An Analytic and Combinatorial-Geometric Proof of a Knopp-Type Identity for Multiple Dedekind Sums

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Abstract: In this paper, we define multiple Dedekind sums by products of Bernoulli functions. From the Fourier expansions of Bernoulli functions, we express the Dedekind sums as series representations. Then by a combinatorial-geometric method, we give a new proof of a Knopp-type identity for the Dedekind sums.

Keywords: Dedekind sums, Knopp's formula, combinatorial

1. Introduction

For $h \in \mathbf{Z}$ and $k \in \mathbf{N}$, the classical Dedekind sum $s(h, k)$ is defined by

$$s(h, k) = \sum_{\alpha \bmod k} \left(\left(\frac{\alpha}{k} \right) \right) \left(\left(\frac{\alpha h}{k} \right) \right), \quad (1)$$

where

$$\left((x) \right) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbf{Z} \\ 0 & \text{if } x \in \mathbf{Z} \end{cases}$$

and the following formulas are well known:

(I) Reciprocity formula (Dedekind [14])

$$12hk\{s(h, k) + s(k, h)\} = h^2 - 3hk + k^2 + 1 \quad (2)$$

for $h, k \in \mathbf{N}$ with $\gcd\{h, k\} = 1$.

(II) Knopp's formula (Knopp [16])

$$\sum_{\substack{ad=N \\ d>0}} \sum_{b=0}^{d-1} s(ah + kb, dk) = \sigma(N)s(h, k) \quad (3)$$

for $N \in \mathbf{N}$, where $\sigma(N) = \sum_{\delta|N} \delta$.

Generalizations of Dedekind sums and formulas (2) and (3) have been studied by many mathematicians

extensively with many methods. For $p \in \mathbf{Z}_{\geq 0}$, let $B_p(X)$ be the p th Bernoulli polynomial and $\bar{B}_p(x)$ the p th Bernoulli function defined by

$$\bar{B}_p(x) = \begin{cases} B_p(x - [x]) & \text{if } p \neq 1 \\ ((x)) & \text{if } p = 1. \end{cases} \quad (4)$$

For $p, q \in \mathbf{Z}_{\geq 0}$, the sum (1) is naturally generalized as

$$s_{p,q}(h, k) = \sum_{\alpha \bmod k} \bar{B}_p\left(\frac{\alpha}{k}\right) \bar{B}_q\left(\frac{\alpha h}{k}\right), \quad (5)$$

which is called a higher-order Dedekind sum ([1],[8],[9]). For $u, v \in \mathbf{R}$, the sum (5) is also generalized as

$$\begin{aligned} s_{p,q}(h, k; u, v) \\ = \sum_{\alpha \bmod k} \bar{B}_p\left(\frac{\alpha + u}{k}\right) \bar{B}_q\left(\frac{(\alpha + u)h}{k} + v\right), \end{aligned} \quad (6)$$

which is often called a Dedekind-Rademacher sum and generalizations of (2) are studied in [11], [12] and [23].

In [15], making use of the Fourier expansion of $\bar{B}_p(x)$, Halbritter expressed the sum (6) as a series representation and obtained generalizations of (2) which connects the sum (6) and transformed ones of (6) by a unimodular matrix.

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Recently, by a combinatorial-geometric method, which is deeply connected with the theory of lattice points in polytopes ([3], [7]), Beck gave new proofs of (2) and some of its generalizations ([5], [6], [21]). In [19] and [20], motivated by the method, the author studied generalizations of (3) and obtained results including many preceding ones shown in [16], [22], [1], [24], [4], [17] and [18].

In the present paper, in addition to the combinatorial-geometric method, we also make use of the Fourier expansion of $\bar{B}_p(x)$ as in [15] and study generalizations of (3) for multiple Dedekind sums attached to Dirichlet characters. The main result is contained in that of [20] as an important example and the significance of this paper can be regarded as giving a new proof of it.

Let us give a description of each section.

In Section 2, we define multiple Dedekind sums by making use of certain generalized Bernoulli functions attached to Dirichlet characters and state the main result. In Section 3, making use of the Fourier expansions of Bernoulli functions, we deduce a series representation of the multiple Dedekind sum. In Section 4, we prove the main result for the special case $s_{p,q}(h,k)$ in order to provide a good overview. In Section 5, we give a complete proof for the general case.

2. Definitions and the Main Result

As in the introduction, let $B_p(X)$ be the p th Bernoulli polynomial defined by

$$\frac{te^{tX}}{e^t - 1} = \sum_{p=0}^{\infty} B_p(X) \frac{t^p}{p!}.$$

and $\bar{B}_p(x)$ the p th Bernoulli function defined by (4).

Let χ be a Dirichlet characters (not necessarily primitive) defined modulo $\mathcal{N}_\chi$. The usual p th Bernoulli polynomial $B_{p,\chi}(X)$ attached to χ is defined by

$$\sum_{a=0}^{\mathcal{N}_\chi-1} \frac{\chi(a)te^{(X+a)t}}{e^{\mathcal{N}_\chi t} - 1} = \sum_{p=0}^{\infty} B_{p,\chi}(X) \frac{t^p}{p!},$$

which is equivalent to

$$B_{p,\chi}(\mathcal{N}_\chi X) = \mathcal{N}_\chi^{p-1} \sum_{a \bmod \mathcal{N}_\chi} \chi(a) B_p \left(X + \frac{a}{\mathcal{N}_\chi} \right).$$

On the basis of this equation, we define a function

$$\beta_{p,\chi}(x) = \sum_{a \bmod \mathcal{N}_\chi} \chi(a) \bar{B}_p \left(x + \frac{a}{\mathcal{N}_\chi} \right).$$

For $k = (k_1, \dots, k_r) \in \mathbf{N}^r$ we denote by $R_m(k) \subset \mathbf{Z}^r$ an arbitrarily fixed complete set of representatives of the residue class group $(\mathbf{Z}/k_1\mathbf{Z}) \times \dots \times (\mathbf{Z}/k_r\mathbf{Z})$. If $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbf{Z}^r$, we put

$$\frac{\alpha}{k} = \left(\frac{\alpha_1}{k_1}, \dots, \frac{\alpha_r}{k_r} \right).$$

Let $r, s \in \mathbf{N}$ and let $P = (p_1, \dots, p_r) \in \mathbf{Z}_{\geq 0}^r$ and $Q = (q_1, \dots, q_s) \in \mathbf{Z}_{\geq 0}^s$. Let $\Phi = (\phi_1, \dots, \phi_r)$ and $\Psi = (\psi_1, \dots, \psi_s)$ be tuples of Dirichlet characters. For $x = (x_1, \dots, x_r) \in \mathbf{R}^r$ and $y = (y_1, \dots, y_s) \in \mathbf{R}^s$, we put

$$F(x, y; P, Q, \Phi, \Psi) =$$

$$\left(\prod_{i=1}^r \beta_{p_i, \phi_i}(x_i) \right) \prod_{j=1}^s \beta_{q_j, \psi_j}(y_j).$$

Let $H = (h_{ij})_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}}$ be an $r \times s$ matrix with

$h_{ij} \in \mathbf{Z}$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$. Note that if $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbf{Z}^r$, then

$$F\left(\frac{\alpha}{k}, \frac{\alpha}{k} H; P, Q, \Phi, \Psi\right) = \left(\prod_{i=1}^r \beta_{p_i, \phi_i} \left(\frac{\alpha_i}{k_i} \right) \right) \times \prod_{j=1}^s \beta_{q_j, \psi_j} \left(\frac{\alpha_1 h_{1j}}{k_1} + \dots + \frac{\alpha_r h_{rj}}{k_r} \right).$$

We define a multiple Dedekind sum as

$$S(H, k; P, Q, \Phi, \Psi) = \sum_{\alpha \in R_m(k)} F\left(\frac{\alpha}{k}, \frac{\alpha}{k} H; P, Q, \Phi, \Psi\right). \quad (7)$$

The sum (7) is an important example of a generalized Dedekind type sum defined as (2.3) in [20]

and includes (1), (5) and some of the generalized Dedekind sums appearing in [4], [24], [18], [19] etc. Note that if we define a twisted p th Bernoulli function $\tilde{B}_{p,\chi}(x)$ for $x \in \mathbf{Q}$ with denominators relatively prime to $\mathcal{N}_\chi$ by

$$\tilde{B}_{p,\chi}(x) = \mathcal{N}_\chi^{p-1} \sum_{a \bmod \mathcal{N}_\chi} \chi(x+a) \bar{B}_p\left(\frac{x+a}{\mathcal{N}_\chi}\right)$$

as in Section 2 of [20], then we have $\tilde{B}_{p,\chi}(\mathcal{N}_\chi x) = \mathcal{N}_\chi^{p-1} \beta_{p,\chi}(x)$. If we put

$$F(x, y) = \left(\prod_{i=1}^r \tilde{B}_{p_i, \phi_i}(x_i) \right) \prod_{j=1}^s \tilde{B}_{q_j, \psi_j}(y_j),$$

then the relation between (7) and $S(F: H, k)$ in [20] is verified as follows:

Suppose $\gcd\{\mathcal{N}_{\phi_i} \mathcal{N}_{\psi_j}, k_1 \cdots k_r\} = 1$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$ and let $H' = (h'_{ij})_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}}$

be an $r \times s$ matrix with $h'_{ij} \in \mathbf{Z}$ such that $\mathcal{N}_{\phi_i} h'_{ij} \equiv \mathcal{N}_{\psi_j} h_{ij} \pmod{k_i}$ holds for all $1 \leq i \leq r$ and $1 \leq j \leq s$. Then it is easy to see that

$$\begin{aligned} S(H, k: P, Q, \Phi, \Psi) \\ = \left(\prod_{i=1}^r \mathcal{N}_{\phi_i}^{1-p_i} \right) \left(\prod_{j=1}^s \mathcal{N}_{\psi_j}^{1-q_j} \right) S(F: H', k). \end{aligned}$$

Now for any Dirichlet character χ and any $l, N \in \mathbf{N}$, we put

$$\sigma_{l,\chi}(N) = \sum_{\delta|N} \chi(\delta) \delta^l.$$

For any $d \in \mathbf{N}$, let $\mathcal{I}_d(r, s)$ be a set of $r \times s$ matrices such as

$$\mathcal{I}_d(r, s) = \{B = (b_{ij})_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} \mid b_{ij} \in \mathbf{Z}, 0 \leq b_{ij} \leq d-1\}$$

for all $1 \leq i \leq r$ and $1 \leq j \leq s$.

For $k = (k_1, \dots, k_r) \in \mathbf{N}^r$ and $B \in \mathcal{I}_d(r, s)$, we put

$$k \circ B = \begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_r \end{pmatrix} B,$$

which is a matrix obtained from B by multiplying the i th row by k_i for each $1 \leq i \leq r$. Put

$$s(P) = \sum_{i=1}^r p_i, s(Q) = \sum_{j=1}^s q_j,$$

$$s(P, Q) = s(P) + s(Q),$$

$$\phi = \prod_{i=1}^r \phi_i \text{ and } \psi = \prod_{j=1}^s \psi_j.$$

Then the main result is the following.

Theorem 2.1 Let $N \in \mathbf{N}$ with

$$\gcd\{N, \mathcal{N}_{\phi_i}\} = \gcd\{N, \mathcal{N}_{\psi_j}\} = 1 \text{ for all } 1 \leq i \leq r$$

and $1 \leq j \leq s$. Then we have

$$\begin{aligned} N^{s(P)-r} \phi(N) \sum_{\substack{ad=N \\ d>0}} \sum_{B \in \mathcal{I}_d(r, s)} d^{s(Q)-sr} \psi(d) \\ \times S(aH + k \circ B, dk: P, Q, \Phi, \Psi) \\ = \sigma_{s(P, Q)-r, \phi\psi}(N) S(H, k: P, Q, \Phi, \Psi). \end{aligned}$$

3. Series Representations of Dedekind Sums

For $p \in \mathbf{N}$, the Fourier expansion of $\bar{B}_p(x)$ is expressed as

$$\bar{B}_p(x) = -\frac{p!}{(2\pi\sqrt{-1})^p} \lim_{N \rightarrow \infty} \sum_{\substack{n=-N \\ n \neq 0}}^N \frac{\exp(2\pi\sqrt{-1}nx)}{n^p}.$$

In order to include the case of $p = 0$, we put

$$c_p(n) = \begin{cases} n^{-p} & \text{if } p \geq 1 \text{ and } n \in \mathbf{Z} \text{ with } n \neq 0 \\ -1 & \text{if } p = 0 \text{ and } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We put further

$$e(x) = \exp(2\pi\sqrt{-1}x).$$

Then for all $p \geq 0$, we have

$$\bar{B}_p(x) = -\frac{p!}{(2\pi\sqrt{-1})^p} \lim_{N \rightarrow \infty} \sum_{n=-N}^N c_p(n) e(nx). \quad (8)$$

In addition, for any fixed $N_1, N_2 \in \mathbf{Z}$, we also

have

$$\bar{B}_p(x) = -\frac{p!}{(2\pi\sqrt{-1})^p} \lim_{N \rightarrow \infty} \sum_{n=-N+N_1}^{N+N_2} c_p(n) e(nx). \quad (9)$$

As in Section 2, let χ be a Dirichlet character defined modulo N_χ . Put $\zeta_\chi = e(1/N_\chi)$ and

$$\tau(\chi, \zeta_\chi^n) = \sum_{b \bmod N_\chi} \chi(b) \zeta_\chi^{nb}$$

for $n \in \mathbf{Z}$ and define

$$c_{p,\chi}(n) = \tau(\chi, \zeta_\chi^n) c_p(n).$$

Then, direct calculation shows that

$$\beta_{p,\chi}(x) = -\frac{p!}{(2\pi\sqrt{-1})^p} \lim_{N \rightarrow \infty} \sum_{n=-N+N_1}^{N+N_2} c_{p,\chi}(n) e(nx)$$

for any fixed $N_1, N_2 \in \mathbf{Z}$. If χ is primitive, we have $c_{p,\chi}(n) = \chi^{-1}(n) \tau(\chi, \zeta_\chi) c_p(n)$ for all $n \in \mathbf{Z}$ and so,

$$\beta_{p,\chi}(x) = -\frac{p!}{(2\pi\sqrt{-1})^p} \tau(\chi, \zeta_\chi)$$

$$\times \lim_{N \rightarrow \infty} \sum_{n=-N+N_1}^{N+N_2} \chi^{-1}(n) c_p(n) e(nx).$$

For $P = (p_1, \dots, p_r) \in \mathbf{Z}_{\geq 0}^r$ and $Q = (q_1, \dots, q_s) \in \mathbf{Z}_{\geq 0}^s$, we put

$$A(P, Q) = (-1)^{r+s} \left(\prod_{i=1}^r p_i! \prod_{j=1}^s q_j! \right)^{-1} (2\pi\sqrt{-1})^{s(P, Q)}$$

and

$$T(H, \mathbf{k} : P, Q, \Phi, \Psi) = A(P, Q) S(H, \mathbf{k} : P, Q, \Phi, \Psi).$$

Especially, we put

$$t_{p,q}(h, k) = (p!q!)^{-1} (2\pi\sqrt{-1})^{p+q} s_{p,q}(h, k)$$

for $k \in \mathbf{N}, h \in \mathbf{Z}$ and $p, q \in \mathbf{Z}_{\geq 0}$. Then we have

the following.

Lemma 3.1 We have

$$t_{p,q}(h, k) = k \lim_{M \rightarrow \infty} \sum_{m=-M}^M c_q(m) \left(\lim_{L \rightarrow \infty} \sum_{l=-L}^L c_p(lk - mh) \right). \quad (10)$$

and

$$\begin{aligned} T(H, \mathbf{k} : P, Q, \Phi, \Psi) &= k_1 \cdots k_r \lim_{M \rightarrow \infty} \sum_{\substack{-M \leq m_j \leq M \\ (1 \leq j \leq s)}} \left(\prod_{j=1}^s c_{q_j, \psi_j}(m_j) \right) \\ &\times \left(\lim_{L \rightarrow \infty} \sum_{\substack{-L \leq l_i \leq L \\ (1 \leq i \leq r)}} \prod_{i=1}^r c_{p_i, \phi_i}(l_i k_i - m_1 h_{i1} - \cdots - m_s h_{is}) \right). \end{aligned} \quad (11)$$

Proof: By (8) and the definition of $t_{p,q}(h, k)$, we

see that

$$\begin{aligned} t_{p,q}(h, k) &= \lim_{M \rightarrow \infty} \sum_{m=-M}^M c_q(m) \\ &\times \left(\lim_{N \rightarrow \infty} \sum_{n=-N}^N c_p(n) \sum_{\alpha \bmod k} e\left(\frac{(n+mh)\alpha}{k}\right) \right). \end{aligned}$$

Note that

$$\sum_{\alpha \bmod k} e\left(\frac{(n+mh)\alpha}{k}\right) = \begin{cases} k & \text{if } n \equiv -mh \pmod{k} \\ 0 & \text{otherwise} \end{cases}$$

Hence, taking (9) into account, we obtain (10). In the same way, we also obtain (11).

Remark.

(1) The formula (10) is contained in equation (13) of [15], which is the series representation of (6).

(2) In the summations of the right hand side of (10), we can replace m and l by $-m$ and $-l$, respectively and see that

$$\begin{aligned} t_{p,q}(h, k) &= k \lim_{M \rightarrow \infty} \sum_{m=-M}^M c_q(-m) \\ &\times \left(\lim_{L \rightarrow \infty} \sum_{l=-L}^L c_p(-lk + mh) \right) \\ &= (-1)^{p+q} t_{p,q}(h, k). \end{aligned}$$

In particular, if $p \not\equiv q \pmod{2}$, we have $t_{p,q}(h, k) = 0$. In the same way, we have

$$T(H, \mathbf{k} : P, Q, \Phi, \Psi) = (-1)^{s(P, Q)} S(H, \mathbf{k} : P, Q, \Phi, \Psi).$$

and if $s(P) \not\equiv s(Q) \pmod{2}$, we have

$$T(H, \mathbf{k} : P, Q, \Phi, \Psi) = 0.$$

4. Proof of Theorem 2.1 in a Special Case

In this section, we prove theorem 2.1 in a special case of $r = s = 1$ and $\phi = \psi = 1$. In this case, theorem 2.1 reduces to the following formula:

$$N^{p-1} \sum_{\substack{ad=N \\ d>0}} d^{q-1} \sum_{b=0}^{d-1} t_{p,q}(ah + kb, dk) \quad (12)$$

$$= \sigma_{p+q-1}(N) t_{p,q}(h, k).$$

In [19] and [20], we introduced an additive subgroup of $\mathbf{Z}^2$ such as

$$\mathcal{A}(a, d : b) = (a, -b)\mathbf{Z} + (0, d)\mathbf{Z}$$

for $a, d \in \mathbf{N}$ and $b \in \mathbf{Z}$ and by considering conditions for $(m, l) \in \mathbf{Z}^2$ to be in $\mathcal{A}(a, d : b)$, we deduced the following (lemma 3.3 of [20]).

Lemma 4.1 Let R be a ring extension of $\mathbf{Q}$ and let $\mathcal{F}$ be a map from $\mathbf{Z}^2$ to R . Then for any finite subset F of $\mathbf{Z}^2$ and for any $N \in \mathbf{N}$, we have

$$\begin{aligned} & \sum_{\substack{ad=N \\ d>0}} \sum_{b=0}^{d-1} \sum_{(m,l) \in \mathcal{A}(a,d:b) \cap F} \mathcal{F}(m, l) \\ &= \sum_{\delta|N} \delta \sum_{(m,l) \in \delta \mathbf{Z}^2 \cap F} \mathcal{F}(m, l). \end{aligned}$$

Let us deduce (12) from (10) and lemma 4.1. Put

$$\mathcal{F}(m, l : h, k) = c_q(m) c_p(lk - mh).$$

Then

$$t_{p,q}(h, k) = k \lim_{M \rightarrow \infty} \left(\lim_{L \rightarrow \infty} \sum_{m=-M}^M \sum_{l=-L}^L \mathcal{F}(m, l : h, k) \right).$$

Note that we have

$$\mathcal{F}(\delta m, \delta l : h, k) = \delta^{-p-q} \mathcal{F}(m, l : h, k) \quad (13)$$

for any $\delta \in \mathbf{N}$ and that for any fixed L_1 and $L'_1 \in \mathbf{Z}$, we also have

$$t_{p,q}(h, k) = k \lim_{M \rightarrow \infty} \left(\lim_{L \rightarrow \infty} \sum_{m=-M}^M \sum_{l=-L+L_1}^{L+L_1} \mathcal{F}(m, l : h, k) \right). \quad (14)$$

For $N, M, L \in \mathbf{N}$, we put

$$\begin{aligned} \mathcal{R}_N(M, L) &= \{(m, l) \in \mathbf{Z}^2 \mid -MN \leq m \leq MN, \\ &\quad -LN \leq l \leq LN\}. \end{aligned}$$

If $(m, l) \in \mathcal{A}(a, d : b)$, we can express

$$(m, l) = (a, -b)\mu + (0, d)\nu = (a\mu, -b\mu + d\nu)$$

with $\mu, \nu \in \mathbf{Z}$. Put $N = ad$. Then

$$\begin{aligned} \mathcal{F}(m, l : h, k) &= c_q(a\mu) c_p((-b\mu + d\nu)k - a\mu h) \\ &= a^{-q} c_q(\mu) c_p(vdk - (ah + kb)\mu) \\ &= \frac{d^q}{N^q} \mathcal{F}(\mu, \nu : ah + kb, dk). \end{aligned}$$

Taking $F = \mathcal{R}_N(M, L)$, we see that

$$\begin{aligned} & \sum_{\substack{ad=N \\ d>0}} \sum_{b=0}^{d-1} \sum_{(m,l) \in \mathcal{A}(a,d:b) \cap F} \mathcal{F}(m, l : h, k) \\ &= \frac{1}{N^q} \sum_{\substack{ad=N \\ d>0}} d^q \sum_{b=0}^{d-1} \sum_{\mu=-Md}^{Md} \\ &\quad \times \sum_{\substack{\nu \in \mathbf{Z} \text{ with} \\ -LN \leq -b\mu + d\nu \leq LN}} \mathcal{F}(\mu, \nu : ah + kb, dk) \\ &= \frac{1}{N^q} \sum_{\substack{ad=N \\ d>0}} d^q \sum_{b=0}^{d-1} \sum_{\mu=-Md}^{Md} \\ &\quad \times \sum_{\nu=-La-[-b\mu/d]}^{La+[b\mu/d]} \mathcal{F}(\mu, \nu : ah + kb, dk). \quad (15) \end{aligned}$$

We also see from (13) that

$$\begin{aligned} & \sum_{\delta|N} \delta \sum_{(m,l) \in \delta \mathbf{Z}^2 \cap F} \mathcal{F}(m, l : h, k) \\ &= \sum_{\delta|N} \delta \sum_{m=-MN/\delta}^{MN/\delta} \sum_{l=-LN/\delta}^{LN/\delta} \mathcal{F}(\delta m, \delta l : h, k) \\ &= \sum_{\delta|N} \delta^{1-p-q} \sum_{m=-MN/\delta}^{MN/\delta} \sum_{l=-LN/\delta}^{LN/\delta} \mathcal{F}(m, l : h, k). \end{aligned}$$

Note that if δ ranges the divisors of N , then so does N/δ . It follows that

$$\begin{aligned} & \sum_{\delta|N} \delta \sum_{(m,l) \in \delta \mathbf{Z}^2 \cap F} \mathcal{F}(m, l : h, k) \\ &= N^{1-p-q} \sum_{\delta|N} \delta^{p+q-1} \sum_{m=-M\delta}^{M\delta} \sum_{l=-L\delta}^{L\delta} \mathcal{F}(m, l : h, k). \end{aligned} \quad (16)$$

Taking limits for (15) and (16), we see from (14) that

$$\begin{aligned} & \lim_{M \rightarrow \infty} \left(\lim_{L \rightarrow \infty} \sum_{\substack{ad=N \\ d>0}} \sum_{b=0}^{d-1} \sum_{(m,l) \in A(a,d;b) \cap F} \mathcal{F}(m, l : h, k) \right) \\ &= \frac{1}{kN^q} \sum_{\substack{ad=N \\ d>0}} d^{q-1} \sum_{b=0}^{d-1} t_{p,q}(ah + kb, dk) \end{aligned} \quad (17)$$

and

$$\begin{aligned} & \lim_{M \rightarrow \infty} \left(\lim_{L \rightarrow \infty} \sum_{\delta|N} \delta \sum_{(m,l) \in \delta \mathbf{Z}^2 \cap F} \mathcal{F}(m, l : h, k) \right) \\ &= \frac{N^{1-p-q}}{k} \sigma_{p+q-1}(N) t_{p,q}(h, k). \end{aligned} \quad (18)$$

By lemma 4.1, (17) equals (18). Hence, we obtain (12).

5. Proof of Theorem 2.1 in the General Case

The method in the previous section is naturally generalized to the general case.

For $a, d \in \mathbf{N}$ and $b \in \mathbf{Z}$, the additive group $A(a, d : b)$ in the previous section is also expressed as

$$A(a, d : b) = \left\{ (\mu \ v) \begin{pmatrix} a & -b \\ 0 & d \end{pmatrix} \mid \mu, v \in \mathbf{Z} \right\}.$$

Let B be an $s \times r$ matrix with components in

$\mathbf{Z}$. We define an additive subgroup of $\mathbf{Z}^s \times \mathbf{Z}^r$ as

$$\begin{aligned} & A(a, d : B) \\ &= \left\{ (\mu \ v) \begin{pmatrix} aE_s & -B \\ 0 & dE_r \end{pmatrix} \mid \mu \in \mathbf{Z}^s, v \in \mathbf{Z}^r \right\}, \end{aligned}$$

where E_s and E_r are unit matrices of degrees s and r , respectively. As shown in [20], lemma 4.1 is generalized as the following (lemma 4.3 of [20]).

Lemma 5.1 Let R be a ring extension of $\mathbf{Q}$. Let $\mathcal{F}$ be a map from $\mathbf{Z}^s \times \mathbf{Z}^r$ to R . Then for any finite subset F of $\mathbf{Z}^s \times \mathbf{Z}^r$ and for any $N \in \mathbf{N}$, we have

$$\begin{aligned} & \sum_{\substack{ad=N \\ d>0}} \sum_{B \in I_d(s,r)} \sum_{(m,l) \in A(a,d:B) \cap F} a^{(s-1)r} \mathcal{F}(m, l) \\ &= N^{(s-1)r} \sum_{\delta|N} \delta^r \sum_{(m,l) \in (\delta \mathbf{Z}^s \times \delta \mathbf{Z}^r) \cap F} \mathcal{F}(m, l). \end{aligned}$$

Let us deduce theorem 2.1 from (11) and lemma 5.1. For $\mathbf{m} = (m_1, \dots, m_s) \in \mathbf{Z}^s, \mathbf{l} = (l_1, \dots, l_r) \in \mathbf{Z}^r$ and $\mathbf{k} = (k_1, \dots, k_r) \in \mathbf{N}^r$, we put

$$\begin{aligned} & F(\mathbf{m}, \mathbf{l} : H, \mathbf{k}) = \left(\prod_{j=1}^s c_{q_j, \psi_j}(m_j) \right) \\ & \times \prod_{i=1}^r c_{p_i, \phi_i}(l_i k_i - m_1 h_{i1} - \dots - m_s h_{is}). \end{aligned}$$

Then

$$T(H, \mathbf{k} : P, Q, \Phi, \Psi) = k_1 \cdots k_r$$

$$\times \lim_{M \rightarrow \infty} \left(\lim_{L \rightarrow \infty} \sum_{\substack{-M \leq m_j \leq M \\ (1 \leq j \leq s)}} \sum_{\substack{-L \leq l_i \leq L \\ (1 \leq i \leq r)}} \mathcal{F}(m, l : H, \mathbf{k}) \right).$$

Note that we have

$$\begin{aligned} & \mathcal{F}(\delta \mathbf{m}, \delta \mathbf{l} : H, \mathbf{k}) = \\ & \left(\delta^{s(P,Q)} \phi \psi(\delta) \right)^{-1} \mathcal{F}(\mathbf{m}, \mathbf{l} : H, \mathbf{k}) \end{aligned} \quad (19)$$

for any $\delta \in \mathbf{N}$ and that for any fixed $L_1, L'_1, \dots, L_r, L'_r \in \mathbf{Z}$, we also have

$$\begin{aligned} & T(H, \mathbf{k} : P, Q, \Phi, \Psi) = k_1 \cdots k_r \\ & \times \lim_{M \rightarrow \infty} \left(\lim_{L \rightarrow \infty} \sum_{\substack{-M \leq m_j \leq M \\ (1 \leq j \leq s)}} \sum_{\substack{-L+L'_i \leq l_i \leq L+L'_i \\ (1 \leq i \leq r)}} \mathcal{F}(m, l : H, \mathbf{k}) \right). \end{aligned} \quad (20)$$

For $N, M, L \in \mathbb{N}$, we put

$$\begin{aligned} \mathcal{R}_N(M, L) &= \{(\mathbf{m}, \mathbf{l}) \in \mathbf{Z}^s \times \mathbf{Z}^r \mid \mathbf{m} \\ &= (m_1, \dots, m_s), \mathbf{l} = (l_1, \dots, l_r) \\ &\text{with } -MN \leq m_j \leq MN, -LN \leq l_i \leq LN \\ &\text{for } 1 \leq j \leq s, 1 \leq i \leq r\}. \end{aligned}$$

If $B = (b_{ij})_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}}$ is an $r \times s$ matrix with

components in $\mathbf{Z}$, then the transposed matrix ${}^t B$ of B is an $s \times r$ matrix and $(\mathbf{m} \mid \mathbf{l}) \in \mathcal{A}(a, d : {}^t B)$ is expressed as

$$(\mathbf{m} \mid \mathbf{l}) = (\boldsymbol{\mu} \mid \mathbf{v}) \begin{pmatrix} aE_s & -{}^t B \\ O & dE_r \end{pmatrix}$$

with $\boldsymbol{\mu} \in \mathbf{Z}^s$ and $\mathbf{v} \in \mathbf{Z}^r$. If $\mathbf{m} = (m_1, \dots, m_s)$, $\mathbf{l} = (l_1, \dots, l_r)$, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_s)$ and $\mathbf{v} = (v_1, \dots, v_r)$, then

$$m_j = a\mu_j \quad \text{and} \quad l_i = -\mu_1 b_{i1} - \dots - \mu_s b_{is} + v_i d$$

for $1 \leq j \leq s$ and $1 \leq i \leq r$. Hence, by direct calculations, we see that

$$\mathcal{F}(\mathbf{m}, \mathbf{l} : H, \mathbf{k}) =$$

$$\frac{d^{s(Q)} \psi(d)}{N^{s(Q)} \psi(N)} \mathcal{F}(\boldsymbol{\mu}, \mathbf{v} : aH + \mathbf{k} \circ B, d\mathbf{k}),$$

where $N = ad$. Taking $F = \mathcal{R}_N(M, L)$, we see that

$$\begin{aligned} &\sum_{\substack{ad=N \\ d>0}} \sum_{B \in \mathcal{I}_d(r, s)} \sum_{(\mathbf{m} \mid \mathbf{l}) \in \mathcal{A}(a, d : {}^t B) \cap F} a^{(s-1)r} \mathcal{F}(\mathbf{m}, \mathbf{l} : H, \mathbf{k}) \\ &= \frac{1}{N^{s(Q)-(s-1)r} \psi(N)} \sum_{\substack{ad=N \\ d>0}} d^{s(Q)-(s-1)r} \psi(d) \\ &\quad \times \sum_{B \in \mathcal{I}_d(r, s)} \sum_{\boldsymbol{\mu}} \sum_{\mathbf{v}} F(\boldsymbol{\mu}, \mathbf{v} : aH + \mathbf{k} \circ B, d\mathbf{k}), \end{aligned} \quad (21)$$

where in the above summations, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_s)$ ranges in $\mathbf{Z}^s$ such that $-Md \leq \mu_j \leq Md$ for $1 \leq j \leq s$ and $\mathbf{v} = (v_1, \dots, v_r)$ in $\mathbf{Z}^r$ such that $-LN \leq -\mu_1 b_{i1} - \dots - \mu_s b_{is} + v_i d \leq LN$ for $1 \leq i \leq r$, namely

$$\begin{aligned} &-La - [-(\mu_1 b_{i1} + \dots + \mu_s b_{is}) / d] \\ &\leq v_i \leq La + [(\mu_1 b_{i1} + \dots + \mu_s b_{is}) / d]. \end{aligned}$$

We also see that

$$\begin{aligned} &N^{(s-1)r} \sum_{\delta \mid N} \delta^r \sum_{(\mathbf{m} \mid \mathbf{l}) \in (\delta \mathbf{Z}^s \times \delta \mathbf{Z}^r) \cap F} \mathcal{F}(\mathbf{m}, \mathbf{l} : H, \mathbf{k}) \\ &= N^{(s-1)r} \sum_{\delta \mid N} \delta^r \sum_{(\mathbf{m} \mid \mathbf{l}) \in (\delta \mathbf{Z}^s \times \mathbf{Z}^r) \cap \frac{1}{\delta} F} \mathcal{F}(\delta \mathbf{m} \mid \delta \mathbf{l} : H, \mathbf{k}) \\ &= N^{(s-1)r} \sum_{\delta \mid N} \delta^{r-s(P, Q)} (\phi \psi(\delta))^{-1} \\ &\quad \times \sum_{(\mathbf{m} \mid \mathbf{l}) \in (\mathbf{Z}^s \times \mathbf{Z}^r) \cap \frac{1}{\delta} F} \mathcal{F}(\mathbf{m} \mid \mathbf{l} : H, \mathbf{k}) \text{ (by (19))} \\ &= \frac{N^{sr-s(P, Q)}}{\phi \psi(N)} \sum_{\delta \mid N} \delta^{s(P, Q)-r} \phi \psi(\delta) \\ &\quad \times \sum_{(\mathbf{m} \mid \mathbf{l}) \in (\mathbf{Z}^s \times \mathbf{Z}^r) \cap \frac{\delta}{N} F} \mathcal{F}(\mathbf{m}, \mathbf{l} : H, \mathbf{k}). \end{aligned} \quad (22)$$

Taking limits for (21) and (22), we see from (20)

that

$$\begin{aligned} &\lim_{M \rightarrow \infty} \left(\lim_{L \rightarrow \infty} \sum_{\substack{ad=N \\ d>0}} \sum_{B \in \mathcal{I}_d(r, s)} a^{(s-1)r} \mathcal{F}(\mathbf{m} \mid \mathbf{l} : H, \mathbf{k}) \right) \\ &= \frac{1}{N^{s(Q)-(s-1)r} \psi(N)} \sum_{\substack{ad=N \\ d>0}} \frac{d^{s(Q)-(s-1)r} \psi(d)}{d^r k_1 \cdots k_r} \\ &\quad \times \sum_{B \in \mathcal{I}_d(r, s)} T(aH + \mathbf{k} \circ B, d\mathbf{k} : P, Q, \Phi, \Psi) \end{aligned} \quad (23)$$

and

$$\begin{aligned} &\lim_{M \rightarrow \infty} \left(\lim_{L \rightarrow \infty} N^{(s-1)r} \sum_{\delta \mid N} \delta^r \right. \\ &\quad \times \sum_{(\mathbf{m} \mid \mathbf{l}) \in (\delta \mathbf{Z}^s \times \delta \mathbf{Z}^r) \cap F} \mathcal{F}(\mathbf{m} \mid \mathbf{l} : H, \mathbf{k}) \Big) \\ &= \frac{N^{sr-s(P, Q)}}{\phi \psi(N) k_1 \cdots k_r} \sigma_{s(P, Q)-r, \phi \psi}(N) \\ &\quad \times T(H, \mathbf{k} : P, Q, \Phi, \Psi). \end{aligned} \quad (24)$$

By lemma 5.1, (23) equals (24). Hence theorem 2.1 follows by the definition of $T(H, \mathbf{k} : P, Q, \Phi, \Psi)$.

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