Solving Linear Fredholm-Stieltjes Integral Equations of the Second Kind by Using the Generalized Midpoint Rule

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Abstract: In this paper, the approximate solution to the linear fredholm-stieltjes integral equations of the second kind (LFSIESK) by using the generalized midpoint rule (GMR) is introduced. A comparison results depending on the number of subintervals “n” are calculated by using Maple 18 and presented. These results are demonstrated graphically in a particular numerical example. An algorithm of this application is given by using Maple 18.

Keywords: Approximate solutions, linear fredholm-stieltjes integral equations, midpoint rule.

1. Introduction

An integral equations theory is one of the most active research fields which is based on analysis, functions theory and functional analysis [1]. Moreover, since the partial differential equations can be transformed into integral equations, this also makes integral equations more interesting [1]. Sometimes it may be difficult to solve integral equations analytically, so approximation methods, which has expanded very rapidly during this computer era [2], are used to solve the integral equations. There are many approximation methods to solve linear Fredholm integral equations of the second kind: projection, Nyström [3], Midpoint [4] methods etc. However, there are very few of them that solve LFSIESK: generalized trapezoid rule [5], [6].

2. Solving Lfsiesk by Using the GMR

2.1 Given Lfsiesk

$$u(x) = \int_a^b K(x, y)u(y)dg(y) + f(x), \quad (1)$$

$$x \in [a, b]$$

where $K(x, y) \in C[a, b]$, $g(y)$ is the continuous function on the closed interval $[a, b]$ which can be written as a difference of two strictly increasing functions $\varphi(y), \psi(y)$ on the closed interval $[a, b]$, $f(x)$ given function and $u(x)$ is the unknown function to be determined. Now, instead of $g(y)$ in (1), if the difference $\varphi(y) - \psi(y)$ is substituted, then it becomes of the form

$$u(x) = \int_a^b K(x, y)u(y)dg(y) + f(x) =$$

$$\underbrace{\int_a^b K(x, y)u(y)d\varphi(y)}_I - \underbrace{\int_a^b K(x, y)u(y)d\psi(y)}_{II} + f(x) \quad (2)$$

Here the integrals in (2) can be calculated separately by using the GMR [7],

$$I = \int_a^b K(x, y)u(y)d\varphi(y) \approx$$

$$\sum_{i=1}^n K(x, x_{2i-1}^*)u(x_{2i-1}^*)[\varphi(x_{2i}) - \varphi(x_{2i-2})]$$

where

$$x_{2i-1}^* = \varphi^{-1} \left[\frac{\varphi(x_{2i}) + \varphi(x_{2i-2})}{2} \right]$$

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and

$$II = \int_a^b K(x, y) u(y) d\psi(y) \approx \sum_{i=1}^n K(x, x_{2i-1}^{**}) u(x_{2i-1}^{**}) [\psi(x_{2i}) - \psi(x_{2i-2})]$$

$$\text{where } x_{2i-1}^{**} = \psi^{-1} \left[\frac{\psi(x_{2i}) + \psi(x_{2i-2})}{2} \right]$$

Therefore, the integral equation (2) becomes

$$\bar{u}(x) = \sum_{i=1}^n \begin{pmatrix} K(x, x_{2i-1}^*) \bar{u}(x_{2i-1}^*) \times \\ [\varphi(x_{2i}) - \varphi(x_{2i-2})] - \\ K(x, x_{2i-1}^{**}) \bar{u}(x_{2i-1}^{**}) \times \\ [\psi(x_{2i}) - \psi(x_{2i-2})] \end{pmatrix} + f(x) \quad (3)$$

where $\bar{u}(x)$ is the approximate solution of (2).

Now, in (3), if we use the following substitution:

$$A_i(x) = K(x, x_{2i-1}^*) [\varphi(x_{2i}) - \varphi(x_{2i-2})], \\ B_i(x) = -K(x, x_{2i-1}^{**}) [\psi(x_{2i}) - \psi(x_{2i-2})],$$

for $i = 1, 2, 3, \dots, n$,

then the equation (3) can be written as

$$\bar{u}(x) = \sum_{i=1}^n \begin{pmatrix} A_i(x) \bar{u}(x_{2i-1}^*) + \\ B_i(x) \bar{u}(x_{2i-1}^{**}) \end{pmatrix} + f(x) \quad (4)$$

If x_{2i-1}^* and x_{2i-1}^{**} for $i = 1, 2, 3, \dots, n$ are substituted into (4), then we get a following system of linear equations

$$\bar{u}(x_{2j-1}^*) = \sum_{i=1}^n \begin{pmatrix} A_i(x_{2j-1}^*) \bar{u}(x_{2i-1}^*) + \\ B_i(x_{2j-1}^*) \bar{u}(x_{2i-1}^{**}) \end{pmatrix} + f(x_{2j-1}^*) \\ \bar{u}(x_{2j-1}^{**}) = \sum_{i=1}^n \begin{pmatrix} A_i(x_{2j-1}^{**}) \bar{u}(x_{2i-1}^*) + \\ B_i(x_{2j-1}^{**}) \bar{u}(x_{2i-1}^{**}) \end{pmatrix} + f(x_{2j-1}^{**})$$

$$\text{For } j = 1, 2, \dots, n \quad (5)$$

If the system of linear equations (5) is converted into matrix form, then

$$\begin{pmatrix} I - \underbrace{\begin{pmatrix} A_1(x_{2j-1}^*) & \cdots & B_1(x_{2j-1}^*) \\ \vdots & \ddots & \vdots \\ A_n(x_{2j-1}^{**}) & \cdots & B_n(x_{2j-1}^{**}) \end{pmatrix}}_A \end{pmatrix} \underbrace{\begin{pmatrix} \bar{u}(x_{2j-1}^*) \\ \vdots \\ \bar{u}(x_{2j-1}^{**}) \end{pmatrix}}_U = \underbrace{\begin{pmatrix} f(x_{2j-1}^*) \\ \vdots \\ f(x_{2j-1}^{**}) \end{pmatrix}}_F$$

$$\text{for } i, j = 1, 2, \dots, n \quad (6)$$

Now, the system (6) $(I - A) \cdot U = F$ has a unique solution $U = (I - A)^{-1} \cdot F$ if and only if $\det(I - A) \neq 0$.

Now, let $\varphi(x) \in C^\alpha[a, b]$, $\psi(x) \in C^\beta[a, b]$ where $0 < \alpha \leq 1$, $0 < \beta \leq 1$.

Then in the

$$A_i(x) = K(x, x_{2i-1}^*) [\varphi(x_{2i}) - \varphi(x_{2i-2})] \quad \text{and} \\ B_i(x) = -K(x, x_{2i-1}^{**}) [\psi(x_{2i}) - \psi(x_{2i-2})], \text{ the terms} \\ [\varphi(x_{2i}) - \varphi(x_{2i-2})], [\psi(x_{2i}) - \psi(x_{2i-2})]$$

approaches 0 as $x_{2i} - x_{2i-2}$ approaches 0, at least as fast as $|x_{2i} - x_{2i-2}|^\alpha$ approaches 0.

Therefore, $A_i(x)$ and $B_i(x)$ approaches 0 as the number of subintervals "n" increases for all $i = 1, 2, 3, \dots, n$. [8], [9].

So, the coefficient matrix will be of the form

$$A \approx \begin{pmatrix} \approx 0 & \cdots & \approx 0 \\ \vdots & \ddots & \vdots \\ \approx 0 & \cdots & \approx 0 \end{pmatrix} \text{ and we can conclude that}$$

$$\det(I - A) \neq 0.$$

Therefore, the system (6) has a unique solution, namely $U = (I - A)^{-1} \cdot F$.

Thus, if the solution of the system of linear equations (6) is substituted back into the (4), then the general solution is defined as

$$\begin{aligned} \bar{u}(x) = & \sum_{i=1}^n U_i \cdot A_i(x) + \\ & \sum_{i=1}^n U_{n+i} \cdot B_i(x) + f(x) \end{aligned} \quad (7)$$

Numerical example:

Let us consider the following LFSIESK

$$\begin{aligned} u(x) = & \int_0^1 (1+x^2s)u(s) d(\ln(1+\sqrt{s})) \\ & - \frac{x^2}{4} - \frac{x}{6} + x\sqrt{x} \end{aligned} \quad (8)$$

here

$$K(x, s) = 1 + x^2s, \varphi(x) = \ln(1 + \sqrt{x}) \quad \psi(x) = 0$$

and

$$f(x) = -\frac{x^2}{4} - \frac{x}{6} + x\sqrt{x}.$$

Let us take $n = 4$,

then

$$h = \frac{b-a}{2n} = \frac{1-0}{8} = 0.125$$

And

$$x_{2i} = a + 2ih = 0.125 \cdot 2i$$

for

$$i = 1, 2, 3, 4.$$

If it is calculated, then it can be obtained as

$$\begin{aligned} x_0 = 0, \quad x_2 = 0.25, \quad x_4 = 0.5, \\ x_6 = 0.75, \quad x_8 = 1.0 \end{aligned}$$

Then, if the GMR is used to integrate (8),

$$\begin{aligned} I = & \int_0^1 (1+x^2s)u(s) d(\ln(1+\sqrt{s})) \approx \\ & \sum_{i=1}^4 K(x, x_{2i-1}^*) u(x_{2i-1}^*) [\varphi(x_{2i}) - \varphi(x_{2i-2})] \end{aligned} \quad (9)$$

where

$$x_{2i-1}^* = \varphi^{-1} \left[\frac{\varphi(x_{2i}) + \varphi(x_{2i-2})}{2} \right]$$

if calculated, then it can be obtained as

$$\begin{aligned} x_1^* = 0.0505, \quad x_3^* = 0.3602, \\ x_5^* = 0.6159, \quad x_7^* = 0.8683 \end{aligned}$$

So the equation (8) becomes

$$\begin{aligned} \bar{u}(x) = & \sum_{i=1}^4 K(x, x_{2i-1}^*) u(x_{2i-1}^*) [\varphi(x_{2i}) - \varphi(x_{2i-2})] + f(x) \end{aligned} \quad (10)$$

Here

let's

assign

$$A_i(x) = K(x, x_{2i-1}^*) [\varphi(x_{2i}) - \varphi(x_{2i-2})] \quad \text{for}$$

$i = 1, 2, 3, 4$, then (10) becomes

$$\bar{u}(x) = \sum_{i=1}^4 A_i(x) u(x_{2i-1}^*) + f(x) \quad (11)$$

Then if the values $x_1^*, x_3^*, x_5^*, x_7^*$ are substituted into the equation (11), then the following system is obtained and solution is found by using Maple 18 as follows

$$\begin{pmatrix} 0.5945 & -0.1295 & -0.0892 & -0.0695 \\ -0.4081 & 0.8646 & -0.0961 & -0.0772 \\ -0.4132 & -0.1470 & 0.8902 & -0.0922 \\ -0.4209 & -0.1645 & -0.1303 & 0.8853 \end{pmatrix} \cdot \begin{pmatrix} u(x_1^*) \\ u(x_3^*) \\ u(x_5^*) \\ u(x_7^*) \end{pmatrix} = \begin{pmatrix} 0.0023 \\ 0.1237 \\ 0.2859 \\ 0.4759 \end{pmatrix} \Rightarrow \begin{pmatrix} u(x_1^*) \\ u(x_3^*) \\ u(x_5^*) \\ u(x_7^*) \end{pmatrix} = \begin{pmatrix} 0.2853 \\ 0.4207 \\ 0.6102 \\ 0.8413 \end{pmatrix} \quad (12)$$

Then, this solution is substituted back into (11) and simplified by Maple 18 to get

$$\begin{aligned} \bar{u}(x) = & 0.282737088933138 - \\ & 0.140452902234602 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x} \end{aligned} \quad (13)$$

which is pretty close to the exact solution

$$\begin{aligned} u(x) = & 0.3058111302 - \\ & 0.1289085929 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x} \end{aligned}$$

As the number of subintervals “n” increased, the accuracy in the approximate solution increases and the

Table 1 Comparison determinant of the coefficient matrix in (12) and the approximate solution, as n increases.

n	$\det(A)$	$\bar{u}(x)$	$u(x)$
4	0.2658	$0.2827 - 0.14045 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$	$0.3058 - 0.1289 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$
16	0.2625	$0.3041 - 0.1300 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$	$0.3058 - 0.1289 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$
64	0.2622	$0.3057 - 0.1290 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$	$0.3058 - 0.1289 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$
256	0.2622	$0.3058 - 0.1289 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$	$0.3058 - 0.1289 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$

Fig. 1 Comparison graphs of the approximate solutions as n increases.

error decreases. The following table 1 shows how the approximate solution approaches the exact solution as the number of subintervals “ n ” increases.

In the following figure 1, we have Maple 18 plot the solutions of the table 1. It can be observed that as the number of subintervals “ n ” increases, the graph of the solutions are accumulating around the exact solution which is close enough to the

solution of $n = 256$.

4. Algorithm for solving Lfsiesk by Using the GMR in Maple 18

The following screenshot is the algorithm of the approximate solution of the written by using Maple 18. Any LFSIESK can be solved using this algorithm by changing its inputs in any accuracy.

```
restart
with(LinearAlgebra) :
with(plots) :
```

Inputs

```
K := (x, s) → 1 + x2 · s : f := x → - $\frac{x^2}{4}$  -  $\frac{x}{6}$  + x · sqrt(x) : p := x → ln(1 + sqrt(x)) : t := x → 0 :
n := 4 : a := 0.0 : b := 1.0 : lambda := 1 :
```

U(x)

```
Solver_General := proc(K, p, t, f, a, b, lambda := 1, n := 100) :: function;
    local invp := x → (solve(x = p(y), y)), h :=  $\frac{b-a}{2 \cdot n}$ , y, x :: list, Id, A, i, j, C, F, Pr, u, U :: list;

    for i from 0 by 2 to 2 · n do xi := a + i · h; end do; for i from 1 to n do x2 · i - 1 := invp( $\frac{p(x_{2 \cdot i}) + p(x_{2 \cdot i - 2})}{2}$ ); end do;
    Id := IdentityMatrix(n); A := Matrix(n);
    for i from 1 to n do for j from 1 to n do A(i, j) := lambda · K(x2 · i - 1, x2 · j - 1) · (p(x2 · j) - p(x2 · j - 2)); end do; end do;
    F := Matrix(n, 1); for i from 1 to n do F(i, 1) := f(x2 · i - 1); end do;
    Pr := Multiply(MatrixInverse(Id - A), F); for i from 1 to n do Ui := Pr(i, 1); end do;
    u := y → add(lambda · K(y, x2 · i - 1) · (p(x2 · i) - p(x2 · i - 2)) · U[i], i = 1 .. n) + f(y); u
end proc;
```

5. Conclusion

In this work, the generalized midpoint rule is applied to solve LFSIESK. In a numerical example, the approximate results are compared with respect to the number of subintervals “n” and plotted their graphs. It has been observed that, as the number of subintervals “n” is increased, a very good accuracy can be obtained.

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