BFGS Method for Linear Programming

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Received: August 20, 2015 / Accepted: September 17, 2015 / Published: December 25, 2015.

Abstract: In this paper we propose an algorithm based on the BFGS Quasi-Newton method to solve a linear program. The choice of this method is justified by its theoretical efficiency, the ease to determine a descent direction and its fast convergence towards an optimal solution. Our proposed method is compared with Newton's method for linear program named *lpnew*, widely used as an optimization algorithm for classification problems.

Key Words: Linear programming, BFGS method, least norm solution, exterior penalty.

1. Introduction

It may seem strange to solve a linear program with nonlinear optimization methods. Indeed, a gradient type algorithm can lead to a numerical explosion, or at best a bifurcation regime. However, and for example, the interior points methods [14], with their ease of implementation and their polynomial complexity, make nonlinear approaches so interesting for linear programming.

Another technique using nonlinear approach is based on the Tikhonov regularization [15] and the fact that the exterior penalty formulation of a linear program provides an exact least 2-norm solution to the dual of the linear program for finite values of the penalty parameter [9]. This solution is used to generate an accurate primal solution. If m (constraints number) $\gg n$ (variables number), a fast globally-convergent and finitely terminating Newton method named *lpnewt* was proposed by the author in [9] to solve the primal linear program. The algorithm, in question, has been successfully used

to solve several problems in Datamining and Machine Learning ([3], [4]).

However, it is to note that when m and n are of the same order, this method gives a solutions of poor quality. Our objective in this paper, is to develop in this situation an algorithm with the same quality as *lpnewt*. In fact, since the gradient of the objective function of the penalized linear program is not differentiable, the author in [9] used a generalized modified Newton method. We propose here, to use a quasi-Newton method of type BFGS (Broyden-Fletcher-Goldfarb-Shanno), wich has never been used in this context, instead of the generalized Newton method. The BFGS quasi-Newton method [13] and its variant are widely regarded as the workhorses of smooth nonlinear optimization due to their combination of computational efficiency and good asymptotic convergence. Byrd and Nocedal [1] showed that, if the objective function is convex and twice continuously differentiable with additional assumptions, we have the assurance that the BFGS method combines global convergence with a rate of superlinear convergence. Since then, several studies have shown that we can have a global convergence with weaker assumptions, and some researchers have

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spoken even about the BFGS method for nonsmooth optimization problems ([16], [7]).

The paper is organized as follows: In the second section, we give the theoretical background showing that the least 2-norm formulation of a linear program, as a strongly convex quadratic program, gives an exact least 2-norm solution for finite parameter values ([11], [5]). In section 3, we supply the theoretical results of our method as well as its theorem of convergence. Comparative numerical simulations with the *lpnewt* method (proposed in [9]) on sparse synthetically-generated linear programs are given to show the superiority and the robustness of our approach. A general conclusion on this work is supplied in the last section.

2. Theoretical Background

We consider the primal linear programming problem

$$\begin{cases} \min c^\top x \\ Ax \leq b \text{ and } x \in \mathbb{R}^n \end{cases} \quad (1)$$

and its dual

$$\begin{cases} -\min b^\top v \\ A^\top v + c = 0 \text{ and } v \in \mathbb{R}_+^m \end{cases} \quad (2)$$

where $A \in \mathbb{R}^{(m,n)}$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. We assume the solvability of problem (1). The least 2-norm formulation [10] of the dual linear program (2) is

$$\begin{cases} -\min b^\top v + \frac{\varepsilon}{2} v^\top v \\ A^\top v + c = 0 \text{ and } v \in \mathbb{R}_+^m \end{cases} \quad (3)$$

Because the objective function of (3) is strongly convex, its solution is unique. The necessary and sufficient Karush-Kuhn-Tucker optimality conditions for problem (3) give the existence of the multiplier $y \in \mathbb{R}^n$ such that

$$\varepsilon v = (Ay - b)_+ \text{ and } A^\top v + c = 0,$$

or equivalently

$$v = \frac{1}{\varepsilon} (Ay - b)_+ \text{ and } A^\top (Ay - b)_+ + \varepsilon c = 0, \quad (4)$$

where the plus function x_+ is defined as

$$(x_+)_i = \max(0, x_i), \quad i = 1, \dots, n.$$

If we consider now the parametric exterior penalty problem of the primal linear program (1) with the penalty parameter ε , given by the function f

$$\begin{cases} \min f(y) = \varepsilon c^\top y + \frac{1}{2} \|(Ay - b)_+\|^2 \\ y \in \mathbb{R}^n, \end{cases} \quad (5)$$

the optimality conditions (4) become

$$v = \frac{1}{\varepsilon} (Ay - b)_+$$

and

$$\nabla f(y) = A^\top (Ay - b)_+ + \varepsilon c = 0$$

That is,

$$v = \frac{1}{\varepsilon} (Ay - b)_+$$

and

$$y = \arg \min_{y \in \mathbb{R}^n} f(y) = \varepsilon c^\top y + \frac{1}{2} \|(Ay - b)_+\|^2$$

Hence, solving the exterior penalty problem (5) for any positive parameter ε provides a solution $v = \frac{1}{\varepsilon} (Ay - b)_+$ to (3). In addition, if $\varepsilon \in]0, \bar{\varepsilon}[$ for some positive $\bar{\varepsilon}$, we will have the following results.

Theorem 1. [10] The unique least 2-norm solution to the dual linear program (3) is given by

$$v = \frac{1}{\varepsilon} (Ay - b)_+ \quad (6)$$

where y is a solution of the primal penalty problem (5) for any finite value of the penalty parameter $\varepsilon \in]0, \bar{\varepsilon}[$, for some positive $\bar{\varepsilon}$.

Solving the problem 5 gives a solution y . We can then, compute an exact least 2-norm solution v to the dual linear program (2) from (6), and if we make the assumption that the submatrix A_S has linearly independent columns, where

$$S = \{j : v_j > 0\}. \quad (7)$$

As a consequence of the complementarity conditions, we have

$$A_j z = b_j, \text{ for } j \in S, \quad (8)$$

and assumption (7) yields an exact solution of the primal linear program (1). From the theoretical point of view, it is not easy to find $\bar{\varepsilon}$. However, if for two successive values of ε , $\varepsilon^1 > \varepsilon^2$, the corresponding solutions of problem (3) v^1 and v^2 , are equal, then under certain assumptions [8], $v = v^1 = v^2$, is the solution of problem (2). Computationally this can be treated effectively.

3. Linear Programming BFGS Algorithm

If we dispose of a fast method for solving the problem (5), the scheme proposed in section 2, can be very successful in solving large-scale linear programming.

3.1 Linear Programming Newton Algorithm

In [9], the author proposed a modified Newton method with Armijo step size for solving the problem (5), and since the gradient of the objective function of problem (5) is not differentiable, he used the generalized Hessian [6], which is the $n \times n$ symmetric positive-semidefinite matrix

$$\partial^2 f(x) = A^\top \text{diag}(Ax - b)_* A, \quad (9)$$

where $\text{diag}(Ax - b)_*$ denotes an $m \times m$ diagonal matrix with diagonal elements $\text{diag}(A_i x - b_i)_*$, $i = 1, \dots, m$, while x_* denotes the subgradient of x_* which is the step function defined as

$$\begin{aligned} (x_*)_i &= 1, \text{ if } x_i > 0 \\ (x_*)_i &= 0, \text{ if } x_i < 0 \\ (x_*)_i &\in [0, 1] \text{ if } x_i = 0, i = 1, \dots, n. \end{aligned}$$

The Linear programming primal algorithm *lpnewt*, given in [9] was faster than CPLEX 6.5 [2] on a class of synthetically generated sparse linear

programs with as many as two million constraints such that $m \gg n$. However, the method is relatively slow if n approach m , and often gives a solution of bad quality especially when the matrix A is dense. In fact, *lpnewt* utilizes the generalized Hessian instead of the Hessian Matrix $\nabla^2 f$. For a quick computation of the matrix (9) at each iteration, the author in [9] was constrained to take a fixed value for $(x_*)_i$ in the interval $[0, 1]$, when $x_i = 0$, This

necessarily affects the quality of the descent direction. Motivated by the fact that the BFGS method combines good convergence and simple updates for smooth unconstrained minimization problems. We present an algorithm to solve the linear program (1) based on the BFGS method to solve the problem (5).

3.2 Linear Programming BFGS Algorithm (LPBFGS)

We, first, state the *LPBFGS* algorithm and then we guarantee global convergence.

3.2.1 BFGS update formula

The New iterate of the BFGS method applied to the the parametric exterior penalty problem of the primal linear program (5) is

$$y_{k+1} = y_k + \alpha_k d_k,$$

where the Quasi-Newton search direction d_k is

$$d_k = -H_k (\nabla f(y_k)), \quad (10)$$

and H_k the approximation of the inverse Hessian matrix is updated at every step by means of Quasi-Newton-BFGS update formula given by

$$\begin{aligned} H_{k+1} &= (I - \rho_k \Delta y_k \Delta f_k^\top) \times \\ &H_k (I - \rho_k \Delta f_k \Delta y_k^\top) + (\rho_k \Delta y_k \Delta y_k^\top) \end{aligned} \quad (11)$$

with

$$\Delta y_k = y_{k+1} - y_k,$$

$$\Delta f_k = \nabla f(y_k) - \nabla f(y_k)$$

and

$$\rho_k = \frac{1}{\Delta f_k \Delta y_k^\top}.$$

The step length α_k is chosen to satisfy the strongly Wolfe conditions [13]:

$$f(y_k + \alpha_k d_k) \leq f(y_k) + c_1 \alpha_k \nabla f(y_k)^\top d_k, \quad (12)$$

and

$$|\nabla f(y_k + \alpha_k d_k)^\top d_k| \leq c_2 |\nabla f(y_k)^\top d_k|, \quad (13)$$

where the constants c_1 and c_2 are chosen such that: $0 < c_1 < c_2 < 1$ (typically for this algorithm; 10^{-3} and 0.6 respectively).

Algorithm: Linear Programming BFGS (LPBFGS)	
1:	Initialize: Set the penalty parameter $\varepsilon = 10^{-3}$, the tolerance $tol = 10^{-12}$, the step size $\alpha_0 = 1$ and iteration limit $k_{\max}$. Star with any $y^{(0)} \in \mathbb{R}^n$ and H_0 (initial approximation of the inverse Hessian matrix). Set the counter $k = 0$.
2:	While ($k < k_{\max}$ and $\ \nabla f(y_k)\ > tol$).
3:	Compute search direction $d_k = -H_k(\nabla f(y_k))$.
4:	Compute α_k to satisfy the Wolfe conditions and (12).
5:	Set $y_{k+1} = y_k + \alpha_k d_k$.
6:	Compute H_{k+1} by relation (11)
7:	$k \leftarrow k + 1$
8:	End while.
9:	Compute the least 2-norm dual solution $v^* = \frac{1}{\varepsilon}(Ay_k - b)_+$
10:	Select the set indice $S = \{j : v_j^* > 0\}$
11:	Solve the linear system 8 to obtain the solution z^*
End.	

3.2.2 Bounded Spectrum of the Inverse Hessian Approximation

When the objective function is twice continuously differentiable, a classic assumption for the convergence of the BFGS method [13], from any starting point y^0 and any initial Hessian approximation H_0 , is that, the Hessian matrix is positive definite on the level set,

$$Lev_f(f(y^0)) = \{y \in \mathbb{R}^n : f(y) \leq f(y^0)\}.$$

This assumption implies that there exist positive constants m and M such that

$$m\|x\|^2 \leq y^T \nabla^2 f(y)^{-1} y \leq M\|x\|^2 \quad (14)$$

for all $x \in \mathbb{R}^n$ and $y \in Lev_f(f(y^0))$. Form the analysis given in [13], we can translate the bounds on the spectrum of H_k into conditions that only involve Δy_k and Δf_k . We get then

$$m \leq m_k = \frac{\Delta f_k^\top \Delta y_k}{\Delta y_k^\top \Delta y_k} \leq \frac{\Delta f_k^\top \Delta f_k}{\Delta f_k^\top \Delta y_k} = M_k \leq M. \quad (15)$$

It is Impossible to guarantee (14) for our problem (5). However, We do not tolerate for *LPBFGS* Algorithm, having a too small m_k and too large M_k . In the first experimental tests we took this into account. But we observed that this condition has never been invoked for the values 10^{-3} and 10^8 of m and M respectively, we therefore omitted.

Theorem 2. Let the sequence $(y_k)_{k \in \mathbb{N}}$ generated by Algorithm.

(i) Then either $(y_k)_{k \in \mathbb{N}}$ terminates at a solution $y^{(k)}$ or every accumulation point y^* of $(y_k)_{k \in \mathbb{N}}$ solve problem (5).

(ii) The corresponding $v^* = \frac{1}{\varepsilon}(Ay^* - b)_+$ is the exact least 2-norm solution to the dual linear program (2), for some sufficiently small ε .

(iii) If the assumption 7 is checked for

$$S = \{j : v_j^* > 0\},$$

an exact solution z^* to the primal linear program (1) is obtained by step 12 of the algorithm.

Proof. We prove only (i), (ii) and (iii) follows immediately from ([9], Theorem 3.1). The solvability of problem (1) involves that of (5). Let the sequence $(y_k)_{k \in \mathbb{N}}$ generated by algorithm 1, and

suppose that $\nabla f(y_k) \neq 0$, there exists a subsequence $(y_{k_n})_{n \in \mathbb{N}}$ that converges to the accumulation point y^* of $(y_k)_{k \in \mathbb{N}}$. By the first Wolfe conditions (12) and the Quasi-Newton direction (10), we get

$$f(y_{k_n}) - f(y_{k_n+1}) \geq c_1 \alpha_k \nabla f(y_{k_n})^\top H_k (\nabla f(y_{k_n})) \geq 0. \quad (16)$$

Then, the sequence $(f(y_{k_n}))_{n \in \mathbb{N}}$ is nonincreasing and converges to the point $f(y^*)$. Since the gradient of f is Lipschitz continuous with constant $\|A^\top\| \|A\|$, we have

$$\|\nabla f(y_{k_n}) - \nabla f(y^*)\| \leq \|A^\top\| \|A\| \|y_{k_n} - y^*\|,$$

we deduce that $(\nabla f(y_{k_n}))_{n \in \mathbb{N}}$ converges to $\nabla f(y^*)$. By letting $n \rightarrow \infty$ in (16) and the fact that H_k always check (15) we get

$$\nabla f(y^*) = 0.$$

Remark 3. In the proof of the last theorem, we have used only the first strong Wolfe condition called the Armijo stepsize or also, the sufficient decrease condition. However, the second strong Wolfe condition called the curvature condition, is sufficient to guarantee the so called secant equation formula, i.e. $H_{k+1} \Delta f_k = \Delta y_k$, which must satisfy the updated approximation H_{k+1} . For more comprehension on the BFGS method, we refer the reader to the classical reference [13].

4. Numerical Tests

The objective of our numerical experiments is to show that the *lpbfgs* method is more effective than *lpnewt* when $m \approx n$. Recall that when $m \gg n$ *lpnewt* was better than CPLEX 6.5 [2] that uses an efficient interior-point method. Firstly, we have implemented the *lpnewt* procedure whose

MATLAB [12] code is given explicitly in ([9], page 13). Secondly we have implemented Algorithm 1 in the same language. Both algorithms was tested on the same very large synthetically generated test problems and on the same Intel (R) Core (TM)2 Duo CPU T5870 @ 2.00 GHz with 3 Go of RAM. The test problem generator, called *lpgen* by the author in [9], generates a random constraint matrix A for given (m, n) , and density d . The elements of A are uniformly distributed between -50 and $+50$. A primal random solution x , with elements in $[-10, 10]$, and a dual solution u , with elements in $[0, 10]$. These solutions are then used to generate an objective function cost vector c and a right-hand side vector b for the linear program (1). For the *lpnewt* the parameter values δ , ε and tolerance *tol* are taken as in [9] (10^{-3} , 10^{-4} , and 10^{-12} respectively). For both procedures, the iteration limit $k_{\max}$ is taken equal to 100, and the starting point $y^{(0)}$ is typically equal to $(\bar{A}^\top \bar{A} + \varepsilon I)^{-1} \bar{A}^\top \bar{b}$, where $\bar{A}$ is an arbitrary $n \times n$ subset of A , and $\bar{b}$ is the corresponding $n \times 1$ subset of b .

In Table 1, we give the comparison between the two methods, when, depending on CPU time (columns 4 and 6) and quality solution (columns 3 and 5); where Z_* denotes the exact solution, Z_{lpbfgs} and Z_{lpnewt} are the solutions given by *LPBFGS* and *lpnewt* algorithms respectively and the norm $\|\cdot\|$ is the Euclidean norm. Column 2 represent the density d of the matrix system A . The best values are in bold. With the exception of the last example, the quality of the solution given by the *LPBFGS* method is much higher than that given by the method *lpnewt*. However, In table 2 we give a comparison when $m = n$. We had to be limited to $n = 10^3$, because beyond *lpgen* is out of memory, and as we see in Table 2, the difference is huge when $m \approx n$.

Table 1 Comparison of LPBFGS and lpnewt when $m \gg n$.

(m, n)	d	$\ Z_* - Z_{lpbfgs}\ $	$Time(lpbfgs)$	$\ Z_* - Z_{lpnewt}\ $	$Time(lpnewt)$
$10^3 \times 10^2$	0.1	0.3548	0.7824	165.6835	0.0585
$10^3 \times 10^2$	1	$1.0822e-013$	1.3813	84.1570	0.1264
$10^4 \times 10^2$	0.1	0.5462	2.4017	31.0228	0.1751
$10^4 \times 10^2$	1	0.0988	6.8536	37.3349	0.8189
$10^5 \times 10^2$	0.1	0.2768	16.6980	243.6130	3.7684
$10^5 \times 10^2$	1	$5.2354e-014$	59.3134	36.2556	16.4920
$10^6 \times 10^2$	0.05	0.3070	607.0350	$1.4492e-014$	195.9422

Table 2 Comparison of LPBFGS and lpnewt when $m = n$.

(m, n)	d	$\ Z_* - Z_{lpbfgs}\ $	$Time(lpbfgs)$	$\ Z_* - Z_{lpnewt}\ $	$Time(lpnewt)$
$10^3 \times 900$	0.1	$9.0547e-014$	$9.4193e+000$	$5.4984e+002$	$2.2633e+001$
$10^3 \times 900$	1	$4.7073e-014$	$1.7154e+001$	$2.3862e+001$	$1.7068e+002$
$10^3 \times 950$	0.1	$4.4578e-014$	$1.0506e+001$	$1.6543e+002$	$2.6286e+001$
$10^3 \times 950$	1	$4.9370e-014$	$1.7268e+001$	$6.6964e+001$	$2.0234e+002$
$10^3 \times 990$	0.1	$3.8192e-014$	$9.6128e+000$	$9.8499e+001$	$2.2491e+001$
$10^3 \times 990$	1	$4.2633e-014$	$1.5928e+001$	$1.0844e+002$	$1.6980e+002$

5. Conclusion

It is known [13] that the *BFGS* formula has very effective self-correcting properties. When the rounding errors that occur infinite-precision implementation, or when the matrix H_k incorrectly estimates the curvature in the objective function, then the Hessian approximation (11) will tend to correct itself within a few steps. This explains the difference in quality between the two methods. Having obtained very satisfactory results, we were merely limited to the BFGS method with strongly Wolfe linear search. However, a method like the Limited-memory BFGS quasi-Newton methods which is useful for solving large problems can also give better results.

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