A Common Fixed Point Theorem for Compatible Mappings of Type (K) in Intuitionistic Fuzzy Metric space

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Received: August 11, 2015 / Accepted: September 06, 2015 / Published: November 25, 2015.

Abstract: The purpose of our paper is to obtain a common fixed point theorem for two pairs of self-mappings of compatible of type (K) in a complete intuitionistic fuzzy Metric space with example. Our result generalized and improves similar other results in literature.

Keywords: Compatible mappings, Compatible mappings of type (K) and common fixed point.

1. Introduction

The fixed point theory is an important area of non-linear functional analysis. The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research during the last three decades. In 1965, the concept of fuzzy set was introduced by Zade [17]. Then, fuzzy metric spaces have been introduced by Kramosil and Michalek [6]. George and Veeramani [3] modified the notion of fuzzy metric spaces with the help of continuous t-norms. In 1998, Y.J. Cho, H.K. Pathak, S.M. Kang and J.S. Jung [2] introduced the concept of compatible mappings in fuzzy metric space. Then, in 2004, intuitionistic fuzzy metric spaces have been introduced by J.H. Park [11] with the help of continuous t-norm and continuous t-conorm as a generalization of fuzzy metric space.

Recently, K. Jha, V. Popa and K. B. Manandhar [4] introduced the concept of compatible mappings of type (K) in metric space. K. B. Manandhar et al. [8,7] extended compatible mappings of type (E) and compatible mappings of type (K) in fuzzy metric space.

The purpose of this paper is to prove a common fixed point theorem involving two pairs of compatible mappings of type (K) in Intuitionistic Fuzzy Metric space with example.

2. Preliminaries

Definition 1. [13] A binary operation \(\ast : [0,1] \times [0,1] \rightarrow [0,1]\) is a continuous t-norm if \(\ast\) is satisfying the following conditions:

(a) \(\ast\) is commutative and associative;
(b) \(\ast\) is continuous;
(c) \(a \ast 1 = a\) for all \(a \in [0,1]\);
(d) \(ab \leq c \ast d\) whenever \(a \leq c\) and \(b \leq d\), and \(a,b,c,d \in [0,1]\).

Definition 2. [13] A binary operation \(\hat{\circ} : [0,1] \times [0,1] \rightarrow [0,1]\) is a continuous t-conorm, if it satisfies the following conditions:

(1) \(\hat{\circ}\) is commutative and associative;
(2) \(\hat{\circ}\) is continuous;
(3) \(a \hat{\circ} 0 = a\) for all \(a \in [0,1]\);
(4) \(a \hat{\circ} b \leq c \hat{\circ} d\) whenever \(a \geq c\) and \(b \geq d\), for each \(a,b,c,d \in [0,1]\).

Definition 3. [1] A 5-tuple \((X, M, N, \ast, \hat{\circ})\) is said to be an intuitionistic fuzzy metric space (shortly IFM-Space) if \(X\) is an arbitrary set, \(\ast\) is a continuous t-norm, \(\hat{\circ}\) is a continuous t-conorm and \(M, N\) are...
fuzzy sets on \( X^2 \times (0, \infty) \) satisfying the following conditions: for all \( x, y, z \in X \) and \( s, t > 0 \):

(IFM-1) \( M(x, y, t) + N(x, y, t) \leq 1 \);

(IFM-2) \( M(x, y, 0) = 0 \);

(IFM-3) \( M(x, y, t) = 1 \) if and only if \( x = y \);

(IFM-4) \( M(x, y, t) = M(y, x, t) \);

(IFM-5) \( M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s) \);

(IFM-6) \( M(x, y, t) : [0, \infty) \to [0,1] \) is left continuous;

(IFM-7) \( \lim_{t \to \infty} M(x, y, t) = 1 \)

(IFM-8) \( N(x, y, 0) = 1 \);

(IFM-9) \( N(x, y, t) = 0 \) if and only if \( x = y \);

(IFM-10) \( N(x, y, t) = N(y, x, t) \);

(IFM-11) \( N(x, y, t) \ast N(y, z, s) \geq N(x, z, t + s) \);

(IFM-12) \( N(x, y, t) : [0, \infty) \to [0,1] \) is right continuous;

(IFM-13) \( \lim_{t \to \infty} N(x, y, t) = 0 \)

Then \( (M, N) \) is called an intuitionistic fuzzy metric on \( X \). The functions \( M(x, y, t) \) and \( N(x, y, t) \) denote the degree of nearness and degree of non-nearness between \( x \) and \( y \) with respect to \( t \), respectively.

**Remark** [16] Every fuzzy metric space \((X, M, \ast)\) is an intuitionistic fuzzy metric space if \( X \) of the form \((X, M, 1 - M, \ast, \hat{\ast})\) such that \( t \)-norm \( \ast \) and \( t \)-conorm \( \hat{\ast} \) are associated, that is, \( x \hat{\ast} y = 1 - ((1 - x) \ast (1 - y)) \) for any \( x, y \in X \). But the converse is not true.

**Example 1.** [6] Let \((X, d)\) be a metric space. Denote \( a \ast b = ab \) and \( a \hat{\ast} b = \min \{1, a + b\} \) for all \( a, b \in [0, 1] \) and let \( M_d \) and \( N_d \) be fuzzy sets on \( X^2 \times (0, \infty) \) defined as follows:

\[
M_d(x, y, t) = \frac{t}{t + d(x, y)},
\]

\[
N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.
\]

Then \((M_d, N_d)\) is an intuitionistic fuzzy metric on \( X \). We call this intuitionistic fuzzy metric induced by a metric \( d \) the standard intuitionistic fuzzy metric.

**Remark** Note the above example holds even with the \( t \)-norm \( a \ast b = \min \{a, b\} \) and the \( t \)-conorm \( a \hat{\ast} b = \max \{a, b\} \) and hence \((M_d, N_d)\) is an intuitionistic fuzzy metric with respect to any continuous \( t \)-norm and continuous \( t \)-conorm.

**Definition 4.** [1] Let \((X, M, N, \ast, \hat{\ast})\) be an intuitionistic fuzzy metric space.

(a) A sequence \([x_n]\) in \( X \) is called cauchy sequence if for each \( t > 0 \) and \( b > 0 \),

\[
\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1
\]

and

\[
\lim_{n \to \infty} N(x_{n+p}, x_n, t) = 0.
\]

(b) A sequence \([x_n]\) in \( X \) is convergent to \( x \in X \) if

\[
\lim_{n \to \infty} M(x_n, x, t) = 1
\]

and

\[
\lim_{n \to \infty} N(x_n, x, t) = 0 \quad \text{for each} \quad t > 0.
\]

(c) An intuitionistic fuzzy metric space is said to be complete if every Cauchy sequence is convergent.

**Definition 5.** [4] The self mappings \( A \) and \( S \) of a fuzzy metric space \((X, M, \ast)\) are said to be compatible of type (K) iff

\[
\lim_{n \to \infty} M(AA x_n, Sx_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} M(SS x_n, Ax_n, t) = 1,
\]

whenever \([x_n]\) is a sequence in \( X \) such that

\[
\lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = x \quad \text{for some} \ x \in X \quad \text{and} \ t > 0.
\]

**Lemma 1.** [11] Let \((X, M, \ast)\) be a fuzzy metric space. Then for all \( x, y \in X \), \( M(x, y, \cdot) \) is non-decreasing.

**Lemma 2.** [15] Let \((X, M, N, \ast, \hat{\ast})\) be an intuitionistic fuzzy metric space. If there exists a constant \( k \in (0, 1) \) such that,

\[
M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)
\]

for every \( t > 0 \) and \( n = 1, 2, ... \), then \([y_n]\) is a Cauchy sequence in \( X \).

**Lemma 3.** [14] Let \((X, M, \ast)\) be a fuzzy metric
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space with the condition: \((FM6)\)
\[ \lim_{n \to \infty} M(x, y, t) = 1 \quad \text{for all } x, y \in X. \]
If there exists \(k \in (0, 1)\) such that \(M(x, y, kt) \geq M(x, y, t)\) then \(x = y.\)

**Proposition 1.** [9] If \(A\) and \(S\) be compatible mappings of type \((K)\) on a intuitionistic fuzzy metric space \((X, M, N, *, \diamond)\) and if one of function is continuous. Then, we have

1. \(A(x) = S(x)\) where \(\lim_{n \to \infty} A x_n = x, \lim_{n \to \infty} S x_n = x,\) for some point \(x \in X\), and sequence \(\{x_n\}\),

2. If these exist \(u \in X\) such that \(A u = S u = x\) then, \(A S u = S A u.\)

**3. Main Results**

**Theorem 1.** Let \(A, B, S\) and \(T\) be self maps of a complete intuitionistic fuzzy metric spaces \((X, M, N, *, \diamond)\) with continuous \(t\)-norm \(*\) and continuous \(t\)-conorm \(\diamond\) defined by \(a * a \geq a\) and \(a \diamond a \leq a\) for all \(a \in [0, 1]\) satisfying the following condition:

(i) \(A(X) \subset T(X)\) and \(B(X) \subset S(X),\)

(ii) the pairs \((A, S)\) and \((B, T)\) are compatible mappings of type \((K),\)

(iii) If \(A, S\) and one of the mapping of pair \((B, T)\) is continuous.

(iv) there exist \(k \in (0, 1)\) such that \(M(A x, B y, k t) \geq M(S x, T y, t) * M(A x, S x, t) * M(B y, T y, t) * M(S x, B x, t)\)

\[ \hat{N}(A x, B y, t) \leq \hat{N}(S x, T y, t) \hat{N}(A x, S x, t) \hat{N}(B y, T y, t) \hat{N}(S x, B x, t) \]

for every \(x, y \in X\) and \(t > 0.\)

Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X.\)

**Proof:** As \(A(X) \subset T(X),\) for any \(x_0 \in X,\) there exists a point \(x_1 \in X\) such that \(A x_0 = T x_1.\) Since \(B(X) \subset S(X),\) for this point \(x_1,\) we can choose a point \(x_2 \in X\) such that \(B x_1 = S x_2.\) Inductively, we can find a sequence \(\{y_n\}\) in \(X\) as follows:

\[ y_{2n-1} = T x_{2n-1} = A x_{2n-2} \]

\[ y_{2n} = S x_{2n} = B x_{2n-1} \]

for \(n = 1, 2, \ldots.\) This can be done by (i). By using contractive condition, we obtain

\[ M(y_{2n+1}, y_{2n+2}, k t) = M(A x_{2n}, B x_{2n+1}, k t) \]

\[ \geq M(S x_{2n}, T x_{2n+1}, t) * M(A x_{2n}, S x_{2n}, t) * M(B x_{2n+1}, T x_{2n+1}, t) * M(A x_{2n}, S x_{2n}, t) \]

\[ = M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t) * M(y_{2n+1}, y_{2n}, t) * M(y_{2n+1}, y_{2n}, t) \]

\[ = M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) * M(y_{2n}, y_{2n+1}, t) \]

\[ = M(y_{2n}, y_{2n+1}, t) * 1 * M(y_{2n+1}, y_{2n}, t) * 1 \]

\[ = M(y_{2n}, y_{2n+1}, t), \text{That is} \]

\[ M(y_{2n+1}, y_{2n+2}, k t) \geq M(y_{2n}, y_{2n+1}, t), \]

Similarly, we have

\[ M(y_{2n}, y_{2n+1}, k t) \geq M(y_{2n-1}, y_{2n}, t), \]

So, we get

\[ M(y_{n+2}, y_{n+1}, k t) \geq M(y_{n+1}, y_{n}, t), \]

Also, we get
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\[ N(y_{2n+1}, y_{2n+2}, kt) = N(Ax_{2n}, Bx_{2n-1}, kt) \]
\[ \leq N(Sx_{2n}, Tx_{2n-1}, t) \odot N(Ax_{2n}, Sx_{2n}, t) \odot N(Bx_{2n-1}, Tx_{2n-1}, t) \]
\[ N(Ax_{2n}, Bx_{2n-1}, t) \odot N(Sx_{2n}, Bx_{2n-1}, t) \]
\[ \leq N(y_{2n}, y_{2n+1}, t) \odot N(y_{2n}, y_{2n}, t) \odot N(y_{2n}, y_{2n+1}, t) \]
\[ N(y_{2n+1}, y_{2n+2}, t) \odot N(y_{2n}, y_{2n+1}, t) \odot N(y_{2n}, y_{2n+1}, t) \]
\[ = N(y_{2n}, y_{2n+1}, t), \text{so,} \]
\[ N(y_{2n+1}, y_{2n+2}, kt) \leq N(y_{2n}, y_{2n+1}, t). \]

Similarly, we have
\[ N(y_{2n}, y_{2n+1}, kt) \leq (y_{2n-1}, y_{2n}, t), \]
So, we get
\[ N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_{n}, t). \] (2)

From (1), (2) and Lemma (2), we get that \( \{y_n\} \) is a Cauchy sequence in \( X \).

Since \( X \) is complete, therefore sequence \( \{y_n\} \) in \( X \) converges to \( z \) for some \( z \) in \( X \) and so the sequences
\[ \{Tx_{2n-1}\}, \{Ax_{2n-2}\}, \{Sx_{2n}\} \] and \( \{Bx_{2n-1}\} \) also converges to \( z \). Since pair (\( A, S \)) is compatible mappings of type (K),
\[ Az = Sz. \] (3)

From (2), we have \( ASx_{2n} \to Az \) and from (3) \( ASx_{2n} \to Sz \). Also, from continuity of \( S \), we have, \( Sx_{2n} \to Sz \).

From (iv), we get
\[ M(ASx_{2n}, Bx_{2n-1}, kt) \geq M(SSx_{2n}, Tx_{2n-1}, t) \ast M(ASx_{2n}, SSx_{2n}, t) \]
\[ \ast M(Bx_{2n-1}, Tx_{2n-1}, t) \ast M(ASx_{2n}, Tx_{2n-1}, t) \text{and} \]
\[ N(ASx_{2n}, Bx_{2n-1}, kt) \leq N(SSx_{2n}, Tx_{2n-1}, t) \odot N(ASx_{2n}, SSx_{2n}, t) \]
\[ \odot N(Bx_{2n-1}, Tx_{2n-1}, t) \odot N(ASx_{2n}, Tx_{2n-1}, t) \]

Proceeding limit as \( n \to \infty \), we have
\[ M(Sz, z, kt) \geq M(Sz, Sz, t) \ast M(z, z, t) \ast = z, \text{ and hence from (3)} \]
\[ Az = Sz = z. \] (4)

Since (\( B, T \)) is compatible mappings of type (K)and one of the mapping is continuous,we get,
\[ Tz = Bz. \] (5)

From (iv),
\[ M(Ax_{2n}, Bz, kt) \geq M(Sx_{2n}, Tz, t) \ast M(Ax_{2n}, Sx_{2n}, t) \ast M(Bz, Tz, t) \]
\[ \ast M(Ax_{2n}, Tz, t) \text{ and} \]
\[ N(Ax_{2n}, Bz, kt) \leq N(Sx_{2n}, Tz, t) \odot N(Ax_{2n}, Sx_{2n}, t) \odot N(Bz, Tz, t) \]
\[ \odot N(Ax_{2n}, Tz, t), \]

Letting \( n \to \infty \), we have
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\[ M(z, Bz, k) \geq M(z, Tz, t) \cdot M(z, z, t) \cdot M(Bz, Tz, t) \cdot M(z, Tz, t) \]
\[ = M(z, Bz, t) \cdot M(z, z, t) \cdot M(Bz, Bz, t) \cdot M(z, Bz, t) \]
\[ \geq M(z, Bz, t) \quad \text{and} \]
\[ N(z, Bz, k) \leq N(z, Tz, t) \cdot N(Bz, Tz, t) \cdot N(z, Tz, t) \]
\[ = N(z, Bz, t) \cdot N(z, z, t) \cdot N(Bz, Bz, t) \cdot N(z, Bz, t) \]
\[ \leq N(z, Bz, t), \]

which implies that \( Bz = z \). From (4) and (5). Therefore, \( Az = Sz = Bz = Tz = z \). Hence \( A, B, S \) and \( T \) have common fixed point \( z \) in \( X \).

For uniqueness, let \( w \) be another common fixed point of \( A, B, S \) and \( T \). Then
\[
M(z, w, k) = M(Az, Bw, qt)
\]
\[
\geq M(Sz, Tw, t) \cdot M(Az, Sz, t) \cdot M(Bw, Tw, t) \cdot M(Az, Tw, t)
\]
\[
\geq M(z, w, t)
\]
\[
N(z, w, k) = N(Az, Bw, qt)
\]
\[
\leq N(Sz, Tw, t) \cdot N(Az, Sz, t) \cdot N(Bw, Tw, t)
\]
\[
\diamond N(Az, Tw, t)
\]
\[
\leq N(z, w, t).
\]

From Lemma (3), we conclude that \( z = w \). Hence \( A, B, S \) and \( T \) have unique common fixed point \( z \) in \( X \).

**Example 2.** Let \( (X, M, N, * , \hat{\diamond} ) \) be an intuitionistic fuzzy metric, where \( X = [2, 20] \)
\[
M(x, y, t) = \frac{1}{t+d(x,y)} \cdot N(x, y, t) = \frac{d(x,y)}{t+d(x,y)}
\]
and \( d \) is the Euclidean metric on \( X \).

Define \( A, B, S \) and \( T : X \rightarrow X \) as follows;
\[
Ax = 2 \quad \text{for all } x;
\]
\[
Bx = 2 \quad \text{if } x < 4 \quad \text{and} \quad \geq 5 \quad Bx = 3 + x \quad \text{if } 4 \leq x < 5;
\]
\[
Sx = \begin{cases} x & \text{if } x \leq 8, \ x = 8 \quad \text{if } \ x > 8; \\
5 + x & \text{if } 4 \leq x < 5. \end{cases}
\]
\[
Tx = 2 \quad \text{if } x < 4 \quad \text{or} \quad \geq 5 \quad Tx = 5 + x \quad \text{if } 4 \leq x < 5.
\]

Then \( A, B, S \) and \( T \) satisfy all the conditions of the above theorem and have a unique common fixed point \( x = 2 \).

**Corollary 1.** Let \( A, B, S \) and \( T \) be self maps of a complete intuitionistic fuzzy metric spaces \( (X, M, N, * , \hat{\diamond} ) \) with continuous \( t \)-norm * and continuous \( t \)-conorm \( \hat{\diamond} \) defined by \( a \ast a \geq a \) and \( a \hat{\diamond} a \leq a \) for all \( a \in [0,1] \) satisfying the following condition:

(1) \( A(X) \subset T(X) \) and \( B(X) \subset S(X) \),

(2) the pairs \( (A, S) \) and \( (B, T) \) are compatible mappings of type \( (K) \),

(3) If \( A, S \) and one of the mapping of pair \( (B, T) \) is continuous.

(4) there exist \( k \in (0,1) \) such that
\[
M(Ax, By, kt) \geq M(Sx, Ty, t) \cdot M(Ax, Sx, t) \cdot M(By, Ty, t) \cdot M(Ax, Ty, t)
\]
\[
M(Sx, By, 2t) \ast M(By, Ty, t) \ast M(Ax, Ty, t)
\]
\[
\hat{\diamond} N(Ax, Sx, t) \quad \hat{\diamond} N(By, Ty, t) \quad \hat{\diamond} N(Ax, Ty, t),
\]
for every \( x, y \in X \) and \( t > 0 \).

Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**Corollary 2.** Let \( A, B, S \) and \( T \) be self maps of a complete intuitionistic fuzzy metric spaces \( (X, M, N, * , \hat{\diamond} ) \) with continuous \( t \)-norm * and continuous \( t \)-conorm \( \hat{\diamond} \) defined by \( a \ast a \geq a \) and \( a \hat{\diamond} a \leq a \) for all \( a \in [0,1] \) satisfying the following condition:

(1) \( A(X) \subset T(X) \) and \( B(X) \subset S(X) \),

(2) the pairs \( (A, S) \) and \( (B, T) \) are compatible mappings of type \( (K) \),

(3) If \( A, S \) and one of the mapping of pair \( (B, T) \) is continuous.

(4) there exist \( k \in (0,1) \) such that
\[
M(Ax, By, kt) \geq M(Sx, Ty, t)
\]
and
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\[ N(Ax, By, t) \leq N(Sx, Ty, t) \]

for every \( x, y \) in \( X \) and \( t > 0 \).

Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

Remarks: Our result is also true for the pair \((A,S)\) and \((B,T)\) are compatible mappings of type \((E)\) in place of compatible mappings of type \((K)\). Our result extends and generalizes the results of S. Manro et al. [10], C. Alaca et al. [1], K.B. Manandhar et. al [9,7] and similar other results of fixed point in the literature.

References