A way finding $r(k,l)$ and $r(3,10)=41$

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Abstract: On basis of two definitions that 1. an induced subgraph by a vertex $v_i \in G$ and its neighbors in G is defined a vertex adjacent closed subgraph denoted by $Q_i (=G[V(Nv_i)])$, with the vertex v_i called the hub; 2. A $r(k,l)$ -1 vertices connected graph is called a (k,l) -Ramsey graph denoted by $RG(k,l)$, if and only if 1. $RG(k,l)$ contains only cliques of degree $k-1$, and its complement contains only cliques of degree $l-1$; 2. the intersect $Q_i \cap Q_j$ of any two nonadjacent vertices v_i and v_j of $RG(k,l)$ contains K_{k-2} , and the intersect $Q_i \cap Q_j$ of any two nonadjacent vertices v_i and v_j of its complement $RG(l,k)$ contains K_{l-2} . Two theorems that theorem1: the biggest clique in G is contained in some Q_i of G , and theorem2: $r(k,l) = |V(RG(k,l))| + 1$ are put forward and proved in this paper. With those definitions and theorems as well as analysis of property of chords a method for quick inspection and building $RG(k,l)$ is proposed. Accordingly, $RG(10,3)$ and its complement are built, which are respectively the strongly 29-regular graph and the strongly 10-regular graph on orders 40. We have tested $RG(10,3)$ and its complement $RG(3,10)$, and gotten $r(3,10)=41$.

Keyword: The vertex adjacent closed subgraph; the hub; the Ramsey number $r(10,3)$; Ramsey graph $RG(10,3)$

1. Introduction

Let G be a simple graph. P vertices of G are numbered clockwise as $i=0,1,\dots,p-1$. A vertex v_i could adjacent to vertices labeled as $i \pm 1, i \pm 2, \dots, \{i \pm (p-1)/2$ (when p is an odd number); $i \pm p/2$ (when p is an even number) (mod p). An edge between two adjacent vertices is called a chord, the at least number of edges between two adjacent vertices (the numeral behind the $\pm$) is called the chord length. According to their values chords are classified into two categories, the odd-length and the even-length chords, called an odd and an even chord for short respectively. The undefined terms and symbols used in this paper can be found in [1].

A clique in G is a subset K of $V(G)$ such that $G[K]$ is complete. The vertex number of K is called the clique degree, and denoted by k . A subset L of $V(G)$ is called an vertex independent set if no two vertices of L are adjacent in G . The vertex number of L is called the independent degree, denoted by l . Clearly, K is a

clique of G if and only if K is an independent set of G 's complement G^c . Therefore, the two concepts are complementary.

If G has no large cliques, then one might expect G to have a large independent set. That this indeed the case was first proved by Ramsey (1930). He showed that given any positive integers k and l , there exists a smallest integer $r(k,l)$ such that every graph on $r(k,l)$ vertices contains either a clique of k vertices or an independent set of l vertex. The number $r(k,l)$ is called the Ramsey number.

At present, the known nine Ramsey numbers are $r(3,3)=6$, $r(3,4)=9$, $r(3,5)=14$, $r(3,6)=18$, $r(3,7)=23$, $r(3,8)=28$, $r(3,9)=36$, $r(4,4)=18$ and $r(4,5)=25$.

The last one, $r(4,5)=25$, was obtained by S. P. Radziszowski and B. D. Mackay in 1995[2]. It nearly cost equivalent 11 years workload of a standard computer before $RG(4,5)$ was constructed and proved. So much computation workload was spent because resolving Ramsey number problems involved finding cliques and independent sets in $RG(4,5)$. They are determining NP-C problems[3]. Although the current computers are quite powerful, they still could do

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nothing for NP-C problems of G with large orders while lacking of effective algorithms. But we have gotten a good method. By which we have manually constructed and validated $RG(3,10)$ in a short time, and have gotten that $r(3,10)=41$.

2. Definitions and Theorems

Definition1: an induced subgraph by $v_i \in G$ and its neighbors in G is defined the vertex adjacent closed subgraph denoted by $Q_i (=G[V(Nv_i)])$, with the vertex v_i called the hub.

Theorem1: the biggest clique in G must be contained in some Q_i of $G[4]$.

Proof end symbol: it can be proved directly by definitions of clique and Q_i .

There is no clear and strict definition of (k,l) -Ramsey graph. Its definition is extended out generally in accordance with the definition of Ramsey number. According to the fuzzy definition, there are multiple (k,l) -Ramsey graphs, one of them is r -regular graph. And even if got all (k,l) -Ramsey graphs, also just get a lower bound of the $r(k,l)$. So that to find Ramsey number is leaded into error way, such that to waste a lot of manpower, material resources to find little significance of lower and upper bounds of $r(k,l)$. And to make it very complex and hard!

We give below (k,l) -Ramsey graph a clear strict definition.

Definition2 A $r(k,l)$ -1 vertices connected graph is called a (k,l) -Ramsey graph, denoted by $RG(k,l)$, if and only if 1. $RG(k,l)$ contains only cliques of $k-1$ vertices, and its complement contains only cliques of $l-1$ vertices; 2. the intersect $Q_i \cap Q_j$ of any two nonadjacent vertices v_i and v_j of $RG(k,l)$ contains K_{k-2} , and the intersect $Q_i \cap Q_j$ of any two nonadjacent vertices v_i and v_j of its complement $RG(l,k)$ contains K_{l-2} .

The essence of definition2 is that a complete graph on $r(k,l)$ -1 vertices is decomposed into two graphs $RG(k,l)$ and its complement $RG(l,k)$ such that $RG(k,l)$ contains only K_{k-1} , its complement $RG(l,k)$ contains only K_{l-1} , and they cannot contain other cliques besides

K_{k-1} and K_{l-1} .

According to the definition and theorem1 as well as analysis of chord property we ask required $RG(k,l)$ must be a r -regular graph, it has a fixed point, and its each Q_i is symmetrical on the hub v_i ; all Q_i is homogeneous.

When $k=l$, maybe, some a $RG(k,k)$ may be not a regular graph. So we study only $RG(k,l)$ when $k \neq l$ in this paper.

Theorem2 $r(k,l) = |V(RG(k,l))| + 1$.

Proof let required $RG(k,l)$ be a r -regular graph, then its complement $RG(l,k)$ must be $(p-r-1)$ -regular graph. Next, if it can be proved that adding a vertex such that either $RG(k,l)$ contains a clique K_k or its complement $RG(l,k)$ contains a clique K_l , then the theorem is established.

Adding a vertex u to $RG(k,l)$ or its complement $RG(l,k)$, the cases that u is adjacent to p vertices in $RG(k,l)$ or its complement $RG(l,k)$ must be the following one of two cases:

Case1, u is adjacent to at least $r+1$ vertices in $RG(k,l)$, or

Case2, u is adjacent to at least $p-r$ vertices in its complement $RG(l,k)$.

In case1, if u is adjacent to $r+1$ vertices of any Q_i of $RG(k,l)$, then this constitutes K_k in $RG(k,l)$, the theorem is established; else if u is adjacent to r vertices of a Q_i except the hub v_i , so that $RG(k,l)$ contains only K_{k-1} , then u is again adjacent to a vertex v_j which does not connect the hub v_i , so that $RG(k,l)$ contains K_k . Because the intersect $Q_i \cap Q_j$ of any two nonadjacent vertices v_i and v_j of $RG(k,l)$ contains K_{k-2} from definition2, that u and v_j are all adjacent to the K_{k-2} makes up K_k .

Similarly, in case2, such the adding vertex u that $RG(k,l)$'s complement $RG(l,k)$ contains a clique K_l .

From what has been discussed above the theorem is established.

3. Analysis and Construction of $RG(k, l)$

According to theorem2 and definition2, we should

construct directly a pair of regular graphs, and need only to construct and to test any Q_i of this pair of regular graphs, can determine the exact value of $r(k,l)$. Don't cost time and effort to test large amounts of lower and up bounds of $r(k,l)$.

Let G be a complete graph on p vertices. When p is an odd number it has $(p-1)/2$ kinds of chords, and the length of the longest chord $= (p-1)/2$, the quantity of chords of each kind is p .

When p is an even number, it has $p/2$ kinds of chords, and the length of the longest chord $= p/2$, and the quantity of the longest chords $= p/2$, the quantity of other chords is p . Endpoint-labels of an even chord both are either two odd, or two even numbers. But for an odd chord's endpoint-labels, one is an odd number the other is an even number.

Regardless of an odd or even chord, using a unique kind of chords can constitute cliques K_2 . But even if using all odd chords, could not constitute any clique greater than K_2 when $p > 3$. But using multiple kinds of even chords can constitute cliques with degree from 3 up to $p/2$.

From definition2 known that $RG(k,l)$ contains only K_{k-1} , its complement contains only K_{l-1} . According to the theorem1, every Q_i of $RG(k,l)$ contains too only K_{k-1} , and every Q_i of its complement contains too only K_{l-1} .

Therefore, we need only consider to construct Q_0 of $RG(k,l)$ in addition to the hub, by which kinds of chords can compose of K_{k-2} to construct $RG(k,l)$, and by which kinds of chords can compose of K_{l-2} to construct $RG(l,k)$ step by step.

3.1 Constitute $RG(10,3)$ and its complement

From above we known that $RG(10,3)$ contains only K_9 , its complement contains only K_2 , while they do not contain other cliques except for K_9 and K_2 . According to theorem1, every Q_i of $RG(10,3)$ contains too only K_9 , and every Q_i of its complement $RG(3,10)$ contains too only K_2 .

So let us consider only Q_0 of $RG(10,3)$ besides the

hub v_0 , by which kinds of chord can compose of K_8 to construct $RG(10,3)$, and by which kinds of chord will disjoin at any vertex to construct $RG(3,10)$ step by step.

First, let $\pm 2, \pm 4, \dots, \pm 14$ fourteen kinds of even chords constitute two cliques K_7 , then eight kinds of odd chords $\pm 13, \pm 15, \pm 17, \pm 19$ and with between of them seven kinds of even chords together form a K_8 . The chords 1, 3, 9, 11 and 16 should be in its complement, for they do not intersect each other. Due to the symmetry, thereby infer that $RG(10,3)$ should be a strongly 29-regular graph, and its complement is a strongly 10-regular graph on order 40. The constructed Q_0 of $RG(10,3)$ is shown in Fig. 1.

It is convenient to use vertex adjacency listing represents $RG(k,l)$ and its complement. After numbering clockwise the vertices with $0, 1, 2, \dots, p-1$, to use each of p linear array represents each vertex adjacency. Its first element is a vertex v_i , and the remaining elements are simply adjacent to the first element, whose label is that the first vertex label $i +$ (or $-$) the length of chords (mod p).

In this representation $RG(10,3)$ will appear as follows:

$i: \pm 2, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 10, \pm 12, \pm 13, \pm 14, \pm 15, \pm 17, \pm 18, \pm 19, \pm 20 \pmod{40}; i = 0, 1, 2, \dots, 39$. See Fig. 2.

The complementary graph of $RG(10,3)$ is represented as below:

$i: \pm 1, \pm 3, \pm 9, \pm 11, \pm 16, i = 0, 1, 2, \dots, 39 \pmod{40}; i = 0, 1, 2, \dots, 39$. See Fig. 3.

4. The testing method that $RG(k,l)$ contains only K_{k-1}

By constructing $RG(k,l)$ and theorem1 we know that all of Q_i of $RG(k,l)$ are isomorphic and each of Q_i contains only K_{k-1} while not contain other cliques. Therefore, let's select arbitrarily a vertex from $RG(k,l)$, delete it and its nonadjacent vertices. As a result the remaining graph will contain only K_{k-2} . Repeating this operation $k-2$ times, the final remains should be

Fig.1 Q_{0-0} of $RG(10,3)$ Fig.2 $RG(10,3)$

isolated vertices. It proves that $RG(k,l)$ contains only K_{k-1} . Because the vertex deleted each time and its nonadjacent vertices could not compose of any clique, but they can only with the rest of vertices every time to compose of K_{k-1} .

4.1 Testing that $RG(10,3)$ contains only K_9

From $RG(10,3)$, we select arbitrarily a vertex, such as v_0 , delete it and its nonadjacent vertices. As a

Fig.3 $RG(3,10)$

result, the remaining graph contains only K_8 (see Fig.1). Next, repeat this operation, from the remained graph select and delete step by step a vertex and its non-adjacent vertices until doing 7 operations, the remains must be isolated vertices. This is the proof that $RG(10,3)$ contains only K_9 .

By above operations, we have tested every vertex in Fig.2, the results are same. That $RG(10,3)$ contains only K_9 is proved.

4.2 Testing that $RG(3,10)$ contains only K_2

By above operation, we delete a vertex and its nonadjacent vertices from $RG(3,10)$ showed in Fig.3. For example, delete v_0 and its nonadjacent vertices. The remaining graph is a null graph, i.e. some isolated vertices. This prove that $RG(3,10)$ contains only K_2 .

4.3 Testing that the Intersect $Q_i \cap Q_j$ of any two Nonadjacent Vertices v_i and v_j of $RG(3,10)$ is Isolated Vertices

Since $RG(3,10)$ contains only K_2 . And by definition2 the intersect $Q_i \cap Q_j$ of any two nonadjacent vertices v_i and v_j of $RG(3,10)$ should be isolated vertices. Due to Q_0 's symmetry we need only to test that the $Q_0 \cap Q_j$ of both v_0 and each of its nonadjacent vertices v_j to be respectively 2,4,5,6,7,8,10,12,13,14,

15,17,18,19,20 is isolated vertices.

The testing results are: $Q_0 \cap Q_2$ is $v_1, v_3, v_{11}, v_{31}, v_{39}$; $Q_0 \cap Q_4$ is v_1, v_3 ; $Q_0 \cap Q_5$ is v_{16}, v_{29} ; $Q_0 \cap Q_6$ is v_3, v_9, v_{37} ; $Q_0 \cap Q_7$ is v_{16}, v_{31} ; $Q_0 \cap Q_8$ is $v_9, v_{11}, v_{37}, v_{39}$; $Q_0 \cap Q_{10}$ is v_1, v_9, v_{11}, v_{39} ; $Q_0 \cap Q_{12}$ is v_1, v_3, v_9, v_{11} ; $Q_0 \cap Q_{13}$ is $v_{16}, v_{24}, v_{29}, v_{37}$; $Q_0 \cap Q_{14}$ is v_3, v_{11} ; $Q_0 \cap Q_{15}$ is $v_{16}, v_{24}, v_{31}, v_{39}$; $Q_0 \cap Q_{17}$ is v_1, v_{16} ; $Q_0 \cap Q_{18}$ is v_9, v_{17}, v_{29} ; $Q_0 \cap Q_{19}$ is v_3, v_{16} and $Q_0 \cap Q_{20}$ is $v_9, v_{11}, v_{29}, v_{31}$.

4.4 Testing that the intersect $Q_i \cap Q_j$ of any two nonadjacent vertices v_i and v_j of $RG(10,3)$ contains a clique K_8

Due to symmetry we need to test that the $Q_0 \cap Q_j$ of vertex 0 and its nonadjacent vertices v_j to be respectively 1,3,9,11,16 contains a clique K_8

The testing results are: $Q_0 \cap Q_1$ contains $K_8(v_5, v_7, v_{13}, v_{15}, v_{19}, v_{27}, v_{33}, v_{35})$; $Q_0 \cap Q_3$ contains $K_8(v_5, v_7, v_{13}, v_{15}, v_{17}, v_{25}, v_{30}, v_{35})$; $Q_0 \cap Q_9$ contains $K_8(v_2, v_4, v_{17}, v_{19}, v_{21}, v_{23}, v_{27}, v_{35})$; $Q_0 \cap Q_{11}$ contains $K_8(v_{13}, v_{15}, v_{17}, v_{19}, v_{21}, v_{23}, v_{25}, v_{38})$ and $Q_0 \cap Q_{16}$ contains $K_8(v_2, v_4, v_6, v_8, v_{10}, v_{12}, v_{14}, v_{34})$.

They all meet the definition2.

5. $r(3,10)=41$

Since $|V(RG(3,10))|=40$, and from theorem 2 $r(k,l)=|V(RG(k,l))|+1$, so we gotten: $r(3,10)=40+1=41$.

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