A Complete Solution to The Problem of Differentiation and Integration of Real Orders for the Gaussian Hypergeometric Function

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Abstract: A complete solution to the problem of finding the n th derivative and the n th anti-derivative of elementary and special functions has been given. It deals with the problem of finding formulas for the n th derivative and the n th anti-derivative of elementary and special functions. We do not limit n to be an integer, it can be a real number. In general, the solution is given through unified formulas in terms of the Fox H -function and the Meijer G -function which, in many cases, can be simplified to less general functions. This, in turn, makes the first real use of these two special functions in the literature and shows the need of such functions. In this talk, we would like to present the idea on the Gauss hypergeometric function which is a well known special function. One of the key points in this work is that the approach does not depend on integration techniques. We adopt the classical definitions for generalization of differentiation and integration. Namely, the n th order of differentiation is found according to the Riemann-Liouville definition

$$f^{(n)}(x) = \frac{1}{\Gamma(k-n)} \frac{d^k}{dx^k} \int_a^x (x-t)^{k-n-1} f(t) dt,$$

where $(k-1 < n < k)$ and $k = \lceil n \rceil$. The generalized Cauchy n -fold integral is adopted for the n th order of integration

$$f^{(-n)}(x) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t) dt, \quad n > 0.$$

1. Introduction

The motivation of this work comes from the area of symbolic computation as well as the area of classical and fractional calculus. The idea is that: Given a function f in a variable x , can we or computer algebra systems (CAS) find a formula for the n th derivative, the n th anti-derivative, or both? This enhances the power of integration and differentiation of CAS. In Maple, the formula corresponds to invoking the command `diff(f(x), x$n)` for the n th derivative and `int(f(x), x$n)` for the n th anti-derivative.

The computation of fractional derivatives and fractional integrals involves evaluating some types of non elementary integrals. In fact, integration techniques

do not work usually. This has led of thinking about different approaches to evaluate them. A series of papers has been established [2] [3] [4] and recently a book [1] by the author of this paper to introduce different approaches to evaluating this type of integrals. This work is a continuation of this series.

2. Riemann-Liouville Fractional Derivative Definition

The most widely known definition of the fractional derivative is The Riemann-Liouville definition (R-L FD) [8] [11] [10]. It appears as a result of unification of the notions of integer-order integration and differentiation. The definition is given by

$$f^{(q)}(x) = \frac{1}{\Gamma(k-q)} \frac{d^k}{dx^k} \int_a^x (x-t)^{k-q-1} f(t) dt, \quad (1)$$

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where $k-1 < q < k$, $k = \lceil q \rceil$, and $f(x)$ is a function with a weak singularity over the interval of integration. If $f(x)$ is continuous over the interval $[a, x]$, then by letting $q \rightarrow k$, one gets $f^{(k)}(x)$.

2.1 The n th Derivative of $(x-a)^m$ of Real Orders

We are interested in the fractional derivative (the n th derivative of real orders) of the function

$$f(x) = (x-a)^m \quad (2)$$

because of its latter use. Substituting (2) in (1) yields

$$\begin{aligned} ((x-a)^m)^{(n)} &= \frac{1}{\Gamma(\alpha)} \frac{d^k}{dx^k} \int_a^x (x-y)^{\alpha-1} (y-a)^m dy \\ &= \frac{\Gamma(m+1)}{\Gamma(m-n+1)} (x-a)^{m-n} \end{aligned}$$

The above formula gives the n th derivative of real orders of the function $(x-a)^m$.

3. The Generalized Cauchy n -fold Integral Definition

The generalized Cauchy n -fold integral is a generalization of the Cauchy n -fold integral of integer order

$$f^{(-n)}(x) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t) dt, \quad n \geq 1.$$

Relaxing the condition on n in the above definition from integer $n \geq 1$ to a real $n > 0$ does not affect the convergence of the integral. $f(x)$ is a function with a weak singularity over the interval of integration.

3.1 The n th Anti-Derivative of $(x-a)^m$ of Real Orders

Again, for our need of the n th anti-derivative of the function $(x-a)^m$, we give it in this section. It is the evaluation of the integral

$$\frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} (t-a)^m dt, \quad n \geq 0$$

Calculating the integral results in the desired

formula

$$\frac{\Gamma(m+1)}{\Gamma(m+n+1)} (x-a)^{m+n} \quad (3)$$

4. The Mellin Transform

As a matter of fact, we need the mellin transform of the Gauss hypergeometric function, as it will be explained later, to find the desired unified formula. So, we introduce the Mellin transform and its inverse through the following two definitions. For more discussion of the Mellin transform; see [9].

Definition 1. The Mellin transform [9] of a locally integrable function $f(x)$ on $(0, \infty)$ is defined by

$$\begin{aligned} M[f; s] &= F(s) = \int_0^\infty x^{s-1} f(x) dx \\ &(\alpha < R(s) < \beta) \end{aligned}$$

where α and β are real constants. The strip $\alpha < R(s) < \beta$ is known as the strip of analyticity.

Definition 2. The inverse Mellin transform [9] is given by

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M[f; s] x^{-s} ds \quad (\alpha < c < \beta),$$

where α and β are the same ones in the last definition.

Using the Mellin transform definition and evaluating the corresponding integral, an explicit expression for the Mellin transform of the Gauss hypergeometric function

$${}_2F_1(a, b; c; x) \quad (4)$$

can be found

$$G(s) = \frac{\Gamma(c)\Gamma(a-s)\Gamma(b-s)\Gamma(s)(-1)^s}{\Gamma(a)\Gamma(b)\Gamma(c-s)} \quad (5)$$

where $\Gamma(s)$ is the gamma function and is defined as

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \quad \text{Re}(s) > 0 \quad (6)$$

5. The Gaussian Hypergeometric Function

The Gauss hypergeometric function is a special function and is a solution of the linear second order

differential equation

$$ab y(x) + ((1+a+b)x - c) \frac{d}{dx} y(x) + (x^2 - x) \frac{d^2}{dx^2} y(x) = 0 \quad (7)$$

One of the solutions of the above differential equation the Gaussian hypergeometric function and is given in terms of a power series

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!} \quad (8)$$

6. The H -function

The H -function is a very general function that encompasses the most of special functions including the Meijer G -function and the generalized hypergeometric function; see [5] [6].

Notation

$$H_{p,q}^{m,n} \left(\begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} \middle| z \right) \equiv H_{p,q}^{m,n} \left(\begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \middle| z \right) \equiv H_{p,q}^{m,n}(z) \equiv H(z).$$

Definition 3. The H -function is defined by the Mellin-Barnes integral [9].

$$H(z) = \frac{1}{2\pi i} \int_C h(s) z^s ds$$

where $h(s)$ is given by

$$h(s) = \frac{\prod_{j=1}^m \Gamma(b_j - B_j s) \prod_{j=1}^n \Gamma(1 - a_j + A_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + B_j s) \prod_{j=n+1}^p \Gamma(a_j - A_j s)} \quad (9)$$

For the above, we require that

- (i) A_j and B_j are positive numbers.
- (ii) a_j and b_j are complex numbers such that

$$A_j(b_k + \nu) \neq B_k(a_j - \lambda - 1)$$

for

$$\nu, \lambda = 0, 1, 2, \dots; j = 1, \dots, n; k = 1, \dots, m.$$

That means the poles of $\Gamma(b_k - B_k s)$ for $k = 1, \dots, m$ and $\Gamma(1 - a_j + A_j s)$ for $j = 1, \dots, n$ do not coincide.

(iii) The contour C separates the poles resulting from $\Gamma(b_k - B_k s)$ ($1 \leq k \leq m$) from those of $\Gamma(1 - a_j + A_j s)$ ($1 \leq j \leq n$).

6.1 Existence Conditions for The H -function

For existence conditions of the H -function, we refer the reader to [6].

6.2 Properties of The H -function

The following are some properties of the H -function that are useful for our purpose (see [6]).

Property 1

$$z^\alpha H_{p,q}^{m,n} \left(\begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \middle| z \right) = H_{p,q}^{m,n} \left(\begin{matrix} (a_p + \alpha A_p, A_p) \\ (b_q + \alpha B_q, B_q) \end{matrix} \middle| z \right)$$

Property 2

$$H_{p,q}^{m,n} \left(\begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \middle| z \right)^\alpha = \frac{1}{\alpha} H_{p,q}^{m,n} \left(\begin{matrix} (a_p, \frac{A_p}{\alpha}) \\ (b_q, \frac{B_q}{\alpha}) \end{matrix} \middle| z \right) \alpha > 0$$

Property 3

$$H_{p,q}^{m,n} \left(\begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \middle| z \right) = H_{q,p}^{n,m} \left(\begin{matrix} (1 - b_q, B_q) \\ (1 - a_p, A_p) \end{matrix} \middle| \frac{1}{z} \right)$$

7. Meijer G -function

The G -function is a special case of the H -function. A large number of special functions are special cases of this function. In this section we give some definitions of the function without any proofs. For a detailed discussion of the G -function, we refer the reader to [7].

Notation

$$G_{p,q}^{m,n} \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) \equiv G_{p,q}^{m,n} \left(\begin{matrix} a_p \\ b_q \end{matrix} \middle| z \right) \equiv G_{p,q}^{m,n}(z) \equiv G(z)$$

These are the standard notations used in the literature. In the following definition an empty

product is interpreted as unity and $0 \leq m \leq q$, $0 \leq n \leq p$. The Meijer G -function with the parameters $a_1, \dots, a_p$ and $b_1, \dots, b_q$ is defined as a Mellin-Barnes type integral as follows [7].

Definition 4.

$$G_{p,q}^{m,n} \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \int_L g(s) z^s ds$$

$$g(s) = \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)}$$

It is clear that the Meijer G -function is a special case of the H -function and it is derived from the latter by letting A_j 's and B_j 's equal one.

7.1 The r th Derivative and The r th Anti-Derivative of The H -function

The closure property of the H -function under real orders of differentiation and integration makes it a very powerful tool for finding unified formulas for the n th derivative and the n th anti-derivative of elementary and special functions. In other words, one can express real order derivatives and integrals of an H -function in terms of a new H -function. This is a very nice property of the H -functions which is not possessed by other special functions. The following lemma illustrates the idea.

Lemma 1. The formula

$$(H_{p,q}^{m,n}(z))^{(r)}(z) = H_{p+1,q+1}^{m,n+1} \left(\begin{matrix} (a_p - rA_p, A_p), (-r, 1) \\ (b_q - rB_q, B_q), (0, 1) \end{matrix} \middle| z \right) \quad (10)$$

gives

- (i) derivatives of arbitrary order if $r > 0$,
- (ii) integrals of arbitrary order if $r < 0$, of the H -function

$$H_{p,q}^{m,n} \left(\begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \middle| z \right)$$

Proof: We give a proof for part (i). The proof for

part (ii) is similar to part(i) and only one needs to use formula (3) for fractional order integration. Recalling the definition of the H -function (9)

$$H(z) = \frac{1}{2\pi i} \int_C h(s) z^s ds$$

where $h(s)$ is given by (9). One can differentiate both sides of the above equation, provided the above integral converges uniformly for some z , using the formula

$$\frac{d^r}{dx^r} x^s = \frac{\Gamma(s+1)}{\Gamma(s-r+1)} x^{s-r}$$

gives

$$(H_{p,q}^{m,n}(z))^{(r)} = \frac{1}{2\pi i} \int_C h(s) z^{s-r} ds$$

where

$$h(s) = \frac{\Gamma(s+1)}{\Gamma(s-r+1)} h(s)$$

$h(s)$ is given by equation (9). Using the notation of the H -function, the above can be written as

$$(H_{p,q}^{m,n}(z))^{(r)}(z) = x^{-r} H_{p+1,q+1}^{m,n+1} \left(\begin{matrix} (a_p, A_p), (0, 1) \\ (b_q, B_q), (r, 1) \end{matrix} \middle| z \right)$$

Property (1) of the H -function simplifies the last equation to a more compact form

$$(H_{p,q}^{m,n}(z))^{(r)}(z) = H_{p+1,q+1}^{m,n+1} \left(\begin{matrix} (a_p - rA_p, A_p), (-r, 1) \\ (b_q - rB_q, B_q), (0, 1) \end{matrix} \middle| z \right)$$

If C is taken as path (1), the existence condition for the above H -function is

$$|\arg(z)| < \frac{\pi}{2} \left(\sum_{j=1}^n A_j + \sum_{j=1}^m B_j - \sum_{j=n+1}^p A_j - \sum_{j=m+1}^q B_j \right)$$

For Part(ii), one needs formula (3)

$$\frac{\Gamma(s+1)}{\Gamma(s+r+1)} x^{s+r}$$

which is derived from the generalized Cauchy formula for the n -fold integral. The rest of the proof is exactly similar to part (i).

Formula (10) is very important for finding integer

and arbitrary order symbolic derivatives and integrals of both elementary and special functions as long as they are representable in terms of the H -function.

8. A Unified Formula for the n th Derivative and the n th Anti-Derivative of Gaussian Hypergeometric Function of Real Orders

Now, we are in a position to give a complete solution of the problem of symbolic differentiation and integration, of the Gaussian hypergeometric

function, of real orders. This will be achieved by introducing a unified formula for the n th derivative and the n th anti-derivative, where n can be any real number. The formula encompasses the two generalized definitions of differentiation and integration. Namely, the Riemann-Liouville definition for fractional differentiation and the generalized Cauchy n -fold integral.

Theorem: Given the Gaussian hypergeometric function, then the formula

$$\frac{\alpha^n e^{n(\ln(\alpha) + i\pi)}}{\gamma} H_{3,3}^{1,3} \left(\begin{matrix} (1-a-\frac{n}{\gamma}, \frac{1}{\gamma}), (1-b-\frac{n}{\gamma}, \frac{1}{\gamma}), (-n, 1) \\ (-\frac{n}{\gamma}, \frac{1}{\gamma}), (1-c-\frac{n}{\gamma}, \frac{1}{\gamma}), (0, 1) \end{matrix} \middle| e^{\frac{i\pi}{\gamma}} (\alpha x + \beta) \right) = \begin{cases} \frac{1}{\Gamma(k-n)} \frac{d^k}{dx^k} \int_0^x (x-t)^{k-n-1} {}_2F_1(a, b, c; (\alpha t + \beta)^\gamma) dt, & \text{if } n > 0 \\ \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1} {}_2F_1(a, b, c; (\alpha t + \beta)^\gamma) dt, & \text{if } n < 0 \end{cases}, \quad (11)$$

where $0 < |\alpha x + \beta| < 1$, $\alpha \neq 0$ and $b, \gamma \in \mathbb{R}$, gives

- (a) The original function if $n = 0$,
- (b) Derivatives of any order if $n > 0$,
- (c) Anti-Derivatives of any order if $n < 0$.

Proof: Representing the hypergeometric function in terms of Mellin-Barnes integral gives

$${}_2F_1(a, b, c; (\alpha x + \beta)^\gamma) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \times \frac{1}{2\pi i} \int_C \frac{(-1)^s \Gamma(a-s) \Gamma(b-s) \Gamma(s)}{\Gamma(c-s)} (\alpha x + \beta)^{-\gamma s} ds$$

where C is a suitable contour. Replacing s by $-s$ in the last equation yields

$${}_2F_1(a, b, c; (\alpha x + \beta)^\gamma) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_C \times \frac{\Gamma(a+s) \Gamma(b+s) \Gamma(-s)}{\Gamma(c+s)} \left(e^{\frac{i\pi}{\gamma}} (\alpha x + \beta) \right)^{\gamma s} ds$$

with taking in consideration the change in the contour C . Using the H -function notation, we get the H -function representation of our hypergeometric function

$${}_2F_1(a, b, c; (\alpha x + \beta)^\gamma) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \times H_{2,2}^{1,2} \left(\begin{matrix} (1-a, 1), (1-b, 1) \\ (0, 1), (1-c, 1) \end{matrix} \middle| -(\alpha x + \beta)^\gamma \right)$$

By exploiting property (2) of the H -function, we can simplify the above H -function to

$${}_2F_1(a, b, c; (\alpha x + \beta)^\gamma) = \frac{1}{\gamma} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \times H_{2,2}^{1,2} \left(\begin{matrix} (1-a, \frac{1}{\gamma}), (1-b, \frac{1}{\gamma}) \\ (0, \frac{1}{\gamma}), (1-c, \frac{1}{\gamma}) \end{matrix} \middle| e^{\frac{i\pi}{\gamma}} (\alpha x + \beta) \right)$$

Applying lemma (1), for real orders derivatives and anti-derivatives, to the above function gives a unified formula for the n th derivative and the n th anti-derivative of the Bessel function

$$\left({}_2F_1(a, b, c; (\alpha x + \beta)^\gamma) \right)^{(n)} = \frac{\alpha^n (\alpha x + \beta)^{-n}}{\gamma} \times H_{3,3}^{1,3} \left(\begin{matrix} (1-a, \frac{1}{\gamma}), (1-b, \frac{1}{\gamma}), (0, 1) \\ (0, \frac{1}{\gamma}), (1-c, \frac{1}{\gamma}), (n, 1) \end{matrix} \middle| e^{\frac{i\pi}{\gamma}} (\alpha x + \beta) \right) \quad (12)$$

A further simplification can be made to our unified formula by appealing to property (1)

$$\left({}_2F_1(a, b; c; (\alpha x + \beta)^\gamma) \right)^{(n)} = \frac{\alpha^n e^{\frac{i\pi n}{\gamma}}}{\gamma} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} H_{3,3}^{1,3} \left(\begin{matrix} (1-a-\frac{n}{\gamma}, \frac{1}{\gamma}), \\ (1-b-\frac{n}{\gamma}, \frac{1}{\gamma}), \\ (-n, 1) \\ (-\frac{n}{\gamma}, \frac{1}{\gamma}), \\ (1-c-\frac{n}{\gamma}, \frac{1}{\gamma}), \\ (0, 1) \end{matrix} \middle| e^{\frac{i\pi}{\gamma}} (\alpha x + \beta) \right). \quad (13)$$

By considering the appropriate contour an existing condition can be obtained such as $0 < |(\alpha x + \beta)| < 1$.

Example: A unified formula for the gaussian hypergeometric function ${}_2F_1(a, b; c; (\alpha x + \beta)^{-2})$ is given by

$$(-1)^{\frac{n}{2}+1} \alpha^n 2^n \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \times G_{4,4}^{1,4} \left(-(\alpha x + \beta)^{-2} \middle| \begin{matrix} \frac{1}{2}, 1, -a+1+\frac{n}{2}, -b+1+\frac{n}{2} \\ \frac{n}{2}+1, \frac{n}{2}+\frac{1}{2}, \frac{n}{2}, -c+1+\frac{n}{2} \end{matrix} \right) \quad (14)$$

Substituting $n = -\frac{1}{3}$ in the above formula gives the one third anti derivative of the hypergeometric function given in the above example

$$\begin{aligned} & \frac{1}{\Gamma(\frac{1}{3})} \int_0^x (x-t)^{-\frac{2}{3}} {}_2F_1(a, b; c; (\alpha x + \beta)^{-2}) dt \\ &= (-1)^{\frac{n}{2}} \alpha^{-\frac{1}{3}} 2^{-\frac{1}{3}} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \times \\ & G_{4,4}^{1,4} \left(-(\alpha x + \beta)^{-2} \middle| \begin{matrix} \frac{1}{2}, 1, -a+\frac{5}{6}, -b+\frac{5}{6} \\ \frac{5}{6}, \frac{1}{3}, -\frac{1}{6}, -c+\frac{5}{6} \end{matrix} \right) \end{aligned} \quad (15)$$

A further simplification can be made to the latter Meijer G function in terms of the generalized hypergeometric function

$$\begin{aligned} & \frac{1}{\Gamma(\frac{1}{3})} \int_0^x (x-t)^{-\frac{2}{3}} {}_2F_1(a, b; c; (\alpha x + \beta)^{-2}) dt \\ &= -\frac{(-1)^{-\frac{1}{6}} (\sqrt{3} + i)^5 \Gamma(\frac{2}{3}) a b}{48 c \alpha^{\frac{1}{3}} (\alpha x + \beta)^2} \times \\ & {}_4F_3 \left(\frac{5}{6}, \frac{4}{3}, a+1, b+1; \frac{3}{2}, 2, c+1; \frac{1}{(\alpha x + \beta)^2} \right) \end{aligned} \quad (16)$$

9. Conclusion

The main achievement, in this work, can be summarized in the following statements. A complete solution to the problem of finding the n th derivative and the n th anti-derivative of the Gaussian hypergeometric function of real orders is obtained. This can be considered as a big breakthrough in the area of classical and fractional calculus as well as the area of symbolic computation. The difficulty has been in the evaluation of non elementary integrals. The goal has been reached without appealing to integration techniques. The Fox H -function has been introduced, as a main tool, to give a complete solution to the problem in consideration. Another important fact is these unified formulas give a unification for the two classical definitions of generalized differentiation and integration. In the area of symbolic computation, these formulas enhance the power of computer algebra systems since they are plug in formulas. CAS such as Maple and Mathematica have already the Meijer G -function implemented. On the other hand, the Fox H -function has not been implemented yet.

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