

The Formalization of The Arithmetic System on The Ground of The Atomic Logic

T. J. Stepień, L. T. Stepień

The Pedagogical University of Cracow, ul. Podchorążych 2, 30 - 084 Krakow, Poland.

Received: May 21, 2015 / Accepted: June 19, 2015 / Published: September 25, 2015.

Abstract: This paper is a continuation of the paper [16]. Namely, in [16] we have introduced, among others, the definition of the atomic entailment and we have constructed the system $\overset{\Pi}{S}$, which is based on the atomic entailment. In this paper we show that the classical Arithmetic can be based on the Atomic Logic (see [17]).

Keywords: Atomic entailment, Arithmetic System, Peano's Arithmetic System, classical Arithmetic

1. Notational Preliminaries

We assume that $\rightarrow, \sim, \vee, \wedge, \equiv$ denote the connectives of implication, negation, disjunction, conjunction and equivalence, respectively. We use $\Rightarrow, \neg, \Leftrightarrow, \&, \forall, \exists$ as metalogical symbols. Next,

$At_0 = \{p, p_1, p_2, \dots, q, q_1, q_2, \dots, s, s_1, s_2, \dots, t, \dots\}$ denotes the set of all propositional variables. S_0 is the set of all well-formed formulas, which are built in the usual manner from propositional variables and by means of logical connectives. $P_0(\phi)$ denotes the set of all propositional variables occurring in ϕ ($\phi \in S_0$). R_{S_0} denotes the set of all rules over S_0 . Hence, for every $r \in R_{S_0}, \langle \Pi, \phi \rangle \in r$, where $\Pi \subseteq S_0$ and $\phi \in S_0$ and Π is a set of premisses and ϕ is a conclusion. Hence, r_*^0 denotes here the rule of simultaneous substitution for propositional variables. $\langle \{\phi\}, \psi \rangle \in r_*^0 \Leftrightarrow [h^e(\phi) = \psi]$, where h^e is the extension of the mapping $e: At_0 \rightarrow S_0$ ($e \in \varepsilon_*^0$) to endomorphism $h^e: S_0 \rightarrow S_0$, where

1. $h^e(\phi) = e(\phi)$, for $\phi \in At_0$
2. $h^e(\sim\phi) = \sim h^e(\phi)$
3. $h^e(\phi F\psi) = h^e(\phi) F h^e(\psi)$, for $F \in \{\rightarrow, \vee, \wedge, \equiv\}$ and for every $\phi, \psi \in S_0$.

Thus, ε_*^0 is a class of functions $e: At_0 \rightarrow S_0$ (for details, see [5], cf. [2]). r_0^0 denotes here the Modus Ponens rule in propositional calculus. $R_{0*} = \{r_0^0, r_*^0\}$ (for details, see [2], [5]). A logical matrix is a pair $\mathfrak{M} = \{U, U'\}$, U is an abstract algebra and U' is a subset of the universe U , i.e. $U' \subseteq U$. Any $a \in U'$ is called a distinguished element of the matrix \mathfrak{M} . $E(\mathfrak{M})$ is the set of all formulas valid in the matrix \mathfrak{M} . \mathfrak{M}_2 denotes the classical two-valued matrix.

Hence, Z_2 is the set of all formulas valid in the classical matrix \mathfrak{M}_2 (see [2], [5]).

The symbols x_1, x_2, \dots are individual variables. a_1, a_2, \dots are individual constants. V is the set of all individual variables. C is the set of all individual constants. P_i^n ($i, n \in \mathcal{N} = \{1, 2, \dots\}$) are n -ary predicate letters. The symbols f_i^n ($i, n \in \mathcal{N}$) are n -ary function letters. The symbols $\wedge x_k, \vee x_k$ are quantifiers. $\wedge x_k$ is the universal quantifier and $\vee x_k$ is the existential quantifier. The function letters, applied to the individual variables and individual constants, generate terms. The symbols t_1, t_2, \dots are terms. T is the set of all terms. $V \cup C \subseteq T$.

The predicate letters, applied to terms, yield simple formulas, i.e. if P_i^k is a predicate letter and t_1, \dots, t_k are terms, then $P_i^k(t_1, \dots, t_k)$ is a simple formula.

Corresponding author: L. T. Stepień, The Pedagogical University of Cracow, Kraków, Poland. E-mail: sfstepie@cyf-kr.edu.pl, <http://www.ltstepien.up.krakow.pl>.

Smp is the set of all simple formulas. Next, At_1 is the set of all atomic formulas, where $At_1 = \{P_i^k(x_{j_1}, \dots, x_{j_k}) : k, i, j_1, \dots, j_k \in \mathcal{N}\}$. At last, S_1 is the set of all well-formed formulas. $FV(\phi)$ denotes the set of all free variables occurring in ϕ , where $\phi \in S_1$. $x_k \in Ff(t_m, \phi)$ expresses that x_k is free for term t_m in ϕ . By x_k/t_m we denote the substitution of the term t_m for the individual variable x_k . $P_1(\phi)$ denotes the set of all predicate letters occurring in ϕ ($\phi \in S_1$). If $FV(\phi) = \{x_1, \dots, x_k\}$, then $\wedge \phi = \wedge x_1 \dots \wedge x_k \phi$.

R_{S_1} denotes the set of all rules over S_1 . Hence, for every $r \in R_{S_1}$, $\langle \Pi, \phi \rangle \in r$, where $\Pi \subseteq S_1$ and $\phi \in S_1$ and Π is a set of premisses and ϕ is a conclusion. Hence, r_*^1 denotes here the rule of simultaneous substitution for predicate letters. $\langle \{\phi\}, \psi \rangle \in r_*^1 \Leftrightarrow [h^e(\phi) = \psi]$, where h^e is the extension of the mapping $e: Smp \rightarrow S_1$ ($e \in \varepsilon_*^1$) to endomorphism $h^e: S_1 \rightarrow S_1$, where

1. $h^e(\phi) = e(\phi)$, for $\phi \in Smp$
2. $h^e(\sim \phi) = \sim h^e(\phi)$
3. $h^e(\phi F \psi) = h^e(\phi) F h^e(\psi)$,

for $F \in \{\rightarrow, \vee, \wedge, \equiv\}$

4. $h^e(\wedge x_k \phi) = \wedge x_k h^e(\phi)$
5. $h^e(\vee x_k \phi) = \vee x_k h^e(\phi)$ for every

$\phi, \psi \in S_1$ and $k \in \mathcal{N}$ (for details, see [6], [7]).

Next, r_0^1 denotes the Modus Ponens rule in predicate calculus, r_+^1 denotes the generalization rule. $R_{0+} = \{r_0^1, r_+^1\}$, $R_{0*+} = \{r_0^1, r_*^1, r_+^1\}$. We write $X \subset Y$, if $X \subseteq Y$ and $X \neq Y$.

We assume here that for every $\alpha \in S_1$, if $FV(\alpha) = \{x_1, \dots, x_n\}$, then $\alpha^* = \vee x_1 \dots \vee x_n \sim \alpha$. Hence, for every $\alpha \in S_1$, if $FV(\alpha) = \emptyset$, then $\alpha^* = \sim \alpha$. Analogically, for every $\alpha \in S_0$, $\alpha^* = \sim \alpha$.

Finally, for any $X \subseteq S_i$ and $R \subseteq R_{S_i}$, $Cn_i(R, X)$ is the smallest subset of S_i , containing X and closed under the rules $R \subseteq R_{S_i}$, where $i \in \{0, 1\}$. The couple $\langle R, X \rangle$ is called a system, whenever $R \subseteq R_{S_i}$ and $X \subseteq S_i$ and $i \in \{0, 1\}$. $Syst \cap A_0$ denotes here the class of all systems $\langle R, X \rangle$, which are based on an atomic entailment, where $R \subseteq R_{S_0}$ and $X \subseteq S_0$.

$Syst \cap A_1$ denotes here the class of all systems $\langle R, X \rangle$, which are based on an atomic entailment, where $R \subseteq R_{S_1}$ and $X \subseteq S_1$. $Syst \cap C_1$ denotes here the class of all systems $\langle R, X \rangle$, which are based on a classical entailment, where $R \subseteq R_{S_1}$ and $X \subseteq S_1$.

$\phi \left| \frac{A_0}{R, X} \right. \psi$ denotes that ψ results atomically from ϕ , on the ground of the system $\langle R, X \rangle$, where $R \subseteq R_{S_0}$ and $X \subseteq S_0$. Next, $\phi \left| \frac{A_1}{R, X} \right. \psi$ denotes that ψ results atomically from ϕ , on the ground of the system $\langle R, X \rangle$, where $R \subseteq R_{S_1}$ and $X \subseteq S_1$. At last, $\phi \left| \frac{C_1}{R, X} \right. \psi$ denotes that ψ results classically from ϕ , on the ground of the system $\langle R, X \rangle$, where $R \subseteq R_{S_1}$ and $X \subseteq S_1$ (see [16]).

Definition 1.1. The function $j: S_1 \rightarrow S_0$, is defined, as follows (see [16]):

- (1) $j(P_k^n(t_1, \dots, t_n)) = p_k$ ($p_k \in At_0$)
- (2) $j(\sim \phi) = \sim j(\phi)$
- (3) $j(\phi F \psi) = j(\phi) F j(\psi)$, for $F \in \{\rightarrow, \vee, \wedge, \equiv\}$
- (4) $j(\wedge x_k \phi) = j(\vee x_n \phi) = j(\phi)$.

2. Classical Entailment

Definition 2.1. Let $Cn_1(R, X) = L \neq \emptyset$ and

$\phi, \psi \in S_1$. Then $\phi \left| \frac{C_1}{R, X} \right. \psi$ iff the following

conditions are satisfied, [14], [16]:

- (1) $(\forall e \in \varepsilon_*^1)[h^e(\wedge \phi) \in L \Rightarrow h^e(\psi) \in L]$
- (2) $(\forall e \in \varepsilon_*^1)[h^e((\psi^* \rightarrow \phi^*) \rightarrow \phi^*) \in L \Rightarrow h^e(\phi^*) \in L]$.

Definition 2.2. $\langle R, X \rangle \in Syst \cap C_1$ iff the following condition is satisfied, [14], [16]:

$$(\forall \phi, \psi \in S_1) \left[\wedge \phi \rightarrow \psi \in Cn_1(R, X) \Leftrightarrow \phi \left| \frac{C_1}{R, X} \right. \psi \right].$$

3. The Classical Logic

Let L_2 denote the set of all formulas valid in the classical predicate calculus.

Thus, (cf. [6] pp. 68 – 74):

Theorem 3.1. $Cn_1(R_{0*+}, L_2) = L_2$.

4. Atomic Entailment

In [14], [15] and [16], we have introduced the following definitions (cf. [12], [13]):

Definition 4.1. Let $Cn_0(R, X) = L \neq \emptyset$ and

$\phi, \psi \in S_0$. Then $\phi \Big|_{R, X}^{A_0} \psi$ iff the following

conditions are satisfied:

- (1) $(\forall e \in \varepsilon_*^0)[h^e(\phi) \in L \Rightarrow h^e(\psi) \in L$
 $\& P_0(h^e(\phi)) \subseteq P_0(h^e(\psi))]$
- (2) $(\forall e \in \varepsilon_*^0)[h^e((\psi^* \rightarrow \phi^*) \rightarrow \phi^*) \in L \Rightarrow$
 $h^e(\phi^*) \in L \& P_0(h^e(\psi^*)) \subseteq P_0(h^e(\phi^*))]$.

Definition 4.2. $\langle R, X \rangle \in Syst \cap A_0$ iff the following condition is satisfied:

$$(\forall \phi, \psi \in S_0) \left[\phi \rightarrow \psi \in Cn_0(R, X) \Leftrightarrow \phi \Big|_{R, X}^{A_0} \psi \right].$$

Definition 4.3. Let $Cn_1(R, X) = L \neq \emptyset$ and

$\phi, \psi \in S_1$. Then $\phi \Big|_{R, X}^{A_1} \psi$ iff the following

conditions are satisfied:

- (1) $(\forall e \in \varepsilon_*^1)[h^e(\wedge \phi) \in L \Rightarrow h^e(\psi) \in L$
 $\& P_1(h^e(\wedge \phi)) \subseteq P_1(h^e(\psi))]$
- (2) $(\forall e \in \varepsilon_*^1)[h^e((\psi^* \rightarrow \phi^*) \rightarrow \phi^*) \in L \Rightarrow$
 $h^e(\phi^*) \in L \& P_1(h^e(\psi^*)) \subseteq P_1(h^e(\phi^*))]$.

Definition 4.4. $\langle R, X \rangle \in Syst \cap A_1$ iff the following condition is satisfied:

$$(\forall \phi, \psi \in S_1) \left[\wedge \phi \rightarrow \psi \in Cn_1(R, X) \Leftrightarrow \phi \Big|_{R, X}^{A_1} \psi \right].$$

5. The Atomic Logic

Let us take the matrix (see [16])

$\mathfrak{M}_D = \langle \{0, 1, 2\}, \{1, 2\}, f_D^{\rightarrow}, f_D^{\equiv}, f_D^{\vee}, f_D^{\wedge}, f_D^{\sim} \rangle$, where:

f_D^{\rightarrow}	0	1	2
0	1	1	1
1	0	1	0
2	0	1	2

f_D^{\equiv}	0	1	2
0	1	0	0
1	0	1	0
2	0	0	2

f_D^{\vee}	0	1	2
0	0	1	0
1	1	1	1
2	0	1	2

f_D^{\wedge}	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

f_D^{\sim}	
0	1
1	0
2	2

It should be noticed here that the matrix $\mathfrak{M}'_D = \langle \{0, 1, 2\}, \{1, 2\}, f_D^{\rightarrow}, f_D^{\sim} \rangle$ was investigated by B. Sobocinski (see [9], [10]).

Next, we define T_D , putting:

Definition 5.1. $T_D = E(\mathfrak{M}'_D)$.

In [10] (see [11]) we have proved the following:

Theorem 5.2. The system $\langle R_{0*}, T_D \rangle$ is axiomatizable.

Theorem 5.3. Let $\phi, \psi \in S_0$ and

$(\exists e \in \varepsilon_*^0)[h^e(\phi) \in T_D]$.

Then $\phi \rightarrow \psi \in Cn_0(R_{0*}, T_D)$ iff

$$(\forall e \in \varepsilon_*^0)[h^e(\phi) \in T_D \Rightarrow h^e(\psi) \in T_D$$

$$\& P_0(h^e(\phi)) \subseteq P_0(h^e(\psi))].$$

Now we define the system \hat{S} of the Atomic Logic,

as follows:

Definition 5.4. $\hat{S} = \langle R_0, T_D \rangle$.

Next, we define the set L_D , putting (see [16]):

Definition 5.5. $L_D = \{ \phi \in L_2 : j(\phi) \in T_D \}$.

The system $\overset{\square}{S}$ of the Atomic Logic,

is defined, as follows (see [16]):

Definition 5.6. $\overset{\square}{S} = \langle R_{0+}, L_D \rangle$.

6. The Fundamental Properties of the Systems

$\langle R_0, T_D \rangle$, $\langle R_{0+}, L_D \rangle$ and $\langle R_{0+}, L_2 \rangle$

In [16] we have proved the following:

Theorem 6.1. $\langle R_0, T_D \rangle \in Syst \cap A_0$.

Theorem 6.2. $\langle R_{0+}, L_D \rangle \in Syst \cap A_1$.

Theorem 6.3. $\langle R_{0+}, L_2 \rangle \in Syst \cap C_1$.

7. The Main Result

Arithmetic terminology.

Let S_A denote the set of all well-formed formulas of the Arithmetic System. $FV_A(\phi)$ denotes the set of all free variables occurring in ϕ , where $\phi \in S_A$. Hence, $\bar{S}_A = \{\phi \in S_A : FV_A(\phi) = \emptyset\}$. $Pr(\phi)$ denotes the set of all predicate letters occurring in ϕ , where $\phi \in S_A$. R_{S_A} denotes the set of all rules over S_A . For any $X \subseteq S_A$ and for any $R \subseteq R_{S_A}$, $Cn(R, X)$ is the smallest subset of S_A , containing X and closed under the rules of R . The couple $\langle R, X \rangle$ is called a system, whenever $R \subseteq R_{S_A}$ and $X \subseteq S_A$. $R_{0+}^P = \{r_0^P, r_+^P\}$, where $\{r_0^P, r_+^P\} \subseteq R_{S_A}$. r_0^P, r_+^P are Modus Ponens and generalization rule in the Arithmetic System, respectively. Next (cf. [3], [4], [8]):

- (1) $\psi^1: \Lambda x_1 x_1 = x_1$
- (2) $\psi^2: \Lambda x_1 \Lambda x_2 (x_1 = x_2 \rightarrow x_2 = x_1)$
- (3) $\psi^3: \Lambda x_1 \Lambda x_2 \Lambda x_3 (x_1 = x_2 \rightarrow (x_2 = x_3 \rightarrow x_1 = x_3))$
- (4) $\psi^4: \Lambda x_1 \Lambda x_2 \Lambda x_3 \Lambda x_4 (x_1 = x_2 \rightarrow (x_3 = x_4 \rightarrow (x_1 + x_3 = x_2 + x_4)))$
- (5) $\psi^5: \Lambda x_1 \Lambda x_2 \Lambda x_3 \Lambda x_4 (x_1 = x_2 \rightarrow (x_3 = x_4 \rightarrow (x_1 \cdot x_3 = x_2 \cdot x_4)))$
- (6) $\psi^6: \Lambda x_1 \Lambda x_2 \Lambda x_3 \Lambda x_4 (x_1 = x_2 \rightarrow (x_3 = x_4 \rightarrow (x_1 < x_3 \rightarrow x_2 < x_4)))$
- (7) $\psi^7: \Lambda x_1 \sim(1 = x_1 + 1)$
- (8) $\psi^8: \Lambda x_1 \Lambda x_2 (x_1 + 1 = x_2 + 1 \rightarrow x_1 = x_2)$
- (9) $\psi^9: \Lambda x_1 \Lambda x_2 (x_1 + (x_2 + 1) = (x_1 + x_2) + 1)$
- (10) $\psi^{10}: \Lambda x_1 (x_1 \cdot 1 = x_1)$
- (11) $\psi^{11}: \Lambda x_1 \Lambda x_2 [x_1 \cdot (x_2 + 1) = (x_1 \cdot x_2) + x_1]$
- (12) $\psi^{12}: \Lambda x_1 \Lambda x_2 [x_1 < x_2 \equiv \forall x_3 (x_1 + x_3 = x_2)]$
- (13) $\psi^{13}: (A(1) \wedge \Lambda x_1 (A(x_1) \rightarrow A(x_1 + 1))) \rightarrow$

$\Lambda x_1 A(x_1)$,

where $A(1), A(x_1), A(x_1 + 1) \in S_A$.

Next, $X_P = \{\psi^1, \psi^2, \psi^3, \psi^4, \psi^5, \psi^6, \psi^7, \psi^8, \psi^9, \psi^{10}, \psi^{11}, \psi^{12}\}$.

Y_P denotes here the set of all axioms of induction. At last, L_2^r and X_r denote the set of all logical axioms of the Arithmetic System and the set of all specific axioms of the Arithmetic System, where

$L_2^r, X_r \subseteq S_A$, respectively. Sx denotes here the successor of x (see [1]).

ψ^{14} denotes the formula (see [3])

$\Lambda x_1 \Lambda x_2 [\forall x_3 (Sx_3 + x_1 = x_2) \equiv (x_1 < x_2)]$.

Definition 7.1. The function $i: S_A \rightarrow S_0$, is defined, as follows:

- (1) $i(t_n = t_m) = p_k (p_k \in At_0)$
- (2) $i(t_n < t_m) = p_s (p_s \in At_0)$
- (3) $i(\sim \phi) = \sim i(\phi)$
- (4) $i(\phi F \psi) = i(\phi) F i(\psi)$, for $F \in \{\rightarrow, \vee, \wedge, \equiv\}$
- (5) $i(\Lambda x_k \phi) = i(\forall x_n \phi) = i(\phi)$,

where $\phi, \psi \in S_A$.

Definition 7.2. $\langle R_{0+}^P, L_2^r \cup X_r \rangle$ is the Arithmetic System, where $X_r = X_P \cup Y_P$ (see [3], [4], [8]).

In [8], one can read that the System $\langle R_{0+}^P, L_2^r \cup X_r \rangle$ is a modification of Peano's Arithmetic System.

Next,

Definition 7.3. $L_D^r = \{\phi \in L_2^r : i(\phi) \in T_D\}$.

Theorem 7.4.

$Cn(R_{0+}^P, L_D^r \cup X_r) = Cn(R_{0+}^P, L_2^r \cup X_r)$, where $X_r = X_P \cup Y_P$.

Proof. Let

- (1) $\alpha \in L_2^r - L_D^r$

and

- (2) $X_r = X_P \cup Y_P$.

From **Definition 5.1.**, **Definition 5.5.**, **Definition 7.1.** and **Definition 7.3.**, it follows that

- (3) $(\forall \phi \in L_2^r) [Pr(\phi) \subseteq \{=\} \Rightarrow \phi \in L_D^r]$

and

- (4) $(\forall \phi \in L_2^r) [Pr(\phi) \subseteq \{<\} \Rightarrow \phi \in L_D^r]$.

From (1), (3) and (4), it follows that

- (5) $< \in Pr(\alpha)$

and

- (6) $= \in Pr(\alpha)$.

From (1), (3), (4), (5), (6), **Definition 5.1.**,

Definition 5.5., **Definition 7.1.**, and the definition of the formula ψ^{12} , it follows that

$$(7) i(\psi^{12} \rightarrow \alpha) \in T_D$$

and

$$(8) \psi^{12} \rightarrow \alpha \in L_2^r.$$

Hence, from (1), (5), (6), by **Definition 5.1.**,

Definition 5.5. and **Definition 7.3.**, it follows that

$$(9) \psi^{12} \rightarrow \alpha \in L_D^r.$$

Hence, from (1), (2), (3), (4), **Definition 7.2.** and

Definition 7.3., it follows that

$$(10) Cn(R_{0+}^P, L_D^r \cup X_r) = Cn(R_{0+}^P, L_2^r \cup X_r). \quad \square$$

Thus, by the proof of **Theorem 7.4.**, one can obtain

(see [17]):

Conclusion 7.5.

Every Arithmetic System $\langle R_{0+}^P, L_2^r \cup X_r \rangle$ can be based on the system of the Atomic Logic $\langle R_{0+}, L_D^r \rangle$, where

$$\psi^{12} \in Cn(R_{0+}^P, L_2^r \cup X_r)$$

or

$$\psi^{14} \in Cn(R_{0+}^P, L_2^r \cup X_r)$$

and $(\forall \phi \in L_2^r \cup X_r)[Pr(\phi) \subseteq \{=, <\}]$,

(cf. [1] p.530 – 541).

References

- [1] B. Buldt. The Scope of Gödel's First Incompleteness Theorem, *Logica Universalis*, 8:499 – 552, 2014.
- [2] Yu. L. Ershov and E. A. Palyutin. *Mathematical Logic*. Mir Publishers, Moscow, 1984.
- [3] A. Grzegorzczuk. *An Outline of Mathematical Logic*. *Fundamental Results and Notions Explained with All Details*, D. Reidel Publishing Company, Dordrecht-Holland/Boston-USA, PWN, Warszawa, 1974.
- [4] R. Murawski. *Recursive Functions And Metamathematics. Problems of Completeness and Decidability, Gödel's Theorems*. Springer Science+Business Media Dordrecht, 1999.
- [5] W. A. Pogorzelski. *The Classical Propositional Calculus*. PWN, Warszawa, 1975.
- [6] W. A. Pogorzelski. *The Classical Calculus of Quantifiers*. PWN, Warszawa, 1981.
- [7] W. A. Pogorzelski and T. Prucnal. The substitution rule for predicate letters in the first-order predicate calculus. *Reports on Mathematical Logic*, 5:77 – 90, 1975.
- [8] H. Rasiowa. *Introduction to Modern Mathematics*. North-Holland Publishing Company, 1973.
- [9] B. Sobocinski. Axiomatization of partial system of three-valued calculus of propositions. *The Journal of Computing Systems*, 1:23 – 55, 1952.
- [10] T. Stepien. System \bar{S} . *Reports on Mathematical Logic*, 15:59 – 65, 1983.
- [11] T. Stepien. System \bar{S} . *Zentralblatt für Mathematik*, 471, 1983.
- [12] T. Stepien. Logic based on atomic entailment. *Bulletin Of The Section Of Logic*, 14:65-71, 1985.
- [13] T. Stepien. Logic Based On Atomic Entailment And Paraconsistency. *11th International Congress Of Logic, Methodology And Philosophy Of Science* (August 1999, Krakow, Poland).
- [14] T. J. Stepien and L. T. Stepien. Atomic Entailment and Classical Entailment, *The Bulletin of Symbolic Logic*, 17:317 – 318, 2011.
- [15] T. J. Stepien and L. T. Stepien. Atomic Entailment and Atomic Inconsistency, *6th International Conference "Non-classical logics. Theory & Applications"* (4 – 6 September 2013, Lodz, Poland).
- [16] T. J. Stepien and L. T. Stepien. Atomic Entailment and Atomic Inconsistency and Classical Entailment. *Journal of Mathematics and System Science*, 5:60 – 71, 2015.
- [17] T. J. Stepien and L. T. Stepien. The formalization of the arithmetic system on the ground of atomic logic. (Logic Colloquium 2015, 3 – 8 August 2015, Helsinki, Finland), to appear in *The Bulletin of Symbolic Logic*.