Modified Kantorovich Operators Providing a Better Error Estimation

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Abstract: The approximation operator for every kind of the objective function is different in approximation theory. For Lebesgue function, we introduce a kind of modified Kantorovich operators, which preserve the test functions 1 and x^2 . This type of modification enables better error estimation on the interval $[\frac{\sqrt{3}}{3}, +\infty)$ than the classic ones. Finally, a Voronovskaya-type theorem for these operators is also obtained.

Key words: Kantorovich operators; modulus of continuity; Voronovskaya-type theorem.

1. Introduction

The approximation operator for every kind of the objective function is different in approximation theory. It is well known that many classical operators preserve the test functions 1 and x but do not preserve the test function x^2 .

King-type approximation operators [1-7] preserving the test functions 1 and x^2 , and have better approximation properties than the classical ones. Motivated by this, we introduce a kind of modified Kantorovich operators, which preserve the test functions 1 and x^2 and have better error estimation on the interval $[\frac{\sqrt{3}}{3}, +\infty)$ than the classical and the modified Szász-Kantorovich operators discussed in Duman [1].

Then, the classical Szász-Mirakjan-Kantorovich operators are defined by

$$S_n^*(f, x) = n \sum_{k=0}^{\infty} s_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt, \quad (1)$$

where $s_{n,k}(x) = \frac{(nx)^k}{k!} e^{-nx}$,

$$f \in C_\beta[0, +\infty) :=$$

$$\{f \in C[0, +\infty) : |f(t)| \leq M(1+t)^\beta$$

$$\text{for some } M > 0, \beta > 0\}$$

$$\text{Let } f \in C_\beta[0, +\infty), r_n(x) = \frac{-1 + \sqrt{n^2 x^2 + \frac{2}{3}}}{n}, x \geq \frac{\sqrt{3}}{3}$$

and $n \in N$, then we get the following modified positive linear operators:

$$S_n^{**}(f, x) = n \sum_{k=0}^{\infty} s_{n,k}(r_n(x)) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt, \quad (2)$$

We will give the moments and convergence theorem of our operators, which preserving 1 and x^2 .

The main result of this paper is:

Theorem. Let $f \in C_B[0, +\infty)$, the space of all bounded functions on $[0, +\infty)$, for $x \geq \frac{\sqrt{3}}{3}, n \in N$, we have

$$|S_n^{**}(f, x) - f(x)| \leq 2\omega\left(f, \sqrt{\frac{x}{n} - \frac{1}{2n^2}}\right),$$

where the modulus of continuity of f denoted by $\omega(f, \delta_x)$ for $\delta_x > 0$, is defined to be

$$\omega(f, \delta_x) = \sup_{|t-x| \leq \delta_x, t \in [0, +\infty)} |f(t) - f(x)|.$$

Throughout this paper, M denotes a positive constant independent of n and x and not necessarily the same at each occurrence.

2. The Convergence Theorem of the Modified Operators

From Lemma A in [1], we obtain the following result.

Lemma 2.1 For each $x \geq \frac{\sqrt{3}}{3}$, we have

- (1). $S_n^{**}(1, x) = 1$;
- (2). $S_n^{**}(t, x) = \frac{\sqrt{n^2 x^2 + 2/3}}{n} - \frac{1}{2n}$;
- (3). $S_n^{**}(t^2, x) = x^2$.

By Lemma 2.1, it is clear that the operators S_n^{**} given by (2) preserve the test functions 1 and x^2 . Then from Lemma 2.1, one can get the following results for moments.

Lemma 2.2 For each $x \geq \frac{\sqrt{3}}{3}$, we have

- (1). $S_n^{**}(t - x, x) = r_n(x) + \frac{1}{2n} - x = \frac{\sqrt{n^2 x^2 + 2/3}}{n} - \frac{1}{2n} - x$;
- (2). $S_n^{**}((t - x)^2, x) = 2x^2 + \frac{x}{n} - \frac{2x\sqrt{n^2 x^2 + 2/3}}{n^2}$;
- (3). $S_n^{**}((t - x)^2, x) \leq \frac{x}{n} - \frac{1}{2n^2}$.

From Lemma 2.1, 2.2 and with the Korovkin-type property, we have the following convergence theorem.

Theorem 2.1 Let $f \in C_B[0, +\infty)$, $x \geq \frac{\sqrt{3}}{3}$, we have

$$\lim_{n \rightarrow \infty} S_n^{**}(f, x) = f(x).$$

3. Better Error Estimation

Theorem 3.1 Let $f \in C_B[0, +\infty)$, $x \geq \frac{\sqrt{3}}{3}$, $n \in N$, we have

$$|S_n^{**}(f, x) - f(x)| \leq 2\omega(f, \delta_{n,x}),$$

where $\delta_{n,x} = \sqrt{\frac{x}{n} - \frac{1}{2n^2}}$.

Proof. Let $f \in C_B[0, +\infty)$ and $x \geq 0$, using linearity and monotonicity of the operators S_n^{**} , for every $\delta > 0, n \in N$, we get

$$|S_n^{**}(f, x) - f(x)| \leq \omega(f, \delta) \left(1 + \frac{1}{\delta} \sqrt{S_n^{**}((t - x)^2, x)} \right).$$

Applying Lemma 2.2 and choosing $\delta = \delta_{n,x}$, the proof is complete.

Remark 3.1 (1). [8-10] For the Szasz-Kantorovich operators given by (1), we may write that, for $f \in C_B[0, +\infty)$ and $x \geq 0$,

$$|S_n^*(f, x) - f(x)| \leq 2\omega(f, \alpha_{n,x}), \quad (3)$$

where $\alpha_{n,x} = \sqrt{\frac{x}{n} + \frac{1}{3n^2}}$.

(2). [1] For the modified Szasz-Kantorovich operators given by Duman, one has

$$|K_n^*(f, x) - f(x)| \leq 2\omega(f, \delta'_{n,x}), \quad (4)$$

where $\delta'_{n,x} = \sqrt{\frac{x}{n} - \frac{5}{12n^2}}$, and $K_n^*(f, x)$ is defined by

$$K_n^*(f, x) = n \sum_{k=0}^{\infty} s_{n,k} \left(x - \frac{1}{2n} \right) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt, \quad (5)$$

$$x \in \left[\frac{1}{2}, +\infty \right).$$

(3). We can see that the error estimation in Theorem 3.1 is better than that of (3) and (4) provided $f \in C_B[0, +\infty)$, $x \geq \frac{\sqrt{3}}{3}$.

Indeed, it is clear that

$$\frac{x}{n} - \frac{1}{2n^2} < \frac{x}{n} - \frac{5}{12n^2} < \frac{x}{n} + \frac{1}{3n^2}, \quad (6)$$

which guarantees that $\delta_{n,x} < \delta'_{n,x} < \alpha_{n,x}$ for $x \geq \frac{\sqrt{3}}{3}$.

We say that a bounded function $f \in C[0, +\infty)$ belongs to $Lip_M(\alpha)$ if the inequality

$|f(t) - f(x)| \leq M |t - x|^\alpha$ holds for all $t \in [0, +\infty)$.

Theorem 3.2 For every $f \in Lip_M(\alpha)$, $x \geq \frac{\sqrt{3}}{3}$ and $n \in N$, we have

$$|S_n^{**}(f, x) - f(x)| \leq M \left\{ \frac{x}{n} - \frac{1}{2n^2} \right\}^{\frac{\alpha}{2}}.$$

Proof. Since $f \in Lip_M(\alpha)$, $x \geq 0$, using the Hölder inequality with $p = \frac{2}{\alpha}, q = \frac{2}{2-\alpha}$, we have

$$\begin{aligned} |S_n^{**}(f, x) - f(x)| &\leq S_n^{**}(|f(t) - f(x)|, x) \\ &\leq MS_n^{**}(|t - x|^\alpha, x) \\ &\leq M \left(S_n^{**}(|t - x|^2, x) \right)^{\frac{\alpha}{2}} \leq M \left\{ \frac{x}{n} - \frac{1}{2n^2} \right\}^{\frac{\alpha}{2}}. \end{aligned}$$

Remark 3.2 The classical and modified Szász-Kantorovich operators given by (1) and (5) satisfy

$$|S_n^*(f, x) - f(x)| \leq M \left\{ \frac{x}{n} + \frac{1}{3n^2} \right\}^{\frac{\alpha}{2}}, \quad (7)$$

$$|K_n^*(f, x) - f(x)| \leq M \left\{ \frac{x}{n} - \frac{5}{12n^2} \right\}^{\frac{\alpha}{2}}, \quad (8)$$

respectively for $f \in Lip_M(\alpha)$, $x \geq \frac{\sqrt{3}}{3}$ and $n \in N$.

It follows from (6) that the rate of convergence of the operators S_n^{**} for the Lipschitz class functions is better than the error estimation given by (7) and (8) whenever $x \geq \frac{\sqrt{3}}{3}$.

4. A Voronovskaya-type Theorem

Along the same lines of the proof of Theorem 4.2 in [1], we have a Voronovskaya-type theorem for the operators S_n^{**} given by (2).

Theorem 4.1 For every $f \in C_\beta[0, +\infty)$ such that $f', f'' \in C_\beta[0, +\infty)$, we have

$$\lim_{n \rightarrow \infty} n \{ S_n^{**}(f, x) - f(x) \} = -\frac{1}{2} f'(x) + \frac{x}{2} f''(x)$$

uniformly with respect to $x \in [\frac{\sqrt{3}}{3}, b]$ ($b > \frac{\sqrt{3}}{3}$).

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