The Degenerate Form of the Adomian Polynomials in the Power Series Method for Nonlinear Ordinary Differential Equations

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Abstract: In this paper, we propose a new variation of the Adomian polynomials, which we call the degenerate Adomian polynomials, for the power series solutions of nonlinear ordinary differential equations with nonseparable nonlinearities. We establish efficient algorithms for the degenerate Adomian polynomials. Next we compare the results by the Adomian decomposition method using the classic Adomian polynomials with the results by the Rach-Adomian-Meyers modified decomposition method incorporating the degenerate Adomian polynomials, which itself has been shown to be a confluence of the Adomian decomposition method and the power series method. Convergence acceleration techniques including the diagonal Padé approximants are considered, and new numeric algorithms for the multistage decomposition are deduced using the degenerate Adomian polynomials. Our new technique provides a significant advantage for automated calculations when computing the power series form of the solution for nonlinear ordinary differential equations. Several expository examples are investigated to demonstrate its reliability and efficiency.

Keywords: Power series method, Adomian decomposition method, Adomian polynomials, Modified decomposition method, Nonlinear differential equation

1. Introduction

Nonlinear differential equation models have been extensively applied in physics, chemistry, mechanics, hydrology, engineering, economics, biology, epidemiology, etc. For the solution of nonlinear differential equations, apart from various numeric methods such as the finite difference method, finite element method etc, several analytic approximation methods have been proposed, such as the perturbation method [1], Adomian decomposition method [2, 3], homotopy analysis method [4, 5], variational iteration method [6], homotopy perturbation method [7], wavelet method [8], spectral method [9] etc.

The Adomian decomposition method (ADM) [2, 3,

10-20] requires neither linearization nor perturbation, efficiently works for a large class of initial value or boundary value problems, encompassing linear, nonlinear and even stochastic systems. This method decomposes the solution $u(x)$ into a series of components to be determined by recursion, and then decomposes the analytic nonlinearity $Nu(x)$ into the series of the Adomian polynomials [2, 3, 10, 21, 22].

In this work, we investigate the advantages of the Adomian polynomials for nonlinearities which explicitly include the independent variable such as $Nu(x) = f(x, u(x))$ in the ADM and power series method. The considered nonlinearities are designated nonseparable in the sense that they cannot be factored as a product of an analytic function of the independent variables and an analytic function of the dependent variables, e.g. $f_1(x)$ and $f_2(u(x))$, respectively.

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We present a comparison using general deduction and several expository examples. Also, we consider the issues of convergence acceleration and numeric algorithms for solution of nonlinear differential equations.

2. The ADM and the Adomian Polynomials

We shall illustrate the ADM with a nonlinear second-order ordinary differential equation (ODE) written in the form of Adomian's operator-theoretic notation as

$$Lu + Ru + Nu = g, \quad (1)$$

where g is the system input and u is the system output, and where L is the highest order differential operator, in this case $L(\cdot) = \frac{d^2}{dx^2}(\cdot)$, R is the linear remainder operator, and N is the nonlinear operator, which is assumed to be analytic.

When solving second-order initial value problems (IVPs), for example, we take the inverse linear operator as the two-fold definite integral

$$L^{-1}(\cdot) = \int_a^x \int_a^x (\cdot) dx dx, \text{ where } a \text{ is the initial point.}$$

We then apply the inverse linear operator L^{-1} to both sides of Eq. (1) to obtain

$$u = \Phi + L^{-1}g - L^{-1}(Ru + Nu), \quad (2)$$

where Φ incorporates the initial conditions and identically satisfies $L\Phi = 0$.

In the ADM, the solution $u(x)$ is represented by the Adomian decomposition series and the nonlinearity $Nu(x)$ is represented by the series of the Adomian polynomials

$$u = \sum_{n=0}^{\infty} u_n \text{ and } Nu = \sum_{n=0}^{\infty} A_n, \quad (3)$$

respectively, where the Adomian polynomials are dependent upon the solution components from u_0 through u_n , inclusively, and are defined for the simple nonlinearity $Nu = f(u)$ as [3]

$$A_n = A_n(u_0, u_1, \dots, u_n) = \left. \frac{1}{n!} \frac{d^n}{d\lambda^n} f\left(\sum_{k=0}^n \lambda^k u_k(x)\right) \right|_{\lambda=0}, \quad (4)$$

where λ is a grouping parameter of convenience.

For convenient reference, we list the first five Adomian polynomials for the general analytic nonlinearity $Nu(x) = f(u(x))$ as

$$A_0 = f(u_0),$$

$$A_1 = f'(u_0)u_1,$$

$$A_2 = f'(u_0)u_2 + f''(u_0)\frac{u_1^2}{2!},$$

$$A_3 = f'(u_0)u_3 + f''(u_0)u_1u_2 + f'''(u_0)\frac{u_1^3}{3!},$$

$$A_4 = f'(u_0)u_4 + f''(u_0)\frac{u_2^2}{2!} + u_1u_3 +$$

$$f'''(u_0)\frac{u_1^2u_2}{2!} + f^{(4)}(u_0)\frac{u_1^4}{4!}.$$

We observe that

$$A_0 = f(u_0) \text{ and } A_n = \sum_{k=1}^n f^{(k)}(u_0)C_n^k, n \geq 1, \quad (5)$$

where the C_n^k were originally given by Rach's Rule (Page 16 in [11] and Page 51 in [13]): the C_n^k are the sums of all possible products of k components from $u_1, u_2, \dots, u_{n-k+1}$, whose subscripts sum to n , divided by the factorial of the number of repeated subscripts [23]. Equivalently, the Adomian polynomials can be expressed as [24]

$$A_n = \sum_{\sum_{j=1}^n jp_j = n} f^{(p_*)}(u_0) \prod_{j=1}^n \frac{u_j^{p_j}}{p_j!}, \quad (6)$$

where $p_* = \sum_{j=1}^n p_j$,

Several convenient algorithms to readily generate the Adomian polynomials have since been developed by Adomian and Rach [3], Rach [21, 23], Wazwaz [14, 25], Abdelwahid [26], and others [27, 30]. Recently,

Duan [22, 24, 31, 32] has developed several new algorithms and subroutines for fast generation of the one-variable and multivariable Adomian polynomials.

For example, Duan's Corollary 3 algorithm [24] can conveniently generate the coefficients C_n^k and hence the Adomian polynomials quickly and to high orders as

$$C_n^1 = u_n, \text{ for } n \geq 1, \text{ and} \\ C_n^k = \frac{1}{n} \sum_{j=0}^{n-k} (j+1) u_{j+1} C_{n-1-j}^{k-1}, \text{ for } 2 \leq k \leq n. \quad (7)$$

This recursion formula does not involve the differentiation operator, but only requires the analytic operations of addition and multiplication, which is eminently convenient for symbolic implementation by MATHEMATICA, MAPLE or MATLAB as well as for debugging. We list the MATHEMATICA code for this efficient algorithm in the Appendix for convenient reference.

Substituting Eq. (3) into Eq. (2), we obtain the classic Adomian recursion scheme for the solution components

$$u_0 = \Phi + L^{-1}g, \quad (8)$$

$$u_{n+1} = -L^{-1}(Ru_n + A_n), n \geq 0, \quad (9)$$

where the $(n+1)$ -term approximation is $\phi_{n+1}(x) = \sum_{k=0}^n u_k$. For various modified recursion schemes in the ADM, see [14, 22, 33-38].

In the definition of the Adomian polynomials, only the dependent variables are parametrized while the independent variables such as x are not. Thus the Adomian polynomials for the nonlinearity $Nu(x) = f(x, u(x))$ have the same form as those for the nonlinearity $Nu(x) = f(u(x))$ in the ADM.

Multivariable Adomian polynomials

The ADM for a system of coupled nonlinear differential equations, or a multi-order differential nonlinearity in a single nonlinear differential equation, involves the multivariable Adomian polynomials.

Suppose there is a nonlinearity $f(u_1, \dots, u_m)$, where the u_i 's are the dependent variables. Let $u_i, i=1, 2, \dots, m$, and the nonlinear function $f(u_1, \dots, u_m)$ be decomposed as

$$u_i = \sum_{j=0}^{\infty} u_{i,j}, i=1, 2, \dots, m, \quad (10)$$

and

$$f(u_1, \dots, u_m) = \sum_{n=0}^{\infty} A_n, \quad (11)$$

respectively, where the multivariable Adomian polynomials A_n depend on the solution components

$$u_{1,0}, u_{1,1}, \dots, u_{1,n}; u_{2,0}, \dots, u_{2,n}; \dots; u_{m,0}, \dots, u_{m,n},$$

and are defined by the parametrization

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} f \left(\sum_{j=0}^{\infty} u_{1,j} \lambda^j, \dots, \sum_{j=0}^{\infty} u_{m,j} \lambda^j \right) \Big|_{\lambda=0}. \quad (12)$$

The first Adomian polynomial A_0 is

$$A_0 = f(\mathbf{u}_0), \text{ where } \mathbf{u}_0 = (u_{1,0}, \dots, u_{m,0}).$$

Another expression for the multivariable Adomian polynomials [24, 31] is

$$A_n = \sum_{j=1}^n \sum_{i=1}^m \sum_{p_{ij}=n} f^{(p_{1*}, \dots, p_{m*})}(\mathbf{u}_0) \prod_{i=1}^m \prod_{j=1}^n \frac{u_{i,j}^{p_{ij}}}{p_{ij}!}, \quad (13)$$

where $p_{i*} = \sum_{j=1}^n p_{ij}$, and

$$f^{(p_{1*}, \dots, p_{m*})}(\mathbf{u}_0) = \frac{\partial^{p_{1*} + \dots + p_{m*}} f(\mathbf{u}_0)}{\partial u_{1,0}^{p_{1*}} \dots \partial u_{m,0}^{p_{m*}}},$$

denotes the mixed partial derivatives.

Several convenient algorithms for the multivariable Adomian polynomials were proposed in [21, 24, 27, 31]. Here we list the following convenient recursion formula for the m -variable Adomian polynomials A_n [24] as

$$A_n = \frac{1}{n} \sum_{i=1}^m \sum_{k=0}^{n-1} (k+1) u_{i,k+1} \frac{\partial}{\partial u_{i,0}} A_{n-1-k}, n \geq 1. \quad (14)$$

For MATHEMATICA codes which quickly generate the multivariable Adomian polynomials, see [24, 31].

We list the two-variable Adomian polynomials A_1 through A_4 for the abstract analytic function $f(u, v)$ with the decompositions $u = \sum_{n=0}^{\infty} u_n$ and $v = \sum_{n=0}^{\infty} v_n$. Except that, as space is limited, we omit the argument (u_0, v_0) in $f^{(p,q)}(u_0, v_0)$.

$$A_1 = u_1 f^{(1,0)} + v_1 f^{(0,1)},$$

$$A_2 = u_2 f^{(1,0)} + v_2 f^{(0,1)} + \frac{u_1^2}{2} f^{(2,0)} + \frac{v_1^2}{2} f^{(0,2)} + u_1 v_1 f^{(1,1)},$$

$$A_3 = u_3 f^{(1,0)} + v_3 f^{(0,1)} + u_1 u_2 f^{(2,0)} + v_1 v_2 f^{(0,2)} + (u_1 v_2 + u_2 v_1) f^{(1,1)} + \frac{u_1^3}{6} f^{(3,0)} + \frac{v_1^3}{6} f^{(0,3)} + \frac{u_1^2 v_1}{2} f^{(2,1)} + \frac{u_1 v_1^2}{2} f^{(1,2)},$$

$$A_4 = u_4 f^{(1,0)} + v_4 f^{(0,1)} + (u_1 u_3 + \frac{u_2^2}{2}) f^{(2,0)} + (v_1 v_3 + \frac{v_2^2}{2}) f^{(0,2)} + (u_1 v_3 + u_2 v_2 + u_3 v_1) f^{(1,1)} + \frac{u_1^2 u_2}{2} f^{(3,0)} + \frac{v_1^2 v_2}{2} f^{(0,3)} + (u_1 u_2 v_1 + \frac{u_1^2 v_2}{2}) f^{(2,1)} + (u_1 v_1 v_2 + \frac{u_2 v_1^2}{2}) f^{(1,2)} + \frac{u_1^4}{4!} f^{(4,0)} + \frac{u_1^3 v_1}{6} f^{(3,1)} + \frac{u_1^2 v_1^2}{4} f^{(2,2)} + \frac{u_1 v_1^3}{6} f^{(1,3)} + \frac{v_1^4}{4!} f^{(0,4)}.$$

Also, if the nonlinearities explicitly include the independent variables, such as $f(x, t, u_1, \dots, u_m)$, then the Adomian polynomials in the ADM are invariant.

3. The Power Series Method and the Degenerate Adomian Polynomials

The Rach-Adomian-Meyers modified decomposition method (MDM) [39] is both a decelerated ADM [40]

and a developed power series method, which enables us to easily and accurately treat any analytic nonlinearity. The Rach-Adomian-Meyers MDM decomposes the solution into a power series $u(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ and decomposes the nonlinearity into a power series according to the Adomian-Rach theorem [41,42],

$$f(\sum_{n=0}^{\infty} a_n (x - x_0)^n) = \sum_{n=0}^{\infty} A_n (x - x_0)^n, \quad (15)$$

where f is an analytic function and $A_n = A_n(a_0, a_1, \dots, a_n)$ are the Adomian polynomials in terms of the solution coefficients [13, 24, 39, 41, 42] instead of the usual solution components.

The multivariable version of the Adomian-Rach theorem is

$$f(\sum_{n=0}^{\infty} a_{1,n} (x - x_0)^n, \sum_{n=0}^{\infty} a_{2,n} (x - x_0)^n, \dots, \sum_{n=0}^{\infty} a_{m,n} (x - x_0)^n) = \sum_{n=0}^{\infty} A_n (x - x_0)^n, \quad (16)$$

where f is an m -ary analytic function and

$$A_n = A_n(a_{1,0}, a_{1,1}, \dots, a_{1,n}; a_{2,0}, a_{2,1}, \dots, a_{2,n}; \dots; a_{m,0}, a_{m,1}, \dots, a_{m,n})$$

are the m -variable Adomian polynomials [13, 24, 31, 39, 41-43] similarly in terms of the solution coefficients.

We observe that if the nonlinearities do not explicitly include the independent variables, then the Adomian polynomials in Eqs. (15) and (16) have the invariant form as compared with that in the ADM. In the sequel, we investigate those nonlinearities which explicitly depend on the independent variables such as $f(x, u)$.

We consider a nonlinear first-order differential equation

$$\frac{du}{dx} = f(x, u), u(x_0) = a_0, \quad (17)$$

to motivate our derivation and to underscore the substantial advantage that accrues from our stratagem of treating the independent variable as a degenerate case so as to further facilitate automated calculations

in the MDM. If $u(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, then the nonlinearity is accordingly decomposed as

$$\begin{aligned} f(x, u) &= f(x_0 + (x-x_0), \\ \sum_{n=0}^{\infty} a_n (x-x_0)^n &= \sum_{n=0}^{\infty} \mathcal{A}_n (x-x_0)^n, \end{aligned} \quad (18)$$

where the $\mathcal{A}_n$ can be calculated from the two-variable Adomian polynomials A_n as

$$\mathcal{A}_0 = A_0(x_0; a_0),$$

$$\mathcal{A}_1 = A_1(x_0, 1; a_0, a_1),$$

...

$$\mathcal{A}_n = A_n(x_0, 1, 0, \dots, 0; a_0, a_1, \dots, a_n), \quad n \geq 2,$$

where the sequence of sets $(x_0), (x_0, 1), (x_0, 1, 0), \dots, (x_0, 1, 0, \dots, 0)$ represent the coefficients of the respective trivial truncated series expansion of $x_0 + (x-x_0)$ in powers of $(x-x_0)$, that is equivalent to x , which we call the degenerate two-variable Adomian polynomials. Their explicit expressions are

$$\mathcal{A}_0 = f(x_0, a_0),$$

$$\mathcal{A}_1 = f^{(0,1)}(x_0, a_0) a_1 + f^{(1,0)}(x_0, a_0),$$

$$\begin{aligned} \mathcal{A}_2 &= \frac{1}{2} f^{(0,2)}(x_0, a_0) a_1^2 + f^{(1,1)}(x_0, a_0) a_1 \\ &\quad + f^{(0,1)}(x_0, a_0) a_2 + \frac{1}{2} f^{(2,0)}(x_0, a_0), \end{aligned}$$

$$\begin{aligned} \mathcal{A}_3 &= \frac{1}{6} f^{(0,3)}(x_0, a_0) a_1^3 + \frac{1}{2} f^{(1,2)}(x_0, a_0) a_1^2 \\ &\quad + f^{(0,2)}(x_0, a_0) a_1 a_2 + \frac{1}{2} f^{(2,1)}(x_0, a_0) a_1 \\ &\quad + f^{(0,1)}(x_0, a_0) a_3 + f^{(1,1)}(x_0, a_0) a_2 \\ &\quad + \frac{1}{6} f^{(3,0)}(x_0, a_0), \end{aligned}$$

$$\begin{aligned} \mathcal{A}_4 &= \frac{1}{24} f^{(0,4)}(x_0, a_0) a_1^4 + \frac{1}{6} f^{(1,3)}(x_0, a_0) a_1^3 \\ &\quad + \frac{1}{2} f^{(0,3)}(x_0, a_0) a_1^2 a_2 + \frac{1}{4} f^{(2,2)}(x_0, a_0) a_1^2 \\ &\quad + f^{(0,2)}(x_0, a_0) a_1 a_3 + f^{(1,2)}(x_0, a_0) a_1 a_2 \\ &\quad + \frac{1}{6} f^{(3,1)}(x_0, a_0) a_1 + f^{(0,1)}(x_0, a_0) a_4 \\ &\quad + \frac{1}{2} f^{(0,2)}(x_0, a_0) a_2^2 + f^{(1,1)}(x_0, a_0) a_3 \\ &\quad + \frac{1}{2} f^{(2,1)}(x_0, a_0) a_2 + \frac{1}{24} f^{(4,0)}(x_0, a_0), \end{aligned}$$

....

From (6), we deduce the expression of the degenerate two-variable Adomian polynomials after cancelling the vanishing terms as

$$\begin{aligned} \mathcal{A}_n &= A_n(x_0, 1, 0, \dots, 0; a_0, a_1, \dots, a_n) \\ &= \sum_{p_{11} + \sum_{j=1}^n j p_{2j} = n} f^{(p_{11}, p_{2*})}(x_0, a_0) \frac{1^{p_{11}}}{p_{11}!} \prod_{j=1}^n \frac{a_j^{p_{2j}}}{p_{2j}!} \\ &= \sum_{q + \sum_{j=1}^n j p_j = n} f^{(q, p_*)}(x_0, a_0) \frac{1}{q!} \prod_{j=1}^n \frac{a_j^{p_j}}{p_j!}, \end{aligned} \quad (19)$$

where we introduce the new symbols $q = p_{11}$,

$p_j = p_{2j}$ and $p_* = \sum_{j=1}^n p_j$ for convenience. We

remark that the MATHEMATICA command 'Reduce' can be used to solve the Diophantine equation in (19).

Substituting Eq. (18) into the nonlinear differential equation in (17), we obtain

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} (x-x_0)^n = \sum_{n=0}^{\infty} \mathcal{A}_n (x-x_0)^n.$$

Upon equating the coefficients of like powers $(x-x_0)^n$, we obtain the nonlinear recursion formula for the solution coefficients as

$$a_{n+1} = \frac{1}{n+1} \mathcal{A}_n, \quad n \geq 0. \quad (20)$$

The $(n+1)$ -term approximation of the solution is

$$\varphi_{n+1}(x) = \sum_{k=0}^n a_k (x-x_0)^k.$$

For the nonlinearity $f(x, u_1, u_2, \dots, u_m)$ in a

general system of coupled nonlinear ODEs with m unknown functions, if we decompose

$$u_i = \sum_{n=0}^{\infty} a_{i,n} (x - x_0)^n, i = 1, 2, \dots, m, \quad (21)$$

then we have

$$f(x, u_1, u_2, \dots, u_m) = \sum_{n=0}^{\infty} \mathcal{A}_n (x - x_0)^n, \quad (22)$$

where the $\mathcal{A}_n$ are the degenerate $(m+1)$ -variable

$$\mathcal{A}_n = \sum_{q+\sum_{j=1}^m j \sum_{i=1}^m p_{ij} = n} f^{(q, p_{1*}, \dots, p_{m*})}(x_0, a_{1,0}, a_{2,0}, \dots, a_{m,0}) \frac{1}{q!} \prod_{i=1}^m \prod_{j=1}^n \frac{a_{i,j}^{p_{ij}}}{p_{ij}!}, \quad (23)$$

where $p_{i*} = \sum_{j=1}^n p_{ij}$.

In particular, for a differential nonlinearity $Nu = f(x, u, u')$ in a nonlinear second-order ODE, if we decompose the solution $u(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, then we obtain

$$\begin{aligned} Nu &= f \left(x, \sum_{n=0}^{\infty} a_n (x - x_0)^n, \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n \right) \\ &= \sum_{n=0}^{\infty} \mathcal{A}_n (x - x_0)^n, \end{aligned} \quad (24)$$

by the multivariable Adomian-Rach theorem and

$$\mathcal{A}_n = \sum_{q+\sum_{j=1}^n j(p_{1j} + p_{2j}) = n} f^{(q, p_{1*}, p_{2*})}(x_0, a_0, a_1) \frac{1}{q!} \prod_{j=1}^n \frac{a_j^{p_{1j}} ((j+1)a_{j+1})^{p_{2j}}}{p_{1j}! p_{2j}!}, \quad (25)$$

where $p_{i*} = \sum_{j=1}^n p_{ij}$.

Finally, we note that the degenerate one-variable Adomian polynomials equal the coefficients of the Taylor expansion series (power series) of $f(x)$ about the point x_0 . In fact,

$$\begin{aligned} f(x) &= f(x_0 + (x - x_0)) = \\ &= \sum_{n=0}^{\infty} A_n(x_0, x - x_0, 0, \dots, 0) = \\ &= \sum_{n=0}^{\infty} A_n(x_0, 1, 0, \dots, 0)(x - x_0)^n, \end{aligned}$$

Adomian polynomials such that

$$\mathcal{A}_0 = A_0(x_0; a_{1,0}; a_{2,0}; \dots; a_{m,0}),$$

$$\mathcal{A}_1 = A_1(x_0, 1; a_{1,0}, a_{1,1}; a_{2,0}, a_{2,1}; \dots; a_{m,0}, a_{m,1}),$$

$$\mathcal{A}_2 = A_2(x_0, 1, 0; a_{1,0}, a_{1,1}, a_{1,2}; a_{2,0}, a_{2,1}, a_{2,2}; \dots; a_{m,0}, a_{m,1}, a_{m,2}),$$

....

From (6), we further deduce the expression

where the degenerate three-variable Adomian polynomials are

$$\mathcal{A}_0 = A_0(x_0; a_0; a_1) = f(x_0, a_0, a_1),$$

$$\mathcal{A}_1 = A_1(x_0, 1; a_0, a_1; a_1, 2a_2),$$

$$\mathcal{A}_2 = A_2(x_0, 1, 0; a_0, a_1, a_2; a_1, 2a_2, 3a_3),$$

$$\mathcal{A}_n = A_n(x_0, 1, 0, \dots, 0; a_0, a_1, \dots, a_n; a_1, 2a_2, \dots, (n+1)a_{n+1}).$$

Also, we can express the $\mathcal{A}_n$ as

where $A_n(x_0, 1, 0, \dots, 0) = \frac{f^{(n)}(x_0)}{n!}$, which may be readily confirmed upon inspection of the formulas of the one-variable Adomian polynomials for the general analytic nonlinearity, i.e. by simply setting $u_0 = x_0, u_1 = 1$ and $u_n = 0$ for $n \geq 2$.

4. Numeric Examples

In this section, we compare the Adomian polynomials and the calculated results in the ADM with the degenerate Adomian polynomials and the calculated results in the Rach-Adomian-Meyers MDM

by solving and then re-solving several numeric examples. In the two methods, the Adomian polynomials are denoted as A_n and $\mathcal{A}_n$, and the n -term approximations are denoted as $\varphi_n(x)$ and $\phi_n(x)$, respectively.

Example 1. Consider the first-order nonlinear differential equation with a sinusoidal nonlinearity

$$\frac{du}{dx} - \sin(x-u) = 0, \quad (26)$$

$$u(0) = \pi / 2, \quad (27)$$

where the exact solution is $u^*(x) = x - 2 \arctan \frac{x-1}{x+1}$.

ADM:

Applying the integral operator $L^{-1}(\cdot) = \int_0^x (\cdot) dx$

to both sides of Eq. (26) yields

$$u = \frac{\pi}{2} + L^{-1} \sin(x-u).$$

We decompose the solution and the nonlinearity as

$$u = \sum_{n=0}^{\infty} u_n, \sin(x-u) = \sum_{n=0}^{\infty} A_n, \quad (28)$$

where the Adomian polynomials are

$$A_0 = \sin(x-u_0),$$

$$A_1 = -\cos(x-u_0)u_1,$$

$$A_2 = -\frac{1}{2}\sin(x-u_0)u_1^2 - \cos(x-u_0)u_2,$$

$$A_3 = \frac{1}{6}\cos(x-u_0)u_1^3 - \sin(x-u_0)u_1u_2 - \cos(x-u_0)u_3, \\ \dots$$

By the recursion scheme

$$u_0 = \frac{\pi}{2}, u_{n+1} = \int_0^x A_n dx, n \geq 0, \quad (29)$$

we readily calculate

$$u_1 = -\sin(x),$$

$$u_2 = \frac{1}{2}(x - \cos(x)\sin(x)),$$

$$u_3 = \frac{1}{6}(3x \cos(x) - (2 + \cos(2x))\sin(x)),$$

$$u_4 = \frac{1}{32}(4x - 4\sin(2x) - \sin(4x) + 8x \cos(2x)),$$

....

We denote the n -term approximation as

$$\varphi_n(x) = \sum_{k=0}^{n-1} u_k.$$

In Fig. 1, we plot the curves of the exact solution $u^*(x)$ and the approximations $\varphi_n(x)$ for $n=8, 9$ and 10 on the interval $(-1, 1.5]$, where $\varphi_{10}(x)$ and $u^*(x)$ overlap.

MDM:

We decompose the solution into a power series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, a_0 = \pi / 2, \quad (30)$$

and decompose the nonlinearity into the power series where the coefficients are the degenerate two-variable Adomian polynomials $\mathcal{A}_n$.

Alternatively, the degenerate Adomian polynomials can be calculated from the single variable Adomian polynomials A_n for $\sin(u)$ as

$$\sin(x-u) =$$

$$\sin(-a_0 + (1-a_1)x - \sum_{n=2}^{\infty} a_n x^n) = \sum_{n=0}^{\infty} \mathcal{A}_n x^n, \quad (31)$$

where

Fig. 1 The curves of the exact solution $u^*(x)$ (solid line) and the approximations $\varphi_n(x)$ for $n=8$ (dash line), $n=9$ (dot line) and $n=10$ (dot-dash line).

$$\mathcal{A}_0 = A_0(-a_0),$$

$$\mathcal{A}_1 = A_1(-a_0, 1-a_1),$$

$$\mathcal{A}_2 = A_2(-a_0, 1-a_1, -a_2),$$

$$\mathcal{A}_3 = A_3(-a_0, 1-a_1, -a_2, -a_3),$$

....,

where the sequence of sets $(-a_0)$, $(-a_0, 1-a_1)$, $(-a_0, 1-a_1, -a_2)$, ..., $(-a_0, 1-a_1, -a_2, \dots, -a_n)$ represent the coefficients of the respective truncated series expansion of $x-u(x)$ in powers of x . Using the Adomian polynomials for $\sin(u)$, we then obtain

$$\mathcal{A}_0 = -\sin(a_0),$$

$$\mathcal{A}_1 = (1-a_1)\cos(a_0),$$

$$\mathcal{A}_2 = \frac{1}{2}(1-a_1)^2 \sin(a_0) - a_2 \cos(a_0),$$

$$\mathcal{A}_3 = -a_2(1-a_1)\sin(a_0) -$$

$$\frac{1}{6}(1-a_1)^3 \cos(a_0) - a_3 \cos(a_0),$$

....

Consequently, the recursion scheme of the solution coefficients becomes

$$a_0 = \frac{\pi}{2}, a_{n+1} = \frac{1}{n+1} \mathcal{A}_n, \quad (32)$$

from which we can readily calculate the solution coefficients as

$$a_1 = -1, a_2 = 0, a_3 = \frac{2}{3}, a_4 = 0,$$

$$a_5 = -\frac{2}{5}, a_6 = 0, a_7 = \frac{2}{7}, a_8 = 0, \dots$$

In Fig. 2, we plot the curves of the exact solution $u^*(x)$ and the n -term approximations $\phi_n(x) = \sum_{k=0}^{n-1} a_k x^k$ for $n=12, 14$ and 16 on the interval $(-1, 1.5]$.

Example 2. Consider the first-order nonlinear

Fig. 2 The curves of the exact solution $u^*(x)$ (solid line) and the approximations $\phi_n(x)$ for $n=12$ (dash line), $n=14$ (dot line) and $n=16$ (dot-dash line).

differential equation with product, quadratic and negative-power nonlinearities

$$\frac{du}{dx} = \frac{u^2}{1-xu}, x \leq \frac{1}{e}, \quad (33)$$

$$u(0) = 1, \quad (34)$$

We remark that this equation can be solved implicitly as $u = e^{xu}$, for which x is defined up to the maximum value $\frac{1}{e}$, and $u(1/e) = e$. Moreover,

$$\frac{du}{dx} \rightarrow +\infty \text{ as } x \uparrow \frac{1}{e}.$$

ADM:

First, we rewrite the nonlinear IVP in its integral form

$$u = 1 + \int_0^x f(x, u) dx, \quad (35)$$

where

$$f(x, u) = \frac{u^2}{1-xu}. \quad (36)$$

After decomposing the solution and the nonlinearity, then we derive the recursion scheme as

$$u_0 = 1, u_{n+1} = \int_0^x A_n dx, n = 0, 1, \dots, \quad (37)$$

where the Adomian polynomials A_n are

$$\begin{aligned} A_0 &= \frac{u_0^2}{1-xu_0}, \\ A_1 &= \frac{2u_0u_1-xu_0^2u_1}{(1-xu_0)^2}, \\ A_2 &= \frac{u_1^2+2u_0u_2-3xu_0^2u_2+x^2u_0^3u_2}{(1-xu_0)^3}, \\ &\dots \end{aligned}$$

Hence, we compute the solution components accordingly as

$$\begin{aligned} u_1 &= -\ln(1-x), \\ u_2 &= 1 + \frac{1+\ln(1-x)}{-1+x} + \frac{1}{2}\ln^2(1-x), \\ u_3 &= \frac{1}{2} - \frac{1}{2(1-x)^2} - \ln(1-x) + \\ &\quad \frac{(2-x)\ln^2(1-x)}{2(1-x)^2} - \frac{\ln^3(1-x)}{6}, \dots \end{aligned}$$

In Fig. 3, we plot the curves of the implicit exact solution and the n -term approximations

$$\phi_n(x) = \sum_{k=0}^{n-1} u_k \text{ for } n=5, 6, 7 \text{ and } 8.$$

MDM:

We decompose the solution and the nonlinearity into their respective power series

$$\begin{aligned} u(x) &= \sum_{n=0}^{\infty} a_n x^n, \text{ where } a_0 = 1, \\ \frac{u^2}{1-xu} &= \sum_{n=0}^{\infty} \mathcal{A}_n x^n, \end{aligned} \quad (38)$$

where the $\mathcal{A}_n$ are computed from the two-variable Adomian polynomials A_n for the nonlinear function

$$f(w, u) = \frac{u^2}{1-wu} \text{ as}$$

$$\mathcal{A}_0 = A_0(0; a_0),$$

$$\mathcal{A}_1 = A_1(0, 1; a_0, a_1),$$

$$\mathcal{A}_2 = A_2(0, 1, 0; a_0, a_1, a_2),$$

...

where the sequence of sets $(0), (0, 1), (0, 1, 0), \dots, (0, 1, 0, \dots, 0)$ represent the coefficients of the respective truncated series expansion of x in powers of x . From the recursion scheme

$$a_0 = 1, a_{n+1} = \frac{1}{n+1} \mathcal{A}_n, n \geq 0, \quad (39)$$

we readily calculate the solution coefficients as

$$\begin{aligned} a_0 &= 1, a_1 = 1, a_2 = \frac{3}{2}, a_3 = \frac{8}{3}, \\ a_4 &= \frac{125}{24}, a_5 = \frac{54}{5}, a_6 = \frac{16807}{720}, \dots \end{aligned}$$

In Fig. 4, we plot the curves of the implicit exact solution and the n -term approximations

$$\varphi_n(x) = \sum_{k=0}^{n-1} a_k x^k \text{ for } n=11, 14 \text{ and } 17.$$

Fig. 3 The curves of the implicit exact solution (solid line) and the approximations $\varphi_n(x)$ for $n=5$ (dash line), $n=6$ (dot line), $n=7$ (dot-dash line) and $n=8$ (dot-dot-dash line).

Fig. 4 The curves of the implicit exact solution (solid line) and the approximations $\phi_n(x)$ for $n=11$ (dash line), $n=14$ (dot line) and $n=17$ (dot-dash line).

Example 3. Consider the second-order nonlinear differential equation with the product of the derivative and a sinusoidal nonlinearity of the solution

$$\frac{d^2 u}{dx^2} = u' \cos(xu) + x, \quad (40)$$

$$u(0) = 1, u'(0) = -1. \quad (41)$$

ADM:

Applying the integral operator $L^{-1} = \int_0^x \int_0^x (\cdot) dx dx$ to both sides of Eq. (40) yields

$$u = 1 - x + \frac{x^3}{6} + L^{-1} u' \cos(xu). \quad (42)$$

Decomposing the solution and the nonlinearity, we design the modified recursion scheme

$$u_0 = 1, \quad (43)$$

$$u_1 = -x + \frac{x^3}{6} + L^{-1} A_0, \quad (44)$$

$$u_{n+1} = L^{-1} A_n, n \geq 1, \quad (45)$$

where the Adomian polynomials A_n for the nonlinearity $u' \cos(xu)$ are

$$A_0 = \cos(xu_0)u_0,$$

$$A_1 = -x \sin(xu_0)u_1u_0 + \cos(xu_0)u_1,$$

$$A_2 = -\frac{1}{2}x^2 \cos(xu_0)u_1^2u_0 - x \sin(xu_0)u_2u_0 \\ - x \sin(xu_0)u_1u_1 + \cos(xu_0)u_2,$$

...

The calculated solution components are

$$u_1 = -x + \frac{x^3}{6}, -x + \frac{x^3}{6},$$

$$u_2 = -\frac{1}{2}x^2 \cos(x) + 2x \sin(x) + 4 \cos(x) - 4,$$

....

In Fig. 5, we plot the curves of the MATHEMATICA numeric solution and the n -term approximations $\varphi_n(x)$ for $n = 4$ and 5.

Fig. 5 The curves of the MATHEMATICA numeric solution (solid line) and the approximations $\varphi_n(x)$ for $n = 4$ (dash line) and $n = 5$ (dot line).

MDM:

We decompose the solution and then the nonlinearity into their respective power series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, u' \cos(xu) = \sum_{n=0}^{\infty} \mathcal{A}_n x^n, \quad (46)$$

where the degenerate three-variable Adomian polynomials are

$$\mathcal{A}_0 = A_0(0; a_0; a_1),$$

$$\mathcal{A}_1 = A_1(0, 1; a_0, a_1; a_1, 2a_2),$$

$$\mathcal{A}_2 = A_2(0, 1, 0; a_0, a_1, a_2; a_1, 2a_2, 3a_3),$$

...

where the $\mathcal{A}_n$ are the three-variable Adomian polynomials for the nonlinearity $f(w, u, v) = v \cos(wu)$. Substituting (46) into the nonlinear differential equation (40), we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = x + \sum_{n=0}^{\infty} \mathcal{A}_n x^n.$$

From the modified recursion scheme

$$a_0 = 1, a_1 = -1, a_2 = \mathcal{A}_0 / 2, a_3 = (1 + \mathcal{A}_1) / 6,$$

$$a_{n+2} = \frac{1}{(n+2)(n+1)} \mathcal{A}_n, n \geq 2, \quad (47)$$

we readily calculate the solution coefficients as

Fig. 6 The curves of the MATHEMATICA numeric solution (solid line) and the approximations $\phi_n(x)$ for $n=8$ (dash line), $n=9$ (dot line) and $n=10$ (dot-dash line).

$$a_2 = -\frac{1}{2}, a_3 = 0, a_4 = \frac{1}{24}, a_5 = -\frac{1}{60},$$

$$a_6 = -\frac{3}{80}, a_7 = \frac{19}{2520}, a_8 = \frac{223}{13440}, \dots$$

In Fig. 6, we plot the curves of the MATHEMATICA numeric solution and the n -term approximations $\phi_n(x)$ for $n=8, 9$ and 10 .

Example 4. Consider the Lane-Emden type equation with an exponential nonlinearity of the solution with respect to the variable base of $(x+1)$

$$\frac{d^2u}{dx^2} + \frac{2}{x}u'(x) + (x+1)^{u(x)} = 0, \quad (48)$$

$$u(0) = 1, u'(0) = 0. \quad (49)$$

We remark that the built-in numeric solution command in MATHEMATICA cannot solve the nonlinear IVP in this example because of the singularity at the origin.

ADM:

We rewrite Eq. (48) as

$$Lu(x) = -(x+1)^{u(x)}, \quad (50)$$

where $L(\cdot) = \frac{1}{x^2} \frac{d}{dx} (x^2 \frac{d}{dx} (\cdot))$. Applying the

integral operator $L^{-1}(\cdot) = \int_0^x x^{-2} \int_0^x x^2 (\cdot) dx dx$ to

both sides of Eq. (50) yields

$$u(x) = 1 - L^{-1}(x+1)^{u(x)}. \quad (51)$$

Decomposing the solution and the nonlinearity, we obtain the recursion scheme

$$u_0 = 1, \quad (52)$$

$$u_{n+1} = -L^{-1}A_n, n \geq 0, \quad (53)$$

where the Adomian polynomials A_n for the nonlinearity $(x+1)^{u(x)}$

$$A_0 = (x+1)^{u_0},$$

$$A_1 = u_1(x+1)^{u_0} \ln(x+1),$$

$$A_2 = \frac{1}{2}(x+1)^{u_0} \ln(x+1) (u_1^2 \ln(x+1) + 2u_2),$$

$$A_3 = \frac{1}{6}(x+1)^{u_0} \ln(x+1) (u_1^3 \ln^2(x+1)$$

$$+ 6u_2u_1 \ln(x+1) + 6u_3),$$

...

The calculated solution components are

$$u_1 = -\frac{1}{12}x^2(2+x),$$

$$u_2 = \frac{1}{1058400x} (x(3780 + 3990x - 1680x^2 + 1015x^3 - 714x^4 - 2394x^5 - 650x^6)$$

$$+ 420 \left[\frac{-9 - 14x + 21x^5}{+21x^6 + 5x^7} \right] \ln(1+x) \Bigg),$$

....

In Fig. 7, we plot the curves of the n -term approximations $\varphi_n(x)$ for $n=4, 5$ and 6 .

MDM:

We decompose the solution and then the nonlinearity into their respective power series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, (x+1)^{u(x)} = \sum_{n=0}^{\infty} \mathcal{A}_n x^n, \quad (54)$$

where $a_0 = 1$ and $a_1 = 0$, and the $\mathcal{A}_n$ are the degenerate two-variable Adomian polynomials

Fig. 7 The curves of the approximations $\varphi_n(x)$ for $n=4$ (dash line), $n=5$ (dot line) and $n=6$ (solid line).

$$\mathcal{A}_0 = \mathcal{A}_0(0; a_0),$$

$$\mathcal{A}_1 = \mathcal{A}_1(0, 1; a_0, a_1),$$

$$\mathcal{A}_2 = \mathcal{A}_2(0, 1, 0; a_0, a_1, a_2),$$

...,

where the $\mathcal{A}_n$ are the two-variable Adomian polynomials for the nonlinearity $f(w, u) = (w+1)^u$. By substitution, we derive the modified recursion scheme for the solution coefficients as

$$a_0 = 1, a_1 = 0, a_{n+2} = -\frac{\mathcal{A}_n}{(n+2)(n+3)}, n \geq 0. \quad (55)$$

The calculated solution coefficients are

$$a_2 = -\frac{1}{6}, a_3 = -\frac{1}{12}, a_4 = 0, a_5 = \frac{1}{180},$$

$$a_6 = \frac{1}{252}, a_7 = \frac{1}{4032}, a_8 = -\frac{7}{25920}, \dots$$

In Fig. 8, we plot the curves of the n -term approximations $\phi_n(x)$ for $n=11, 12$ and 13 .

5. Convergence Acceleration, Multistage Decomposition and Numeric Algorithms

As with all methods that generate infinite series solutions, we remark that the domain of convergence for the decomposition series solution may not always be sufficiently large for engineering purposes. But we

can readily handle this problem by means of one of several common convergence acceleration techniques [44], such as the diagonal Padé approximants [13, 14, 45, 46] or the iterated Shanks transform [13, 47]. Rach and Duan [46] presented the combined solution of the near-field and far-field approximations by the Adomian and asymptotic decomposition methods, where the Padé approximants technique was only used as necessary to calculate the mid-field approximation in order to bridge the near- and far-field approximations. In the ADM, the parametrized recursion scheme [22, 36, 37] was proposed in order to obtain decomposition solutions with large effective regions of convergence.

The multistage ADM and its numeric algorithms [48-56] can also be used to greatly expand the effective region of solution. In [54] Duan and Rach have considered one-step numeric algorithms for nonlinear IVPs for ODEs based on the ADM and the Rach-Adomian-Meyers MDM, respectively. Therein algorithms are parametrized by the order of the method, so that we can easily and rapidly generate a higher-order algorithm for numerical integration at will, whereas it is extremely difficult and exceptionally tedious to develop even a 6th-order Runge-Kutta algorithm. In fact, there is no common algorithm to generate higher orders of explicit Runge-Kutta formulas. In [55], higher-order numeric

Fig. 8 The curves of the approximations $\phi_n(x)$ for $n=11$ (dash line), $n=12$ (dot line) and $n=13$ (solid line).

algorithms based on the Wazwaz-El-Sayed modified ADM were also proposed.

Ghosh and Roy [57] considered the numeric-analytic form of the ADM for two-point boundary value problems (BVPs). The multistage ADM for BVPs including even Robin boundary conditions has been considered in [58], where a multistage ADM for BVPs has been developed through partitioning the domain into two, or more, subdomains, where a separate series was computed in each subdomain using the Duan-Rach modified recursion scheme for nonlinear BVPs [36]. The

sub-solutions are combined to form the decomposition spline by applying the conditions of continuity at the interior boundary points in analogy to the multistage ADM for IVPs [49].

In the sequel, we first consider the Padé approximation of the MDM solution in Example 2 and then the numeric solution for Example 4.

The MDM solution in Example 2 has a radius of convergence of only $\frac{1}{e}$. The $[5/5]$ Padé approximation of the 11-term approximation $\phi_{11}(x)$ is

$$[5/5]\{\phi_{11}(x)\} = \left(1 - \frac{64958411931481x}{10796179523602} + \frac{602627559979987x^2}{48582807856209} - \frac{3917670006940357x^3}{388662462849672} + \frac{4808893764386087x^4}{1813758159965136} - \frac{10585540152954929x^5}{163238234396862240}\right) / \left(1 - \frac{75754591455083x}{10796179523602} + \frac{870649009743547x^2}{48582807856209} - \frac{869838597842227x^3}{43184718094408} + \frac{5692390072282141x^4}{604586053321712} - \frac{213859991604212971x^5}{163238234396862240}\right).$$

In Fig. 9, we plot the curves of the implicit exact solution and the diagonal Padé approximation $[5/5]\{\phi_{11}(x)\}$ on the interval $[-10, 1/e]$, where the two curves overlap.

Next, we consider the numeric solution on the interval $[0, X]$ for Example 4. We take a sufficiently small positive number as x_1 . Here we select $x_1 = 0.1$. Using the MDM solution in Example 4, e.g. the 13-term approximation $\phi_{13}(x)$, we calculate

$$\begin{aligned}\hat{u}_1 &:= \phi_{13}(x_1) = 0.9982500595, \\ \dot{\hat{u}}_1 &:= \phi_{13}'(x_1) = -0.03583031596.\end{aligned}$$

Next, we solve a new nonlinear IVP on the interval $[x_1, X]$,

$$\frac{d^2u}{dx^2} + \frac{2}{x}u'(x) + (x+1)^{u(x)} = 0, \quad (56)$$

$$u(x_1) = \hat{u}_1, u'(x_1) = \dot{\hat{u}}_1, \quad (57)$$

where there is now no singularity on the considered interval. In order to develop the numeric algorithm, we derive the MDM solution of Eq. (56) with the initial conditions

$$u(\xi) = a_0, u'(\xi) = a_1, \xi > 0, \quad (58)$$

Fig. 9 The curves of the implicit exact solution (solid line) and the Padé approximation $[5/5]\{\phi_{11}(x)\}$ (dash line).

where ξ, a_0 and a_1 are regarded as three parameters.

We decompose the solution and then the nonlinearity into their respective power series

$$\begin{aligned} u(x) &= \sum_{n=0}^{\infty} a_n (x - \xi)^n, \\ (x+1)^{u(x)} &= \sum_{n=0}^{\infty} \mathcal{A}_n (x - \xi)^n, \end{aligned} \quad (59)$$

where the degenerate Adomian polynomials are

$$\mathcal{A}_0 = A_0(\xi; a_0),$$

$$\mathcal{A}_1 = A_1(\xi, 1; a_0, a_1),$$

$$\mathcal{A}_2 = A_2(\xi, 1, 0; a_0, a_1, a_2),$$

...

$$\begin{aligned} \mathcal{A}_3 = \frac{1}{6}(\xi+1)^{a_0} &\left(\frac{(a_0-2)(a_0-1)a_0}{(\xi+1)^3} + a_1^3 \ln^3(\xi+1) + 6a_2a_1 \ln^2(\xi+1) + \frac{3a_1^2 \ln(\xi+1)(a_0 \ln(\xi+1) + 2)}{\xi+1} \right. \\ &\left. + \frac{3a_1(a_0(a_0 \ln(\xi+1) - \ln(\xi+1) + 2) - 1)}{(\xi+1)^2} + \frac{6a_2(a_0 \ln(\xi+1) + 1)}{\xi+1} + 6a_3 \ln(\xi+1) \right), \end{aligned}$$

...

We further decompose the system coefficient $\frac{2}{x}$ into the compatible series

$$\frac{2}{x} = \frac{2}{\xi} \sum_{n=0}^{\infty} \left(-\frac{x-\xi}{\xi} \right)^n. \quad (60)$$

Substituting (59) and (60) into Eq. (56), we obtain

$$a_2 = -\frac{1}{2}(1+\xi)^{a_0} - \frac{a_1}{\xi},$$

$$a_3 = \frac{1}{6} \left(-(1+\xi)^{-1+a_0} a_0 + \frac{2\xi(1+\xi)^{a_0} + (6 - \xi^2(1+\xi)^{a_0} \ln(1+\xi))a_1}{\xi^2} \right),$$

$$\begin{aligned} a_4 = &-\frac{1}{24\xi^3(1+\xi)^2} \left(\xi^3(1+\xi)^{a_0} a_0^2 + \xi^2(1+\xi)^{a_0} a_0 (-2 - 3\xi + 2\xi(1+\xi) \ln(1+\xi))a_1 \right. \\ &- (1+\xi) \left(\xi(1+\xi)^{1+a_0} (-8 + \xi^2(1+\xi)^{a_0} \ln(1+\xi)) + (-2(12 + 12\xi + \xi^3(1+\xi)^{a_0}) \right. \\ &\left. \left. + 4\xi^2(1+\xi)^{1+a_0} \ln(1+\xi))a_1 - \xi^3(1+\xi)^{1+a_0} \ln(1+\xi)^2 a_1^2 \right) \right), \end{aligned}$$

...

where the sequence of sets $(\xi), (\xi, 1), (\xi, 1, 0), \dots, (\xi, 1, 0, \dots, 0)$ represent the coefficients of the respective truncated series expansion of x in powers of $x - \xi$, and where the $\mathcal{A}_n$ are the two-variable Adomian polynomials for the nonlinear function $f(w, u) = (w+1)^{u(x)}$. Calculating the Adomian polynomials, we have

$$\mathcal{A}_0 = (\xi+1)^{a_0},$$

$$\mathcal{A}_1 = (\xi+1)^{a_0-1} (a_1(\xi+1) \ln(\xi+1) + a_0),$$

$$\begin{aligned} \mathcal{A}_2 = &\frac{1}{2}(\xi+1)^{a_0-2} (2a_1(\xi+1) + \\ &(\xi+1)^2 \ln(\xi+1) (a_1^2 \ln(\xi+1) + 2a_2) \\ &+ a_0(2a_1(\xi+1) \ln(\xi+1) - 1) + a_0^2), \end{aligned}$$

the modified recursion scheme

$$a_{n+2} = -\frac{1}{(n+1)(n+2)} \left[\mathcal{A}_n + \frac{2}{\xi} \sum_{k=0}^n \left(-\frac{1}{\xi} \right)^{n-k} (k+1)a_{k+1} \right], \quad n \geq 0. \quad (61)$$

Using the Adomian polynomials, we derive

where all of the a_k , for $k \geq 2$, depend only on ξ, a_0 and a_1 . Thus we obtain the $(n+1)$ -term approximation of the nonlinear IVP (56) and (58) as

$$\varphi_{n+1}(x, \xi, a_0, a_1) = \sum_{k=0}^n a_k (x - \xi)^k, \quad (62)$$

which we use to generate the n th-order numeric solutions as follows. Partitioning the interval $[x_1, X]$ into $x_1 < x_2 < \dots < x_N = X$, then the algorithm for the n th-order numeric solution as generated by $\varphi_{n+1}(x, \xi, a_0, a_1)$ is

$$\hat{u}_1 := u(x_1), \hat{u}_1' := u'(x_1) \quad (63)$$

$$\begin{aligned} \hat{u}_{k+1} &:= \varphi_{n+1}(x_{k+1}, x_k, \hat{u}_k, \hat{u}_k') \\ &= \varphi'_{n+1}(x, x_k, \hat{u}_k, \hat{u}_k')|_{x=x_{k+1}}, \end{aligned} \quad (64)$$

$$k = 1, 2, \dots, N-1,$$

where the $\hat{u}_k$, $k = 1, 2, \dots, N$, are the n th-order numeric solutions.

If we take an equal step-size partition, i.e. $x_{k+1} = x_1 + kh$, $k = 1, 2, \dots, N-1$, then the calculation will be further simplified. We take $X = 30$, generating the 6th-order numeric solutions for the equal step-size $h = 0.4$ and 0.2 , respectively. In Fig. 10, the two numeric solutions nearly coincide, which demonstrates a high degree of accuracy.

6. Conclusion

In this work, we have developed the concept of the degenerate multivariable Adomian polynomials in conjunction with the Rach-Adomian-Meyers MDM. Our new technique provides a significant advantage for automated calculations for computing the power series form of the solution for nonlinear ODEs with intrinsically nonseparable nonlinearities with respect to the independent and dependent variables. We establish efficient algorithms for the degenerate Adomian polynomials derived from the classic multivariable Adomian polynomials.

Fig. 10 The 6th-order numeric solutions for the equal step-size $h = 0.4$ (plus) and $h = 0.2$ (dot), respectively.

We have demonstrated the high accuracy and efficiency of our new technique by comparison of the ADM using the classic Adomian polynomials with the Rach-Adomian-Meyers MDM incorporating the degenerate Adomian polynomials. In the Rach-Adomian-Meyers MDM, the recursion scheme of the solution coefficients does not involve the integration operation, but the degenerate multivariable Adomian polynomials. We investigated the solutions of four numeric nonlinear examples. Our expository examples display the practicality, ease of use, and efficacy of our new modification of the Rach-Adomian-Meyers MDM.

Next we considered the issues of convergence acceleration by such classic techniques as the diagonal Padé approximants and the iterated Shanks transform, then the multistage ADM and MDM and finally numeric algorithms that implement our new approach. To illustrate our discussion, we applied the diagonal Padé approximants, which is readily achieved within MATHEMATICA, to the approximate solution for Example 2 and then the multistage MDM to Example 4 by developing its detailed numeric algorithm. The results show that the effective regions for the solutions are greatly extended.

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**Appendix. MATHEMATICA code for the one-variable Adomian polynomials
based on the Corollary 3 algorithm in [24]**

```

AP[f_, M_] := Module[{c, n, k, j, der},
Table[c[n, k], {n, 1, M}, {k, 1, n}];
der = Table[D[f[Subscript[u, 0]], {Subscript[u, 0], k}], {k, 1, M}];
A[0] = f[Subscript[u, 0]];
For[n = 1, n <= M, n++, c[n, 1] = Subscript[u, n];
For[k = 2, k <= n, k++,
c[n, k] = Expand[1/n* Sum[(j + 1)*Subscript[u, j+1]* c[n-1-j, k-1],
{j, 0, n-k}]] ];
A[n] = Take[der, n].Table[c[n, k], {k, 1, n} ]];
Table[A[n], {n, 0, M}]] ]

```