Elementary Functions of a Quaternion Variable and Some Applications

Bimurat Sagindykov

Candidate (PhD) of Physical and Mathematical sciences, KazNTU

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Abstract: In this paper, an effective formula for the calculation of the elementary functions of a quaternion variable obtained using the methods of differential equations. Also the elementary functions are obtained from the quaternion matrices.

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1. Quaternion Algebra

According to Hamilton, quaternion is a mathematical object which we can write in the form [1]

$$q \equiv q_0 + iq_1 + jq_2 + kq_3, \quad (1)$$

where q_0, q_1, q_2, q_3 are real numbers, q is called quaternion's components, the basis element «1» is the identity element of q_0 , and i, j, k are three imaginary units. Quaternion product is denoted by « $\circ$ » sign; and defined by the following rules for quaternion units multiplication given by Hamilton's definition:

$$i i = j j = k k = -1, \quad i j = k, \quad j k = i, \quad k i = j. \quad (2)$$

We may also recognize i, j, k as unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ of Cartesian basis, and then by analogy with complex numbers, quaternion $q \in H$ can be represented as a formal sum of the scalar part q_0 and the vector part $\vec{q}$:

$$q = q_0 + q_1\vec{e}_1 + q_2\vec{e}_2 + q_3\vec{e}_3 = q_0 + \vec{q}, \quad (3)$$

and the multiplication rules of basis vectors (2) can be expressed in the form of scalar and vector products:

$$\vec{e}_i \vec{e}_j = -(\vec{e}_i \cdot \vec{e}_j) + \vec{e}_i \times \vec{e}_j$$

or

$$\vec{e}_i \vec{e}_j = -(\vec{e}_i \cdot \vec{e}_j) + \varepsilon_{ijk}\vec{e}_k,$$

where ε_{ijk} – Levi-Civita symbols, and $i, j, k = 1, 2, 3$.

These relationships allow us to interpret quaternions multiplication $\Lambda = \lambda_0 + \vec{\lambda}$, $M = \mu_0 + \vec{\mu}$ via scalar and vector products

$$\Lambda M = \lambda_0\mu_0 - (\vec{\lambda} \cdot \vec{\mu}) + \lambda_0\vec{\mu} + \mu_0\vec{\lambda} + \vec{\lambda} \times \vec{\mu}. \quad (4)$$

It follows from multiplication rules for quaternion imaginary units that q multiplication is non-commutative

$$q_1 q_2 \neq q_2 q_1,$$

so, there is a concept of the left and right multiplication although q multiplication is still associative

$$(q_1 q_2) q_3 = q_1 (q_2 q_3).$$

Following the procedure of obtaining conjunction we introduce the operation of quaternion conjunction

$$q^* \equiv q_0 - iq_1 - jq_2 - kq_3$$

and define the modulus of q - number

$$|q| = \sqrt{q q^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

2. Analogue of Euler's Formula

We present an analogue of Euler's formula as an example of quaternion's operations. For this, let us consider quaternion's exponential function in the form:

$$e^{q\varphi} = e^{(q_0 + iq_1 + jq_2 + kq_3)\varphi}, \quad (5)$$

where φ is a variable of the quaternion's variable.

By virtue of the fact that q_0 is a real number, it

commutes with its basis unit as well as with other imaginary units:

$$e^{(q_0+iq_1+jq_2+kq_3)\varphi} = e^{q_0\varphi} \cdot e^{(iq_1+jq_2+kq_3)\varphi}. \quad (6)$$

Assume that the exponential function is given in the next form:

$$e^{(iq_1+jq_2+kq_3)\varphi} = A_0(\varphi) + iA_1(\varphi) + jA_2(\varphi) + kA_3(\varphi).$$

Taking derivative in (6) with respect to φ we obtain the following equation:

$$\begin{aligned} (iq_1 + jq_2 + kq_3)e^{(iq_1+jq_2+kq_3)\varphi} &= A'_0(\varphi) + iA'_1(\varphi) + jA'_2(\varphi) \\ &+ kA'_3(\varphi) \end{aligned}$$

or

$$\begin{aligned} (iq_1 + jq_2 + kq_3)[A_0(\varphi) + iA_1(\varphi) + jA_2(\varphi) \\ + kA_3(\varphi)] &= A'_0(\varphi) + iA'_1(\varphi) + jA'_2(\varphi) \\ &+ kA'_3(\varphi) \end{aligned}$$

Hence it follows from quaternion's equality that

$$\begin{aligned} A'_0(\varphi) &= -a_1A_1(\varphi) - a_2A_2(\varphi) - a_3A_3(\varphi), \\ A'_1(\varphi) &= -a_1A_0(\varphi) - a_3A_2(\varphi) + a_2A_3(\varphi), \\ A'_2(\varphi) &= a_2A_0(\varphi) + a_3A_1(\varphi) - a_1A_3(\varphi), \\ A'_3(\varphi) &= a_3A_0(\varphi) + a_1A_2(\varphi) - a_2A_1(\varphi). \end{aligned} \quad (7)$$

Having differentiated the first equation with respect to φ variable we get

$$\begin{aligned} A''_0(\varphi) &= -a_1A'_1(\varphi) - a_2A'_2(\varphi) - a_3A'_3(\varphi) = \\ &= -a_1[-a_1A_0(\varphi) - a_3A_2(\varphi) + a_2A_3(\varphi)] - \\ &= a_2[a_2A_0(\varphi) + a_3A_1(\varphi) - a_1A_3(\varphi)] - \\ &= a_3[a_3A_0(\varphi) + a_1A_2(\varphi) - a_2A_1(\varphi)] = (-a_1^2 - a_2^2 - \\ &= a_3^2)A_0(\varphi). \end{aligned}$$

то есть

$$A''_0(\varphi) + (a_1^2 + a_2^2 + a_3^2)A_0(\varphi) = 0. \quad (8)$$

We apply initial conditions for equation (8), i.e. at $\varphi = 0$:

$$A_0(\varphi) = 1, A'_0(\varphi) = 0.$$

After differentiating second, third and forth equations in (7) with respect to φ variable we obtain similar equations of the type (8) for $A_1(\varphi)$, $A_2(\varphi)$ and $A_3(\varphi)$ under appropriate initial conditions:

$$A''_1(\varphi) + (a_1^2 + a_2^2 + a_3^2)A_1(\varphi) = 0, \text{ here at } \varphi = 0,$$

$$A_1(\varphi) = 0, A'_1(\varphi) = q_1;$$

$$A''_2(\varphi) + (a_1^2 + a_2^2 + a_3^2)A_2(\varphi) = 0, \text{ here at } \varphi = 0,$$

$$A_2(\varphi) = 0, A'_2(\varphi) = q_2;$$

$$A''_3(\varphi) + (a_1^2 + a_2^2 + a_3^2)A_3(\varphi) = 0, \text{ here at } \varphi = 0,$$

$$A_3(\varphi) = 0, A'_3(\varphi) = q_3.$$

Solving these equations under corresponding initial conditions we have:

$$\begin{aligned} A_0(\varphi) &= \cos \sqrt{q_1^2 + q_2^2 + q_3^2} \varphi, \\ A_1(\varphi) &= \frac{q_1}{\sqrt{q_1^2 + q_2^2 + q_3^2}} \sin \sqrt{q_1^2 + q_2^2 + q_3^2} \varphi, \\ A_2(\varphi) &= \frac{q_2}{\sqrt{q_1^2 + q_2^2 + q_3^2}} \sin \sqrt{q_1^2 + q_2^2 + q_3^2} \varphi, \\ A_3(\varphi) &= \frac{q_3}{\sqrt{q_1^2 + q_2^2 + q_3^2}} \sin \sqrt{q_1^2 + q_2^2 + q_3^2} \varphi. \end{aligned}$$

In turn, an analogue of Euler's formula for quaternion can be written as

$$\begin{aligned} e^{(iq_1+jq_2+kq_3)\varphi} &= \cos \sqrt{q_1^2 + q_2^2 + q_3^2} \varphi + \\ &+ \frac{iq_1+jq_2+kq_3}{\sqrt{q_1^2 + q_2^2 + q_3^2}} \sin \sqrt{q_1^2 + q_2^2 + q_3^2} \varphi. \end{aligned} \quad (9)$$

Hence if the quaternion is used as an argument of elementary function, it can be represented as a conditional complex number with a conditional imaginary unit:

$$\begin{aligned} q &= q_0 + \frac{iq_1+jq_2+kq_3}{\sqrt{q_1^2 + q_2^2 + q_3^2}} \sqrt{q_1^2 + q_2^2 + q_3^2}, \\ q &= Q_0 + IQ_1, \text{ где } Q_0 = q_0, \\ I &= \frac{iq_1+jq_2+kq_3}{\sqrt{q_1^2 + q_2^2 + q_3^2}}, Q_1 = \sqrt{q_1^2 + q_2^2 + q_3^2}. \end{aligned}$$

Here the conditioned imaginary unit has a vector meaning in which it is a unit vector directed along $\vec{I} = \text{Im } q$ vector. In such notation the quaternion retains complex number's properties:

$$\begin{aligned} q^2 &= Q_0^2 + Q_1^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2, \\ q^2 &= q \cdot q = q_0^2 - (q_1^2 + q_2^2 + q_3^2) \\ &+ 2q_0(iq_1 + jq_2 + kq_3) \\ &= Q_0^2 - Q_1^2 + 2IQ_0Q_1 \\ I^2 &= -1. \end{aligned}$$

Using these properties, we can find the elementary functions of a quaternion variable. For this, we

(1) replace the quaternion with a conditional complex number $q \Rightarrow Q_0 + IQ_1$;

(2) expand an elementary function as a function of a complex variable $Q_0 + IQ_1$;

(3) and after that proceed to a converse replacement $Q_0 \Rightarrow q_0$,

$$I \Rightarrow \frac{iq_1 + jq_2 + kq_3}{\sqrt{q_1^2 + q_2^2 + q_3^2}},$$

$$Q_1 \Rightarrow \sqrt{q_1^2 + q_2^2 + q_3^2}.$$

Some elementary functions of quaternion are written below for illustrative purposes.

$$\begin{aligned} \sin q &= \sin(Q_0 + IQ_1) = \\ \sin Q_0 \cos IQ_1 + \cos Q_0 \sin IQ_1 &= \sin Q_0 \operatorname{ch} Q_1 + \\ I \cos Q_0 \operatorname{sh} Q_1 &= \sin q_0 \operatorname{ch} \sqrt{q_1^2 + q_2^2 + q_3^2} + \\ \frac{iq_1 + jq_2 + kq_3}{\sqrt{q_1^2 + q_2^2 + q_3^2}} \cos q_0 \operatorname{sh} \sqrt{q_1^2 + q_2^2 + q_3^2}, \\ \cos q &= \cos(Q_0 + IQ_1) = \cos Q_0 \cos IQ_1 - \\ \sin Q_0 \sin IQ_1 &= \cos Q_0 \operatorname{ch} Q_1 - I \sin Q_0 \sin Q_1 = \\ \cos q_0 \operatorname{ch} \sqrt{q_1^2 + q_2^2 + q_3^2} - \\ \frac{iq_1 + jq_2 + kq_3}{\sqrt{q_1^2 + q_2^2 + q_3^2}} \sin q_0 \operatorname{sh} \sqrt{q_1^2 + q_2^2 + q_3^2}. \end{aligned}$$

3. Matrix Form of Quaternion Algebra

It is interesting to present quaternion multiplication in matrix form [2]. Let

$$\begin{aligned} A &= a_0 + ia_1 + ja_2 + ka_3, \\ B &= b_0 + ib_1 + jb_2 + kb_3. \end{aligned}$$

Then the product of two quaternions gives third quaternion $C = AB$ and the resulting quaternion components are defined by the formula (4):

$$\begin{aligned} C_0 &= a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3, \\ C_1 &= a_1b_0 + a_0b_1 - a_3b_2 + a_2b_3, \\ C_2 &= a_2b_0 + a_0b_2 + a_1b_3 - a_3b_1, \end{aligned}$$

$$\begin{aligned} G_m(A^*) &= G_m^T(A), & G_m(A+B) &= G_m(A) + G_m(B), & m &= 1, 2; \\ G_1(A)G_2(B) &= G_1(A)G_2(B), & G_2(A)G_1(B) &= G_2(A)G_1(B), \\ G_1(A)G_2(B) &= G_2(B)G_1(A), & \det G_m(A) &= \|A\|^4, & m &= 1, 2. \end{aligned}$$

Lemma. For any $A \neq 0$ quaternion the following equations are valid: $G_1(A^{-1}) = G_1^{-1}(A)$, $G_2(A^{-1}) = G_2^{-1}(A)$.

Using G_1 and G_2 matrices we can easily replace equations written in quaternions to equations in matrices. In particular, for $A = BCD$ quaternion the matrix form will be as follows:

$$C_3 = a_3b_0 - a_2b_1 + a_1b_2 + a_0b_3.$$

Next we assign four-dimensional $\vec{V}_A = (a_0, a_1, a_2, a_3)^T$ vector to A quaternion and four-dimensional $\vec{V}_B = (b_0, b_1, b_2, b_3)^T$ vector to B quaternion, respectively.

Then C quaternion can be associated with its own four-dimensional vector defined by as follows:

$$\vec{V}_C = \vec{V}_{AB} = G_1(A) \cdot \vec{V}_B = G_2(B) \cdot \vec{V}_A. \quad (10)$$

Matrix $G_1(A)$ and matrix $G_2(B)$ in the expression (10) equal to

$$G_1(A) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}, \quad G_2(B) = \begin{bmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & b_3 & -b_2 \\ b_2 & -b_3 & b_0 & b_1 \\ b_3 & b_2 & -b_1 & b_0 \end{bmatrix}.$$

For an arbitrary quaternion Q matrices $G_1(A)$ and $G_2(B)$ can be represented as

$$\begin{aligned} G_1(Q) &= \begin{pmatrix} q_0 & -\vec{q}^T \\ \vec{q} & q_0 E_3 + K(\vec{q}) \end{pmatrix}, \\ G_2(Q) &= \begin{pmatrix} q_0 & -\vec{q}^T \\ \vec{q} & q_0 E_3 - K(\vec{q}) \end{pmatrix} \end{aligned}$$

where E_3 is 3-by-3 unity matrix,

$$K(\vec{q}) = \begin{pmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{pmatrix}.$$

Some properties of $K(\vec{a})$ matrix:

$$\begin{aligned} K(\vec{a}) \cdot \vec{r} &= \vec{a} \times \vec{r}, & K(\vec{a}) \cdot \vec{a} &= \vec{a} \times \vec{a} = 0, \\ K^T(\vec{a}) &= -K(\vec{a}), & K(\vec{a})K(\vec{b}) &= \vec{b} \times \vec{a} - \vec{a}^T \cdot \vec{b} \cdot E_3. \end{aligned}$$

Some properties of G_1 and G_2 matrices:

$$A = G_1(B) \cdot G_2(D) \cdot \vec{C},$$

where $\vec{C} = (c_0, c_1, c_2, c_3)^T$ is a four-dimensional vector related to C quaternion.

4. Matrix Exponential

We apply spectral decomposition of matrix function so that to find matrix exponential.

There is an isomorphism between quaternions $q = q_0 + iq_1 + jq_2 + kq_3$ and special form 4-by-4 matrix in terms of quaternion and matrix operations.

$$Q = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix}.$$

Let find characteristic polynomial of Q quaternion matrix

$$\det(Q - \lambda E) = [(q_0 - \lambda)^2 + q_1^2 + q_2^2 + q_3^2]^2.$$

Then the complex values $\lambda = q_0 + i\sqrt{q_1^2 + q_2^2 + q_3^2}$, $\bar{\lambda} = q_0 - i\sqrt{q_1^2 + q_2^2 + q_3^2}$ are the eigenvalues of the quaternion $q = q_0 + iq_1 + jq_2 + kq_3$.

The minimal polynomial of Q quaternion matrix is given by

$$\mu_Q(\lambda) = \frac{(-1)^n \det(Q - \lambda E)}{D_{n-1}(\lambda)},$$

where $D_{n-1}(\lambda)$ is $(Q - \lambda E)$ characteristic matrix's the greatest common divisor of subdeterminant of order $(n - 1)$.

In our case $n = 4$, $D_{n-1} = (q_0 - \lambda)^2 + q_1^2 + q_2^2 + q_3^2$. Then

$$\begin{aligned} \mu_Q(\lambda) &= (q_0 - \lambda)^2 + q_1^2 + q_2^2 + q_3^2 \\ &= (\lambda - q_0 - i\langle q \rangle) \cdot (\lambda - q_0 + i\langle q \rangle) \end{aligned}$$

where $\langle q \rangle = \sqrt{q_1^2 + q_2^2 + q_3^2}$.

Or

$$\begin{aligned} \mu_Q(\lambda) &= [\lambda - (q_0 + i|q - q_0|)] \\ &\quad \cdot [\lambda - (q_0 - i|q - q_0|)] \end{aligned}$$

where $\langle q \rangle = |q - q_0|$. Here $i^2 = -1$.

Then the basic formula for $f(Q)$ is as follows:

$$f(Q) = f(\lambda_1)z_{11} + f(\lambda_2)z_{21},$$

where z_{11} , z_{21} Q matrix components, and

$$\lambda_1 = q_0 + i|q - q_0|,$$

$$\lambda_2 = q_0 - i|q - q_0|.$$

Substituting $\lambda - \lambda_1$, $\lambda - \lambda_2$ consistently in place of $f(\lambda)$ we obtain

$$-2i|q - q_0|z_{21} = Q - \lambda_1 E,$$

$$2i|q - q_0|z_{11} = Q - \lambda_2 E,$$

where E is a unity matrix.

Hence

$$\begin{aligned} 2i|q - q_0|f(Q) &= \\ (Q - \lambda_2 E)f(\lambda_1) - (Q - \lambda_1 E)f(\lambda_2) \end{aligned} \quad (11)$$

Let us consider some applications of this formula.

If $f(\lambda) = \frac{1}{\lambda}$, then $\frac{1}{\lambda_1}$, $\frac{1}{\lambda_2}$ numbers are formula's

value on Q matrix spectrum. Therefore, thus function is defined on Q matrix spectrum.

For this reason the basic formula (11) can be used to find Q^{-1} inverse matrix.

Substituting the values of $f(\lambda_1) = \frac{1}{\lambda_1} = \frac{\lambda_2}{\Delta}$,

$f(\lambda_2) = \frac{1}{\lambda_2} = \frac{\lambda_1}{\Delta}$ into the basic formula $f(Q)$ we

have $Q^{-1} = \frac{1}{\Delta}(-Q + 2q_0 E)$. Here

$$\Delta = q_0^2 + q_1^2 + q_2^2 + q_3^2.$$

Validity of the obtained formula can be verified by direct calculation.

Now we find the exponent of quaternion matrix.

For this, consider $f(\lambda) = e^\lambda$ function, which is also defined Q matrix spectrum

$$\begin{aligned} \exp(Q) &= e^{q_0} \left[\frac{\sin|q - q_0|}{|q - q_0|} Q \right. \\ &\quad \left. + \left(\cos|q - q_0| - \frac{q_0 \sin|q - q_0|}{|q - q_0|} E \right) \right] \end{aligned}$$

By proceeding this process we can get entire spectrum of elementary matrix of Q quaternion matrix.

References

- [1] W.R.Hamilton, Selecta: Optics. Dynamics. Quaternions. Moscow, Nauka, 1994. (Russian edition)
- [2] R.Bellman, Introduction to matrix analysis. Moscow, Nauka, 1969, p. 368. (Russian edition)