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An Introduction to Special Relativity

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This paper is a summary of the lecture the author gives to introduce special relativity in his introductory engineering physics class. It starts with an explanation of three classic paradoxes using a qualitative understanding of time dilation, length contraction, and non-synchronous clocks. Then, it introduces four-vectors, first the space-time four-vector, and then the energy-momentum four-vector. It concludes with a brief discussion of the meaning of $E = mc^2$.

Keywords: special relativity, time dilation, non-synchronous clocks, four-vectors, invariance

Introduction

For most introductory physics classes, the topic of Albert Einstein's theory of special relativity is an elective topic. The author's guess is that many instructors avoid the topic due to its difficulty, not only for students, but also for instructors. This paper is an effort to present an introduction to special relativity that is "relatively" easy to understand by both instructors and students. And the author believes that it is important to include an exposure to special relativity in all introductory physics classes, both because it is an essential topic of modern physics and the students find the topic fascinating and challenging.

A good place to start is with the Michelson-Morley experiment. At the time of the experiment, it was believed that electromagnetic waves were carried on a fluid called ether, just like sound waves were carried on a fluid called air. And given that the earth is spinning, one would expect that the speed of light would depend on the velocity of the earth with respect to the ether and that would change with the time of the day. The observation by Michelson and Morley that the speed of light does not change with the time of the day leads to the fundamental postulate of the theory of special relativity: the speed of light, $c = 3 \times 10^8$ m/s, is the same in all inertial frames of reference (Feldman, 2015). The historical record suggests that Albert Einstein came to this postulate from Maxwell's equations, not from the Michelson-Morley experiment, but that can be discussed during a later class on electromagnetism.

This first postulate leads naturally to time dilation. Consider a stationary light clock consisting of two mirrors separated by a distance, d, with a light pulse bouncing between the two mirrors (see Figure 1a), the time it takes for the pulse of light to travel from one mirror to the other and back is:

$$t_0 = 2d/c \qquad \text{Equation (1)}$$

Now, observe a light clock moving at a velocity v with respect to you (see Figure 1b). You observe the light pulse in that moving light clock trace out diagonals, so that it takes longer for that clock to tick than for your clock to tick. From the Pythagorean theorem:

$$v^2(t/2)^2 + d^2 = c^2(t/2)^2 \qquad \text{Equation (2)}$$

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Solving for t, the time it takes a moving clock to make one tick, gives:

$$t = t_0(1 - v^2/c^2)^{-1/2} \qquad \text{Equation (3)}$$

This is the equation for time dilation, namely, a moving clock takes a longer time to make one tick than a stationary clock. Notice that the key to this result is that the speed of the light pulse, c, is the same whether the clock is stationary or moving—in complete disagreement with your experience with moving objects.

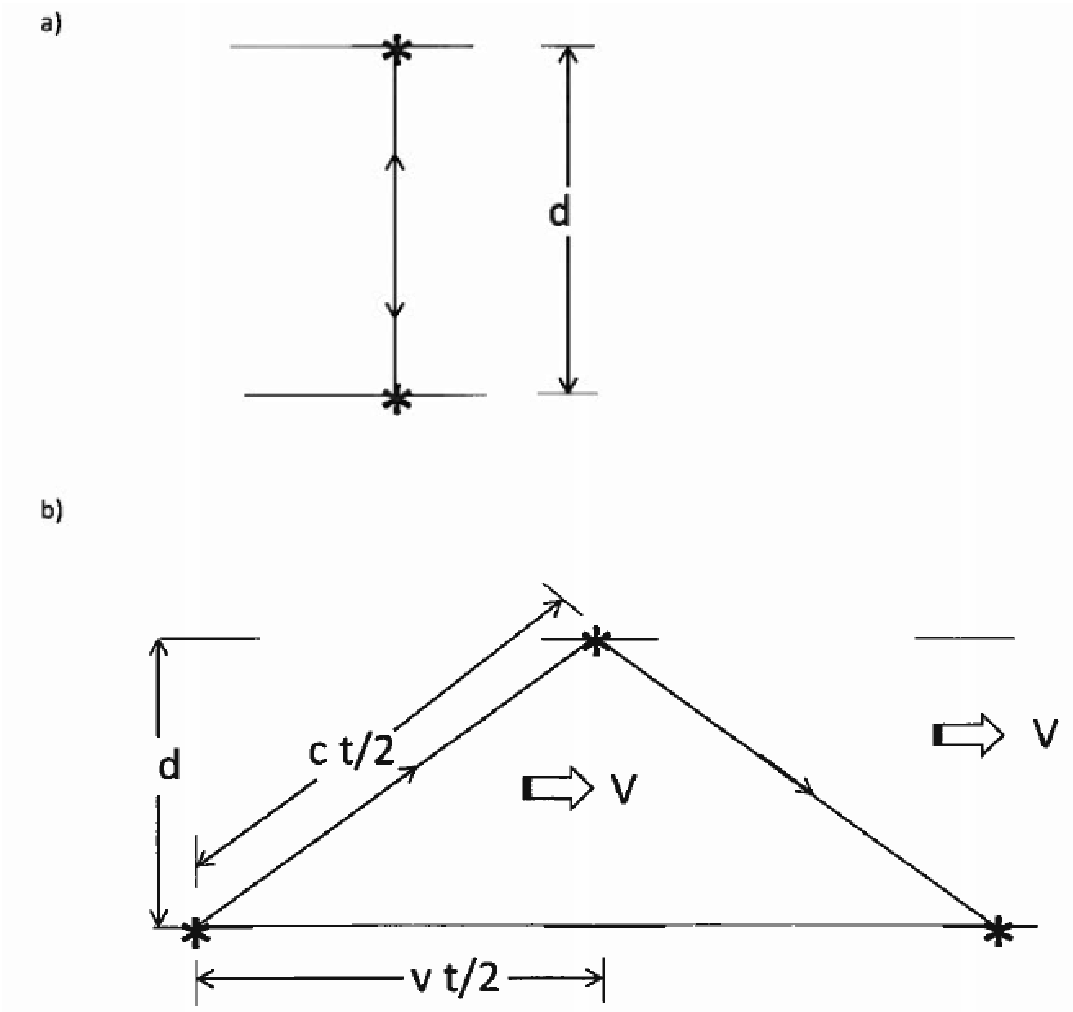

Figure 1. Schematic of light clock (a) at rest and (b) moving with a velocity v.

Paradox One

This leads naturally to the first paradox. You see a moving clock going slower, but the person with the moving clock sees your clock going slower. How can that be?

Start in the frame of reference of two synchronized clocks 1 and 2 watching clock 3 move with a velocity v to the right, such that clock 3 is a factor of 2 slower than clocks 1 and 2, namely, $(1 - v^2/c^2)^{-½} = 2$ (see Figure 2a). When clock 3 passes clock 1, for simplicity, both clocks read 0. When clock 3 reaches clock 2, clock 3 reads 1/4 and clocks 1 and 2 read 1/2, so clock 3 is a factor of 2 slower than clocks 1 and 2. The tough question is: How can clocks 1 and 2 be a factor of 2 slower than clock 3 in clock 3's frame of reference?

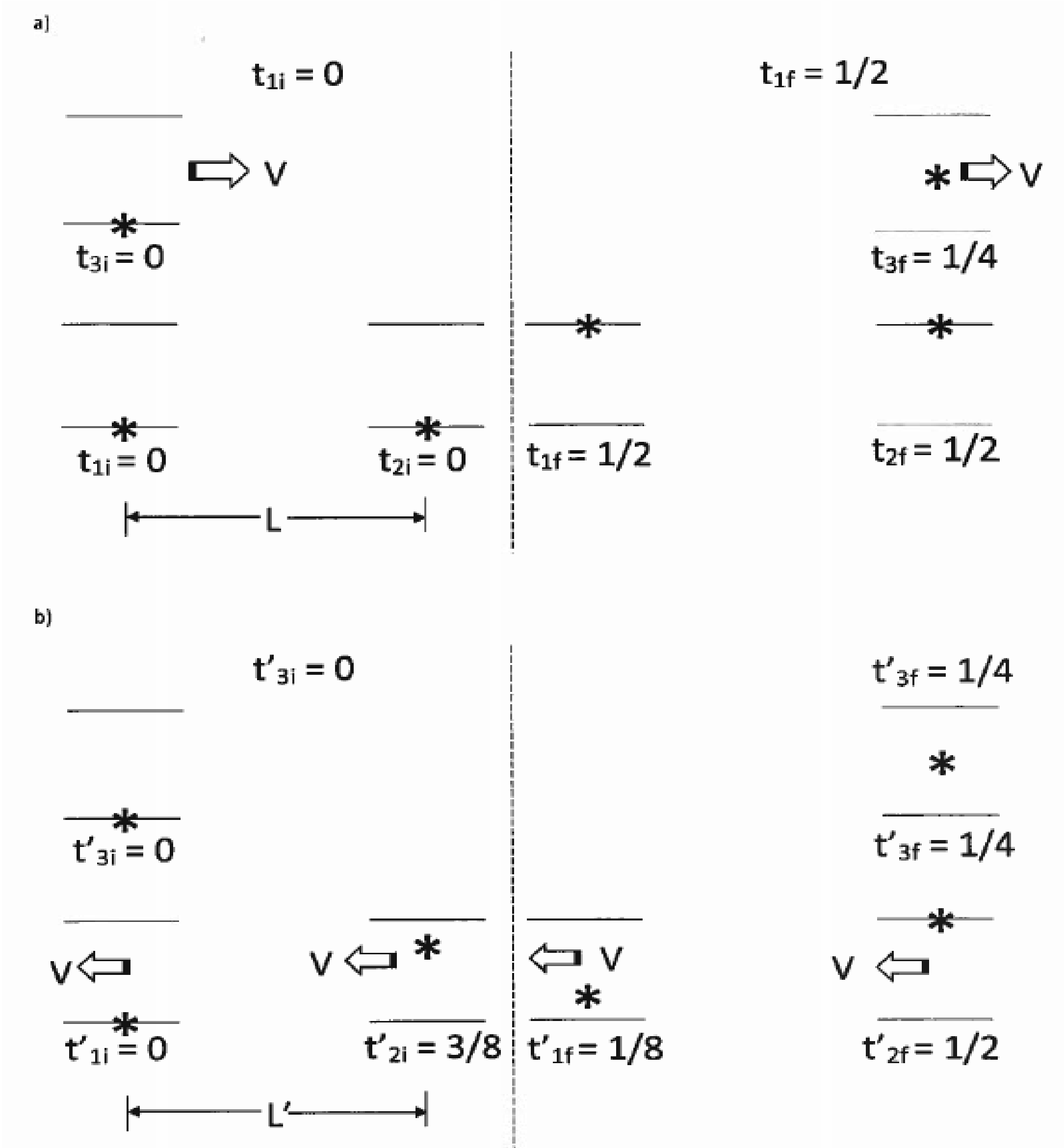

Figure 2. Schematic of three light clocks: (a) In this frame of reference, clocks 1 and 2 are at rest and synchronized and clock 3 is moving at a velocity *v*. The three clocks are shown when clock 3 passes clock 1 and then when clock 3 passes clock 2; and (b) In this frame of reference, clock 3 is at rest and clocks 1 and 2 are moving at velocity, -*v*. The three clocks are shown when clock 1 passes clock 3 and then when clock 2 passes clock 3.

The resolution of all the paradoxes in special relativity, including this one, is with the idea that moving clocks are not synchronized. Let us first talk about how clocks are synchronized. Consider two clocks, 1 and 2, in Figure 2a. To synchronize clocks 1 and 2, the person at rest with respect to clocks 1 and 2 measures the distance between clocks 1 and 2 and divides by two. A pulsed laser is placed at the midpoint between clocks 1 and 2 and it is flashed. When the light flash reaches clocks 1 and 2, those two clocks start ticking and are synchronized ($t_{1i} = t_{2i} = 0$). Now, consider a person at rest with respect of clock 3 (see Figure 2b) watching this synchronization process. Now, clocks 1 and 2 are moving with a velocity v to the left. Notice the clock 2 is moving towards the light flash and clock 1 is moving away from the light flash, so the light flash will reach clock 2 before clock 1, and thus in the frame of reference of clock 3, clock 2 starts before clock 1, and thus, clock 2 ($t'_{2i} = 3/8$) is ahead of clock 1 ($t'_{1i} = 0$).

Now, let us do the analysis in the frame of reference of clock 3 (see Figure 2b). Notice that clock 2 is ahead of clock 1 by 3/8. So, according to clock 3, the stationary clock 3 advances by 1/4, while both moving clocks 1 and 2 advance by only 1/8, a factor of two slower than the stationary clock 3. With non-synchronized moving clocks, both frames of reference can say the moving clocks are slower by a factor of two than stationary clocks.

Now, this paradox can be generalized to the case where clock 3 is moving at a velocity v. Then, in the frame of reference of clocks 1 and 2, the time dilation of clock 3 is:

$$t_{2f}(1 - v^2/c^2)^{1/2} = t_{3f} \qquad \text{Equation (4)}$$

In the frame of reference of clock 3, the time dilation of clock 2 is:

$$t'_{3f}(1 - v^2/c^2)^{1/2} = (t'_{2f} - t'_{2i}) \qquad \text{Equation (5)}$$

Combining these two equations with $t'_{3f} = t_{3f}$, $t'_{2f} = t_{2f}$, and $L' = vt'_{3f}$ gives the time clock 2 is ahead of clock 1 in clock 3's frame of reference:

$$t'_{2i} = (1 - v^2/c^2)^{-1/2}vL'/c^2 \qquad \text{Equation (6)}$$

Paradox Two

Now, let us consider the famous twin paradox. One of the twins (male) remains on earth and the other twin (female) travels to Alpha Centauri and then back to earth. The twin on earth sees his twin as a moving clock, and thus, she is younger than he when she returns to earth. Does not the twin who travels see her stay-at-home twin as a moving clock, and thus, he should be younger than her when she returns to earth. This paradox is identical to paradox one and it is resolved the same way with non-synchronization of clocks. In the case of the traveling twin, when she accelerates to velocity v at the beginning of her trip, the clock on Alpha Centauri is ahead of the clock on earth. So, even though the clock on Alpha Centauri is slow compared to her clock, the time on Alpha Centauri when she arrives will be later than her clock and consequently, her twin on earth is older than her. An identical process occurs on her trip back to earth.

The contraction of moving meter sticks is also easily seen in Figure 2. In the frame of reference of clocks 1 and 2, the velocity of the clock 3 is $v = L/t_{2f}$. In the frame of reference of clock 3, the velocity of clocks 1 and 2 is L'/t'_{3f}. By symmetry, both these velocities must be equal (If you do not believe this, tell me which velocity is greater).

$$L/t_{2f} = L'/t'_{3f} \qquad \text{Equation (7)}$$

Given that $t_{3f} = (1 - v^2/c^2)^{1/2}\, t_{2f}$ and $t_{3f} = t'_{3f}$, then:

$$L' = L(1 - v^2/c^2)^{1/2} \qquad \text{Equation (8)}$$

A moving meter stick of length L' is shorter than stationary meter stick of length L = 1 meter.

Paradox Three

This leads to the third paradox, the train engineer and the tunnel keeper. Both the train and the tunnel are the same length on earth. When the train is traveling at speeds close to c, the tunnel keeper sees the train as shorter than the tunnel—length contraction. At the same time, the train engineer sees the tunnel as traveling in the opposite direction at speeds close to c and thus as shorter than the train—again, length contraction. The train engineer and the tunnel keeper agree to settle this debate by having the train go through the tunnel at speeds close to c, and see if the tunnel keeper can trap the train inside the tunnel. The tunnel keeper installs two powerful gates at each end of his tunnel which he will lower simultaneously the instant the train is inside the tunnel. Question: Is the train captured?

Again, the resolution of this paradox is clock non-synchronization. In the tunnel keepers frame of reference, the train is shorter, the two gates come done simultaneously, and the train is captured inside the tunnel (see Figure 3a). In the train engineer's frame of reference, the two gates are controlled by two clocks which are not synchronized and the farthest one is ahead of the nearer one. The train engineer sees the farthest gate come down first, then after he crashes into that gate, the second gate descends (see Figure 3b).

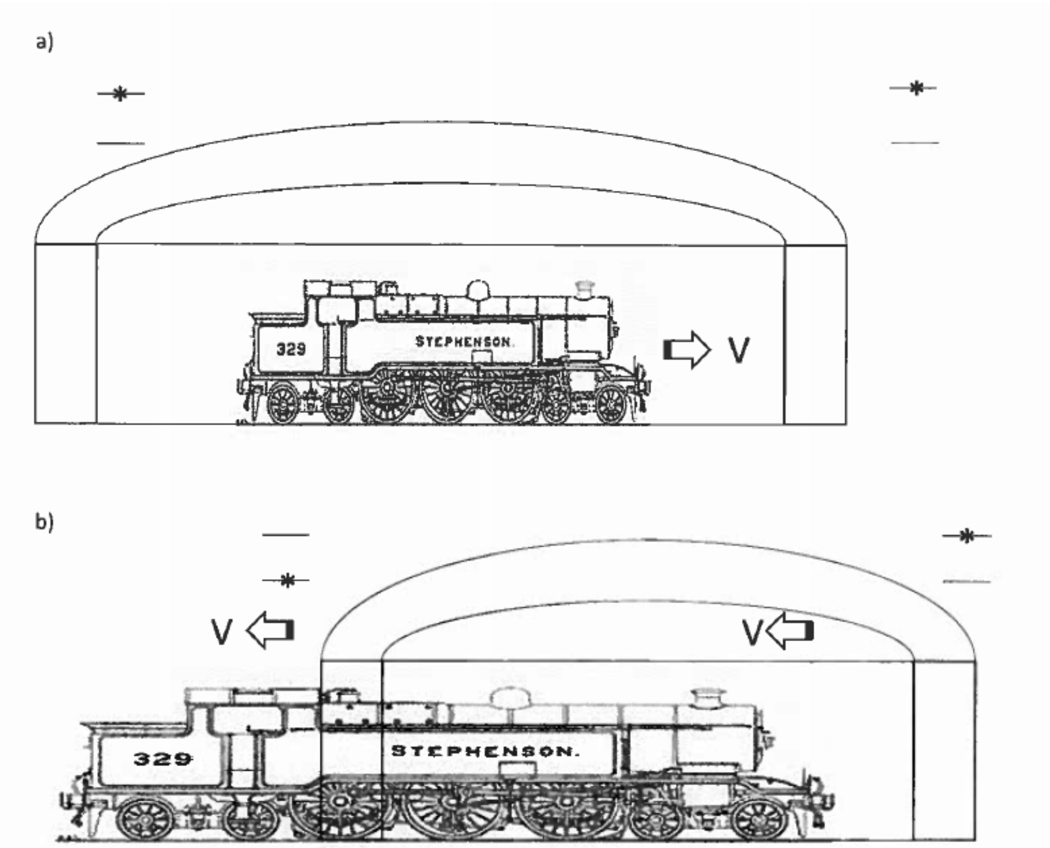

Figure 3. (a) Sketch of the view of the tunnel keeper when both gates come down simultaneously and the train trapped inside of the tunnel; (b) Sketch of the view of the train engineer when the farther gate comes down, the train is part way into the tunnel and the nearest gate is up.

Four-Vectors

These paradoxes lead to a more general discussion of space and time. Notice that two events like the closing of the two gates are separated by only space in the tunnel keeper's frame of reference but are separated by both space and time in the train engineer's frame of reference. Generalizing, space time transforms into space time, or space and time are the same thing and need to be described by a four-component vector, x, y, z, and t.

Consider a pulse of light that starts at $x = 0$, $y = 0$, $z = 0$, and $t = 0$, and at a later time, is at $x = x$, $y = y$, $z = z$, and $t = t$. By the Pythagorean theorem, $x^2 + y^2 + z^2 - c^2t^2 = 0$. In another inertial frame of reference moving at velocity v to the first one, that same pulse of light travels from $x' = 0$, $y' = 0$, $z' = 0$, and $t' = 0$ to $x' =$

x', $y' = y'$, $z' = z'$, and $t' = t'$, and because the velocity of light c is in all inertial frames of reference, $x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$. We now define these two Pythagorean expressions as magnitudes of four-vectors. Then, these two magnitudes are the same, 0, or in physics language, invariant.

Now, let us generalize. Physics is four-dimensional, all invariant physics must be in terms of four-vectors, and the magnitudes of these four-vectors are invariant. Einstein then realized that there is a momentum-energy four-vector, if the linear momentum vector is written as $p = mu(1 - \mathrm{u}^2/\mathrm{c}^2)^{-1/2}$ and energy as $E = mc^2(1 - u^2/c^2)^{-1/2}$, where the u is the velocity vector of the moving mass, m. The magnitude of the energy-momentum four-vector, $p_x^2 + p_y^2 + p_z^2 - (E/c)^2$ is the same in all inertial frames of reference. Also, instead of having two conservation laws—conservation of energy and conservation of linear momentum—we now have one conservation law, conservation of all four components of the energy-momentum four-vector.

Finally, let us examine the famous expression $E = mc^2(1 - u^2/c^2)^{-1/2}$. If we expand this expression in terms of u^2/c^2, $E = mc^2 + 1/2\ mu^2 + 3/8\ mu^4/c^2 + \ldots$. The second term, $1/2\ mu^2$, is just our familiar classical kinetic energy. The first term, mc^2, states that mass is a form of energy, and just like all other forms of energy, it can be transformed from mass energy into other forms of energy. This first term can also accommodate all forms of potential energy. For example, if you examine your periodic table and compare the masses of two hydrogen atoms and two neutrons to one helium atom, you discover that the mass of the helium atom (4.00260 amu) is less than the masses of the two hydrogen atoms and two neutrons (4.03322 amu). This mass difference times c^2 is the nuclear potential energy of a helium nucleus compared to two hydrogen atoms and two neutrons. Another way of saying the same thing is that this difference in mass energy is converted into other forms of energy when two hydrogen atoms and two neutrons fuse into a helium atom, for example, in the core of the sun or in a hydrogen bomb. Gravitational potential energy and chemical potential energy also can be expressed as differences in mass, but in these cases the differences in mass are so small, they cannot be measured.

Conclusion

This concludes the author's introduction to special relativity. It is meant to expose students to some of the key ideas of special relativity, to peak their curiosity, and to challenge their intellect without a great deal of mathematics. When the author teaches this material, he tells his students that he will not test them on this material, to put down their pens and pencils and just try to grasp the ideas. For a more thorough treatment of special relativity, the author recommends the following texts: Serway and Jewett (2004), for an introductory treatment, Tipler and Llewellyn (2012), for an intermediate treatment, and Cheng (2015), for a more advanced undergraduate treatment.

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