Repeating Fractions and Primes

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Abstracts: In base-10 number system, the repeating fractions possess quite lot more properties than could be thought, including natures of primes.

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Fermat's Little Theorem says:

$$a^{p-1} \equiv 1 \pmod{p} \quad (1)$$

p is a prime, and a is a natural number except 1.

i.e.:

$$a^{p-1} - 1 = R \cdot p \quad (2)$$

This is also the expression for repeating decimals, and R is the repeating part. Because a repeating decimal is a convergent geometric series of $R10^{-i(p-1)}$ as the i -th item, $a=10$, and of which the sum is $R \cdot 10^{p-1} / (10^{p-1} - 1)$. For example,

$$10^2 - 1 = 33 \cdot 3, \text{ i.e. } a=10, p=3 \text{ and } R=33$$

$$10^6 - 1 = 142857 \cdot 7, \text{ i.e. } a=10, p=7 \text{ and } R=142857$$

" R " is the integer of the repeating part with " $p-1$ " as its number of digits. Fractions of reciprocals of primes, except 2 and 5, are repeating decimals. Because there are only a limited number of $p-1$ remainders, i.e. the series from 1 to $p-1$, for the p as a divisor. When we talk about decimals they are of base-10 ($2 \cdot 5$) positioned number system unless otherwise, and that's why the primes 2 and 5 are excluded.

Now let's zoom in the repeating part R of a prime p .

The number of digits of R is the same as the number of the remainders of a prime p , i.e. $p-1$, but with different value domains, 0 to 9 for the digits of R and 1 to $p-1$ for the remainders of p . Thanks to the fact that all primes are odd numbers, the remainders' domain can be rewrite as $(1-p)/2$ to $(p-1)/2$, except 0.

From equation (2), the left can be rewritten as below:

$$(a-1)(a^{p-2} + a^{p-3} + \dots + a + 1) \quad (3)$$

The second expression is a $p-1$ digits number of base- a positioned, i.e. $11\dots11$ of $p-1$ digits. It is apparent to say that either $a-1$ or the number $11\dots11$ contains all the primes, for example, assuming $a=10$, the all primes except 2 and 5 are hosted in the numbers of the $11\dots11$ form of $p-1$ digits or a number of digits of one of the $p-1$'s divisible, i.e.

$$11,$$

$$111=3 \cdot 37,$$

$$1111=11 \cdot 101,$$

$$11111=41 \cdot 271,$$

$$111111=3 \cdot 7 \cdot 11 \cdot 13 \cdot 37,$$

$$1111111=239 \cdot 4649,$$

$$11111111=11 \cdot 73 \cdot 13837,$$

$$111111111=3 \cdot 3 \cdot 37 \cdot 333667,$$

$$1111111111=11 \cdot 41 \cdot 271 \cdot 9091,$$

$$11111111111, \text{ not sure whether is a prime or not,}$$

$$111111111111=3 \cdot 7 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \cdot 101 \cdot 9901,$$

$$1111111111111=53 \cdot 79 \cdot 265371653, \text{ etc.}$$

.....

From equation (3), it is also an expression of base- a positioned number, with only two symbols 1 and $a-1$ (e.g. 9 for $a=10$, the same as $1\bar{1}$ of 2-digit number if minus sign "-" is used in number expression). The minimum of a is 2. For base- a positioned numbers, a number of $a/2$ or $(a-1)/2$ symbols in addition to 0 are needed. And it is found that base-10 could be base-2 or base-3 or any base, and base- a positioned numbers can be expressed without 0. For example, let $a=10$,

then base-a positioned number 302 can be expressed as $2a^2$.

Let's assume r_i be the i-th digit of R, and d_i the i-th remainder, then comes the below:

$$d_i = a^i - Q_i p, \quad i=0 \text{ to } p-1, \text{ and } d_{p-1} = d_0 = 1, \quad Q_0 = 0$$

$$d_1 = ad_0 - q_1 p = a - q_1 p$$

$$d_2 = ad_1 - q_2 p$$

.....

$$d_i = ad_{i-1} - q_i p = a^i - p \sum q_j a^{i-j}, \quad Q_i = \sum q_j a^{i-j}, \quad j=1 \text{ to } i \text{ and}$$

$$q_0 = Q_0$$

Sum up both sides of the above equations, then comes the below:

$$\sum d_j = a(\sum d_j - d_i + d_0) - p \sum q_j, \text{ and then,}$$

$$(a-1) \sum d_j = a(d_i - d_0) + p \sum q_j$$

So, suppose d_i 's traverse all the remainders of the prime p as a divisor, i.e. $i=1$ to $p-1$, then $d_i = d_0$, and there, there is the Fermat Little Theorem.

//End in Shenzhen on Apr 2014