(\pi_t, X_t^\pi) - U(\pi_t, X_t^0)] dt + [U(\pi_t, X_t^\pi) - U(\pi_t, X_t^0)] \sigma \pi_t X_t^\pi dW_t. \quad (14)$$

The new solution is changed manifestly only in the control manifold as shown in Fig. 6.

For a sufficiently short time interval (2-5 weeks) and for a sufficiently small time interval in the SDE solver, we can approach the problem by avoiding the determination of the Bellman function. The control is not optimal, but remains suboptimal however. The grounds for such an approach are found in the following theorem [14]:

Theorem 2: Let the function $g(x) \in L_1(\mathbf{R}^d)$, $f(\cdot) \in L_p(\mathbf{R}^d)$ and $\int_{\mathbf{R}^d} g(x) = 1$, $p \geq 1$. Then it is fulfilled $\|g_\varepsilon * f - f\|_p \rightarrow 0$ for $\varepsilon \rightarrow 0$, where $g_\varepsilon(x) = \varepsilon^{-d} g(x/\varepsilon)$. In addition, if $\bar{g}(x) = \sup_{Def |y| \geq |x|} |g(y)| \in L_1$, then for each Lebesgue's point of the function $f(x)$ it is

Fig. 6

fulfilled $\lim_{\varepsilon \rightarrow 0} f * g_\varepsilon(x) = f(x)$, where $(*)$ has the meaning of convolution.

For a summable function $f(\cdot)$, the points that are not Lebesgue's points have measure zero. The role of the function $g_\varepsilon(x)$ is played in this case by the density function of the normal distribution with mathematical expectation zero and variance ε^2 . The function $f(\cdot)$ is replaced by the continuous utility function $U(\cdot)$. Hence the convergence in the above theorem is in every point.

In other words, for a sufficiently small discretization interval ε and by modeling the time-continuous financial market with Wiener process and piecewise continuous utility function $U(\cdot)$ we can replace the conditional mathematical expectation of the Bellman HJB function with the utility itself. Of course, such substitution may be done only for a sufficiently small number of steps which depend on the parameters of the model. This suboptimal approach is demonstrated by the use of the SDE (1) described in the beginning of the paper:

$$\begin{aligned} dX_t^\pi &= \pi X_t^\pi \frac{dS_t}{S_t} + (X_t^\pi - \pi X_t^\pi) \frac{dS_t^0}{S_t^0} = \\ &= (rX_t^\pi + (\mu - r)\pi X_t^\pi)dt + \sigma\pi X_t^\pi dW_t \end{aligned}$$

Let the utility is the single attribute function $U(X_t)$. The Bellman's function $w(\cdot)$ is replaced in HJB equation with the Utility function $U(\cdot)$:

$$\begin{aligned} \frac{\partial w}{\partial t}(t, x) + \sup_{\pi \in [0,1]} [(rx + (\mu - r)\pi x) \frac{\partial w}{\partial t}(t, x) + \\ \frac{1}{2} \sigma^2 \pi^2 x^2 \frac{\partial^2 w}{\partial t^2}(t, x)] = 0. \end{aligned} \quad (15)$$

The determination of the suboptimal control is easy because we avoid the determination of $w(\cdot)$. Now the suboptimal control $\pi(t_i)$ depends on $X_{t_i}^{0, x, \pi}$ and maximizes:

$$\begin{aligned} \sup_{\pi \in [0,1]} [(rx + (\mu - r)\pi x) \frac{\partial U}{\partial x}(t, x) + \\ \frac{1}{2} \sigma^2 \pi^2 x^2 \frac{\partial^2 U}{\partial x^2}(t, x)] \end{aligned} \quad (16)$$

If the maximum belongs to $(0, 1)$ then the sub-optimal solution is:

$$\pi(t_i) = \frac{-(\mu - r)U(X_{t_i}^{0, x, \pi})}{X_{t_i}^{0, x, \pi} U''(X_{t_i}^{0, x, \pi}) \sigma^2} \quad (17)$$

The investment policy $(X_{t_i}, \pi(t_i), \pi(t_i) \in [0, 1])$ in the other two cases is defined by the expressions:

$$\begin{aligned} &\frac{-(\mu - r)U(X_{t_i}^{0, x, \pi})}{U''(X_{t_i}^{0, x, \pi}) \sigma^2} \text{ if } \pi(t_i) \in (0, 1) \\ &\text{or } 0 \text{ or } X_{t_i}^{0, x, \pi} \end{aligned} \quad (18)$$

Fig. 7 shows the obtained solution. This suboptimal control is in accordance with the investor's preferences presented analytically by the utility function.

Figs. 8 and 9 show the solutions with sub-optimal

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Fig. 7

Fig. 8 With control.

Fig. 9 Without control.

control and without control for a long time interval (50 weeks). Even in these conditions the sub-optimal control gives good solution.

5. Conclusions

In the paper, it demonstrated a system engineering value driven approach [15] within the problem of determination of the optimal portfolio allocation modeled with dynamic as Black-Scholes stochastic differential equation. The Black-Scholes optimal portfolio solutions are determined in accordance with the individual consumers' preferences. The optimal control permits deduction of an investment business strategy in agreement with the decision maker's preferences.

The mathematical formulations presented here could serve as basis of tools development. These value evaluations lead to the development of preferences-based decision support in machine learning environments and iterative control design in complex problems.

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