Expanded Relative Operator Entropies and Operator Valued $\alpha$-Divergence

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Received: March 06, 2015 / Accepted: April 03, 2015 / Published: June 25, 2015.

Abstract: For strictly positive operators $A$ and $B$, and for $x \in [0,1]$ and $r \in [-1,1]$, we investigate the operator power mean

$$A \#_{x,r} B \equiv A^{\frac{1}{1-r}} \left( (1-x)I + x \left( A^{-1} B A^{-1} \right)^{\frac{1}{r}} \right)^{\frac{1}{1-r}} A^{\frac{1}{1-r}}.$$ 

If $r = 0$, this is reduced to the geometric operator mean $A \#_0 B \equiv A \frac{1}{2} \left( (I + A^{-1} B A^{-1})^T \right)^{\frac{1}{2}} A \frac{1}{2}$. Since $A \#_{0,r} B = A$ and $A \#_{1,r} B = B$, we regard $A \#_{x,r} B$ as a path combining $A$ and $B$. $S(A|B) \equiv \frac{d}{dx} A \#_x B \bigg|_{x=0}$ is the relative operator entropy given by Fujii and Kamei, and $S_\alpha(A|B) \equiv \frac{d}{dx} A \#_x B \bigg|_{x=t}$ is the generalized one defined by Furuta. We consider these expansion on $A \#_{1,r} B$ and define an expanded relative operator entropy

$$S_{t,r}(A|B) \equiv \frac{d}{dx} A \#_{x,r} B \bigg|_{x=t} = A^{\frac{1}{1-r}} \left[ (1-x)I + t \left( A^{-1} B A^{-1} \right)^{\frac{1}{r}} \right]^{\frac{1}{1-r}} \left( A^{-1} B A^{-1} \right)^{\frac{1}{r}} A^{\frac{1}{1-r}}.$$ 

Our aim is to show the essential properties of $S_{t,r}(A|B)$. The Tsallis relative operator entropy by Yanagi, Kuriyama and Furuichi can also be expanded, and by using this, we can give an expanded operator valued $\alpha$-divergence and obtain its properties.

Key words: Operator power mean, relative operator entropy, Tsallis relative operator entropy, operator valued $\alpha$-divergence

1. Introduction

Throughout this paper, an operator means a bounded linear operator on a Hilbert space $H$. An operator $T$ on $H$ is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and an operator $T$ is said to be strictly positive (denoted by $T > 0$) if $T$ is invertible and positive.

In probability theory and information theory, Kullback-Leibler divergence (relative entropy) is one of notions to measure the difference between two probability distributions. As a quantization of Kullback-Leibler divergence, Umegaki [20] introduced the following (relative) entropy

$$s(A|B) \equiv Tr A (\log A - \log B)$$

for density matrices $A$ and $B$, that is, $A,B > 0$ and $Tr A = Tr B = 1$.

As an operator version of these notions, Nakamura and Umegaki [16] defined operator entropy

$$S(A) \equiv -A \log A,$$

and Fujii and Kamei [4] defined relative operator entropy

$$S(A|B) \equiv A^{\frac{1}{2}} \left( \log A^{-1} B A^{-1} \right) A^{\frac{1}{2}}$$

for strictly positive operators $A$ and $B$. Petz [17] discussed operator versions of Bregman divergence, and he introduced operator divergence $D_{FK}(A|B) \equiv B - A - S(A|B)$ by modifying the definition of
relative operator entropy slightly.

In the field of the statistical physics, Tsallis entropy was defined as an extension of Shannon entropy. As an operator version of Tsallis relative entropy which was discussed in [2,18,19], Yanagi, Kuriyama and Furuichi [21] introduced Tsallis relative operator entropy as follows:

\[ T_t(A|B) = \frac{A \#_t B - A}{t}, t \in (0,1), \]

where \( A \#_x B \equiv A^\frac{1}{\alpha} \left( A^{\frac{1}{\alpha}} B A^{\frac{1}{\alpha}} \right)^{-x} A^\frac{1}{\alpha} \) is the weighted geometric operator mean for \( x \in [0,1] \) (cf. [15]). Since \( \lim_{t \to 0} a^\alpha = \log a \) holds for \( a > 0 \), we have

\[ T_0(A|B) \equiv \lim_{t \to 0} T_t(A|B) = S(A|B). \]

Based on \( \alpha \)-divergence which Amari [1] defined, Fujii [3] introduced operator valued \( \alpha \)-divergence as follows: For strictly positive operators \( A \) and \( B \), and for \( \alpha \in (0,1) \),

\[ D_\alpha(A|B) \equiv \frac{A \nabla_\alpha B - A \#_\alpha B}{\alpha(1-\alpha)}, \]

where \( A \nabla_\alpha B \equiv (1-x)A + xB \) is weighted arithmetic operator mean (\( x \in [0,1] \)). Fujii et al. [7,8] showed the following relations between Petz-Bregman divergence \( D_{FK}(A|B) \) and operator valued \( \alpha \)-divergence at end points for interval (0,1):

\[ D_0(A|B) \equiv \lim_{\alpha \to 0} D_\alpha(A|B) = D_{FK}(A|B), \]

\[ D_1(A|B) \equiv \lim_{\alpha \to 1} D_\alpha(A|B) = D_{FK}(B|A). \]

Recently, we showed a relation between operator valued \( \alpha \)-divergence and Tsallis relative operator entropy [12]:

\[ D_\alpha(A|B) = -T_{1-\alpha}(B|A) - T_\alpha(A|B), \alpha \in [0,1]. \]

For strictly positive operators \( A \) and \( B \), and for \( x \in \mathbb{R} \), a path passing through \( A \) and \( B \) is defined as follows ([5, 6, 14] etc.):

\[ A \#_x B \equiv A^\frac{1}{\alpha} \left( A^{\frac{1}{\alpha}} B A^{\frac{1}{\alpha}} \right)^{-x} A^\frac{1}{\alpha}. \]

If \( x \in [0,1] \), then the path coincides with the weighted geometric operator mean \( A \#_x B \). By the derivative of the path with respect to \( x \) at \( t \in \mathbb{R} \), we can give Furuta’s relative operator entropy [9] (called generalized relative operator entropy) \( S_t(A|B) \) as follows:

\[ S_t(A|B) \equiv \frac{1}{t} \left( \log A^\frac{1}{t} B A^\frac{1}{t} \right) A^\frac{1}{t}. \]

Therefore, we can regard \( S_t(A|B) \) as the slope of the tangent line at \( x = t \) of the path. Since \( S_0(A|B) = S(A|B) \) holds, we can regard \( S(A|B) \) as the slope of the tangent line at \( x = 0 \) of the path.

By replacing \( A \#_x B \) with \( A \#_x B \), Tsallis relative operator entropy can be extended as the notion for \( t \in \mathbb{R} \), and therefore, Tsallis relative operator entropy can be regarded as the average rate of change between \( A \) and \( A \# x B \) on the path.

By such interpretations for relative operator entropies, we can illustrate operator divergences. For instance, Petz-Bregman divergence \( D_{FK}(A|B) \) can be represented as the quantity shown in Fig. 1.

In [14], Kamei showed that relative operator entropy has some kind of additivity as follows: For strictly positive operators \( A \) and \( B \), and for \( s \in \mathbb{R} \),

\[ S(A|A \#_s B) = sS(A|B). \]

\[ \text{Fig. 1  An interpretation for } D_{FK}(A|B). \]
In [11], we gave a viewpoint of operator valued distance for this property. Here, we can regard the lefthand side of the relation \((\ast)\) as relative operator entropy for a fixed point \(A\) and any point on the path.

For strictly positive operators \(A\) and \(B\), and for \(t \in [0,1]\) and \(r \in [-1,1]\), the operator power mean is given as follows:

\[
A \#_{t,r} B \equiv A^\frac{1}{2} \left\{(1-t)I + t \left( A^{-\frac{r}{2}} B A^{-\frac{1}{2}} \right)^r \right\} A^\frac{1}{2} = \left[A \nabla_t \left(A \overset{t}{\bowtie} B\right)\right].
\]

To preserve \((1-t)I + t \left( A^{-\frac{r}{2}} B A^{-\frac{1}{2}} \right)^r \geq 0\), we have to impose \(t \in [0,1]\). Operator power mean interpolates the arithmetic, geometric and harmonic means, that is, the relations shown in Fig. 2 hold. We treat this operator power mean as an expanded path which links point \(A\) with point \(B\). As the corresponding notions to relative operator entropies and operator valued \(\alpha\)-divergence, we introduce expanded relative operator entropy and expanded Tsallis relative operator entropy in section 3, and expanded operator valued \(\alpha\)-divergence in section 4.

In this paper, we aim at getting properties on relative operator entropies and operator valued \(\alpha\)-divergence for two points on the path \(A \overset{s}{\bowtie} B\). We remark that we reconstruct the results in [12] by considering geometrical interpretations like Fig. 1.

To show the results in this section, we prepare the following properties of the path.

**Lemma 2.1.** Let \(A\) and \(B\) be strictly positive operators. Then,

\[
\begin{align*}
(1) & \quad A \overset{t}{\bowtie} \left( A \overset{s}{\bowtie} B \right) = A \overset{s \cdot t}{\bowtie} B, \\
(2) & \quad \left( A \overset{t}{\bowtie} B \right) \overset{s}{\bowtie} A = A \overset{(1-s)t}{\bowtie} B
\end{align*}
\]

hold for \(s,t \in \mathbb{R}\).

**Proof.** (1) This can be shown as follows:

\[
A \overset{t}{\bowtie} \left( A \overset{s}{\bowtie} B \right) = A^\frac{1}{2} \left\{(A^{-\frac{r}{2}} (A \overset{s}{\bowtie} B) A^{-\frac{1}{2}})^r \right\} A^\frac{1}{2} = A^\frac{1}{2} \left( A^{-\frac{r}{2}} B A^{-\frac{1}{2}} \right)^s A^\frac{1}{2} = A \overset{s \cdot t}{\bowtie} B.
\]

(2) By (1), we get

\[
\left( A \overset{t}{\bowtie} B \right) \overset{s}{\bowtie} A = A \overset{(1-s \cdot t)}{\bowtie} \left( A \overset{t}{\bowtie} B \right) = A \overset{(1-s \cdot t)}{\bowtie} B.
\]

\[\square\]

**Lemma 2.2.** Let \(A\) and \(B\) be strictly positive operators. Then,

\[
\begin{align*}
(1) & \quad \left( A \overset{s}{\bowtie} B \right) \overset{t}{\bowtie} \left( A \overset{r}{\bowtie} B \right) = A \overset{(s \cdot t) \cdot (r+s)}{\bowtie} B, \quad t \in \mathbb{R}, \\
(2) & \quad \left( A \overset{r}{\bowtie} B \right) A^{-1} \left( A \overset{s}{\bowtie} B \right) = A \overset{r \cdot s}{\bowtie} B
\end{align*}
\]

hold for \(r,s \in \mathbb{R}\).

**Proof.** (1) By Lemma 4.2 in [13],

\[
T^* (X \overset{u}{\bowtie} Y) T = (T^* X T) \overset{u}{\bowtie} (T^* Y T)
\]

holds for any invertible operator \(T\), positive invertible operators \(X,Y\) and \(u \in \mathbb{R}\). Therefore, we have

\[
\begin{align*}
&\left( A \overset{s}{\bowtie} B \right) \overset{t}{\bowtie} \left( A \overset{r}{\bowtie} B \right) \\
&= \left\{ A^\frac{1}{2} \left( A^{-\frac{r}{2}} (A \overset{s}{\bowtie} B) A^{-\frac{1}{2}} \right)^s \right\} A^\frac{1}{2} = \left\{ A^\frac{1}{2} \left( A^{-\frac{r}{2}} (A \overset{s}{\bowtie} B) A^{-\frac{1}{2}} \right)^s \right\} A^\frac{1}{2}
\end{align*}
\]
For generalized relative operator entropy and Tsallis relative operator entropy, we have the following result corresponding to the relation \((*)\).

**Theorem 2.3.** Let \(A\) and \(B\) be strictly positive operators. Then,

1. \(S_t(A|A \triangledown_s B) = sS_{st}(A|B)\),

2. \(T_t(A|A \triangledown_s B) = sT_{st}(A|B)\)

hold for \(s, t \in \mathbb{R}\).

**Proof.** (1) If \(s = 0\), then it is obvious that the both sides equal zero, and if \(t = 0\), then this equality becomes the relation \((*)\). Otherwise, we get

\[
S_t(A|A \triangledown_s B) = A^\frac{1}{s}(A^{-1}A \triangledown_s B)^{-1}A = A^\frac{1}{s}(A^{-1}A \triangledown_s B)^{-1}A = A^{-1}(A^{-1}A \triangledown_s B)A = sS_{st}(A|B).
\]

(2) By (1) in Lemma 2.1, we have

\[
T_t(A|A \triangledown_s B) = \frac{A \triangledown_s (A \triangledown_s B) - A}{t} = s \frac{A \triangledown_{st} B - A}{st} = sT_{st}(A|B).
\]

For the relation (1) in Theorem 2.3, we can give a geometrical interpretation shown in Fig. 3. We remark that two tangent lines drawn in this figure intersect on the axis of the vertical direction.

In [12], we showed the following result on translation of generalized relative operator entropy.

**Proposition 2.4.** Let \(A\) and \(B\) be strictly positive operators. Then,

\[
S_{u+v}(A|B) = (A \triangledown_v B)A^{-1}S_u(A|B)
\]

holds for \(u, v \in \mathbb{R}\).

**Proof.** This can be shown as follows:

\[
S_{u+v}(A|B) = A^\frac{1}{u+v}(A^{-1}A \triangledown_v B)^{-1}A = A^\frac{1}{u+v}(A^{-1}A \triangledown_v B)^{-1}A = A \triangledown_v B)
\]

When we regard \(S_u(A|B)\) and \(S_{u+v}(A|B)\) as tangent vectors at \(u\) and \(u + v\) on the path \(A \triangledown_x B\), respectively, Proposition 2.4 means that \(S_{u+v}(A|B)\) is parallely transferring \(S_u(A|B)\) by \(v\) along the path.

**Remark 1.** By putting \(u = 0\) in Proposition 2.4, we have \(S_v(A|B) = (A \triangledown_v B)A^{-1}S(A|B)\).

The following is an extension of the relation \((*)\). This is a result of generalized relative operator entropy for any two points on the path.

**Proposition 2.5.** Let \(A\) and \(B\) be strictly positive operators. Then,

\[
S_t(A \triangledown_s B|A \triangledown_s B) = (s - t)S_{(1-t)r+ts}(A|B)
\]

holds for \(s, t, r \in \mathbb{R}\).
Similarly, we get
\[ S_t(A \triangleright \triangleright_B | A \triangleright \triangleright_B) \]
for strictly positive operators \( A, B \) and \( t \in \mathbb{R} \). Similarly, we get
\[ \lim \frac{A \triangleright_{t,v+u} B - A \triangleright_t B}{v} = S_t(A|B) \]
for \( s,t,r \in \mathbb{R} \). Therefore, by (1) in Lemma 2.2, we get
\[ S_t(A \triangleright \triangleright_B | A \triangleright \triangleright_B) = \lim \frac{(A \triangleright_B | B) \triangleright_{t,v+u} (A \triangleright_B | B) - (A \triangleright_B | B) \triangleright_t (A \triangleright_B | B)}{v} \]
\[ = \lim \frac{A \triangleright_{(1-t)v+u+(s-r)} B - A \triangleright_{(1-t)v+u} B}{v} \]
\[ = (s-r)S_{(1-t)v+u}(A|B) \]
for \( s,t,v \in \mathbb{R} \). Therefore, by (1) in Lemma 2.2, we get
\[ S_t(A \triangleright \triangleright_B | A \triangleright \triangleright_B) = \lim \frac{(A \triangleright_B | B) \triangleright_{t,v+u} (A \triangleright_B | B) - (A \triangleright_B | B) \triangleright_t (A \triangleright_B | B)}{v} \]
\[ = \lim \frac{A \triangleright_{(1-t)v+u+(s-r)} B - A \triangleright_{(1-t)v+u} B}{v} \]
\[ = (s-r)S_{(1-t)v+u}(A|B) \]
for \( s,t,v \in \mathbb{R} \). Therefore, by (1) in Lemma 2.2, we get
\[ S_t(A \triangleright \triangleright_B | A \triangleright \triangleright_B) = \lim \frac{(A \triangleright_B | B) \triangleright_{t,v+u} (A \triangleright_B | B) - (A \triangleright_B | B) \triangleright_t (A \triangleright_B | B)}{v} \]
\[ = \lim \frac{A \triangleright_{(1-t)v+u+(s-r)} B - A \triangleright_{(1-t)v+u} B}{v} \]
\[ = (s-r)S_{(1-t)v+u}(A|B) \]
for \( s,t,v \in \mathbb{R} \).

**Proof.** Since \( \lim_{v \to 0} \frac{a^{t+v-a}}{v} = a^t \lim_{v \to 0} \frac{a^{v-1}}{v} = a^t \log a \)
holds for \( a > 0 \), we have
\[ \lim_{v \to 0} \frac{A \triangleright_{t+u} B - A \triangleright_t B}{v} = S_t(A|B) \]
for strictly positive operators \( A, B \) and \( t \in \mathbb{R} \).

(2) By Lemma 2.2, we get
\[ T_t(A \triangleright \triangleright_B | A \triangleright \triangleright_B) = \frac{(A \triangleright_B | B) \triangleright_t (A \triangleright_B | B) - A \triangleright_B}{t} \]
\[ = (s-r) \frac{A \triangleright_{(1-t)v+u+(s-r)} B - A \triangleright_{(1-t)v+u} B}{(s-r)t} \]
\[ = (s-r)(A \triangleright_B | B)A^{-1}T_{(s-r)t}(A|B). \]

(3) This equality can be obtained by putting \( t = 0 \) for (1).

**Remark 2.** We can get Theorem 2.3 by putting \( r = 0 \) for the relations (1) and (2) in Theorem 2.6.

### 3. Expanded Relative Operator Entropies

In this section, we show the results of expanded relative operator entropies for two points on the expanded path \( A \#_{x,r} B \). Similarly to \( S_t(A|B) \), expanded relative operator entropy \( S_{t,r}(A|B) \) is defined by the derivative of expanded path with respect to \( x \) at \( t \) as follows: For strictly positive operators \( A \) and \( B \), and for \( t \in [0,1] \) and \( r \in [-1,1] \),
\[ S_{t,r}(A|B) \equiv \frac{d}{dx} A \#_{x,r} B \bigg\vert_{x=t} = \frac{1}{A^2} \left((1-t)I + t \left(A^{-1}B A^{-1}\right)^{1-1} \right) \]
\[ = \left(A^{-1}B A^{-1}\right)^{-1} \frac{(1-t)I + t \left(A^{-1}B A^{-1}\right)^{1-1} \right) \]
\[ = \left(A^{-1}B A^{-1}\right)^{-1} \frac{(1-t)I + t \left(A^{-1}B A^{-1}\right)^{1-1} \right) \]
\[ = A^{-1}T_{r}(A|B). \]

We remark that expanded relative operator entropy has the relations shown in Fig. 4 ([11]).

By replacing weighted geometric operator mean with operator power mean, we obtain the definition of expanded Tsallis relative operator entropy [11]: For strictly positive operators \( A \) and \( B \), and for \( t \in (0,1] \) and \( r \in [-1,1] \),
Theorem 3.1. Let $A$ and $B$ be strictly positive operators. Then,

$$S_{t,r}(A|B) = (A \#_{t,r} B)\{A \nabla_t (A \uplus_r B)\}^{-1} S_{0,r}(A|B)$$

holds for $t \in [0,1]$ and $r \in [-1,1]$.

Proof. This can be shown as follows:

$$S_{t,r}(A|B) = A^\frac{1}{r} \left\{ (1-t)I + t \left( A^{-1}BA^{-1} \right)^{r} \right\}^{\frac{1}{r}} - A$$

$$= A^\frac{1}{r} \left\{ (1-t)I + t \left( A^{-1}BA^{-1} \right)^{r} \right\}^{\frac{1}{r}} A$$

$$\times A^\frac{1}{r} \left\{ (1-t)I + t \left( A^{-1}BA^{-1} \right)^{r} \right\}^{\frac{1}{r}} A$$

$$= (A \#_{t,r} B)\{A \nabla_t (A \uplus_r B)\}^{-1} S_{0,r}(A|B)$$

As a corresponding relation to (*), we have obtained

$$T_{t,r}(A|A \#_{s,r} B) = sT_{t,r}(A|B)$$

in [14]. Here, we can show the following result corresponding to (1) in Theorem 2.3.

Theorem 3.2. Let $A$ and $B$ be strictly positive operators. Then,

$$S_{t,s,r}(A|A \#_{s,r} B) = s\left( A \#_{s,t,r} B \right)\{A \nabla_{s,t} (A \uplus_r B)\}^{-1} S_{0,s,r}(A|B)$$

holds for $t,s \in [0,1]$ and $r \in [-1,1]$.

In cases of $t \in \{0,1\}$, we get the following relations.

Corollary 3.3. Let $A$ and $B$ be strictly positive operators. Then,

$$\begin{align*}
S_{t,s,r}(A|A \#_{s,r} B) &= s\left( A \#_{s,t,r} B \right)\{A \nabla_{s,t} (A \uplus_r B)\}^{-1} S_{0,s,r}(A|B) \\
&= sS_{s,t,r}(A|B)
\end{align*}$$

hold for $s \in [0,1]$ and $r \in [-1,1]$.

To prove Theorem 3.2, we prepare the following lemma.

Lemma 3.4. Let $A$ and $B$ be strictly positive operators. Then,

$$\begin{align*}
(1) \quad & A \uplus_r (A \#_{t,r} B) = A \nabla_t (A \uplus_r B), \\
(2) \quad & A \#_{t,r} (A \#_{s,r} B) = A \#_{s,t,r} B, \\
(3) \quad & A \#_{t,r} B = B \#_{1-t,r} A
\end{align*}$$
hold for \( t, s \in [0, 1] \) and \( r \in [-1, 1] \).

**Proof.** (1) This can be shown as follows:

\[
A r_{t,r} (A \#_{t,r} B) = A \left[ A^{-1} \left( A^{-1} \left( (1-t)I \right) \right) + t \left( A^{-1}AB^{-1} \right) \right] A^{-1} = A \left( (1-t)A + tA(A \# r_{t,r} B) \right) = A \nabla_t (A \# r_{t,r} B).
\]

(2) Since

\[
A r_{k,k'}(A \# r_{k,k'} B) = A \# r_{kk'} B \quad \text{and} \quad A \nabla_k (A \nabla_k B) = A \nabla_{kk'} B
\]

hold for \( k, k' \in \mathbb{R} \), we get

\[
A \# r_{t,r} (A \# s,r B) = A \left[ A \nabla_t \left( A \# r_{t,r} B \right) \right] = A \nabla_{t} \left( A \# s,r B \right)
\]

(3) By Lemma 4.2 in [13], we have

\[
A \# r_{t,r} B = A \left[ A \nabla_t \left( B \# r_{t-l} A \right) \right] = A \# s_{t,r} B.
\]

For a strictly positive operator \( X \), and \( t \in [0, 1], r \in [-1, 1] \), the following holds:

\[
X r_{t,r} (X \nabla_t X^{1-r}) = X \left[ (1-t)X + tX^{1-r} \right]
\]

\[
= X^2 \left( (1-t)X + tX^{1-r} \right) \left( X^{-1} \right)^{1/2} X^2
\]

\[
= X \left( (1-t)X + tX^{1-r} \right) \left( X^{-1} \right)^{1/2}
\]

\[
= \{(1-t)X^{1-r} + tI\}^{1/2}
\]

Therefore, we have

\[
A \# r_{t,r} B = B^2 \left[ (1-t) \left( B^{-1}AB^{-1} \right) + tI \right] \left( B^{-1} \right)^{1/2}
\]

\[
= B^2 \left[ B^{-1} tB + (1-t) \left( B \# r_{t,r} B \right) \right] \left( B^{-1} \right)^{1/2}
\]

\[
= B B_{t,r} \{ B \nabla_{t-r} (B \# r_{t,r} A) \} = B \# s_{t-r,r} A.
\]

\[\square\]

**Proof of Theorem 3.2.** By Theorem 3.1, Lemma 3.4 and (**), the following holds:

\[
S_{t,r}(A|A \# s,r B) = \left( A \# r_{t,r} (A \# s,r B) \right) \left[ A \nabla_t \left( A \# r_{t,r} (A \# s,r B) \right) \right]^{-1} \times S_{0,r}(A|A \# s,r B)
\]

\[
= (A \# s_{t,r} B) \left( A \nabla_{s} \left( A \# r_{t,r} B \right) \right) \left( S_{0,r}(A|A \# s,r B) \right)
\]

\[
= s(A \# s_{t,r} B) \left( A \nabla_{s} \left( A \# r_{t,r} B \right) \right) \left( S_{0,r}(A|A \# s,r B) \right)
\]

Concerning to \( T_{t,r}(A|B) \), the following holds.

**Theorem 3.5.** Let \( A \) and \( B \) be strictly positive operators. Then,

(1) \( T_{t,r}(A|A \# s,r B) = sT_{s,t,r}(A|B), t \in (0,1), \)

(2) \( T_{0,r}(A|A \# s,r B) = sT_{r}(A|B), \)

(3) \( T_{1,r}(A|A \# s,r B) = sT_{r}(A|B), \)

(4) \( T_{1-r,r}(A \# s,r B|A) = s \frac{T_{s,t,r}(A|B) - T_{s,r}(A|B)}{1-t}, t \in [0,1], \)

hold for \( s \in [0,1] \) and \( r \in [-1,1] \).

**Proof.** (1) By (2) in Lemma 3.4, we have

\[
T_{t,r}(A|A \# s,r B) = A \# r_{t,r} \left( A \# s,r B \right) - A
\]

\[
= X^2 \left[ X^{-1} \left( (1-t)X + tX^{1-r} \right) \right] X^2
\]

\[
= X \left( (1-t)X^{1-r} + tI \right)^{1/2}
\]
$$= s \frac{A \#_{s,t,r} B - A}{st} = sT_{st,r}(A|B).$$

(2), (3) For (1), by putting $t = 0$ and $t = 1$, we can get these relations, respectively.

(4) Since $A \#_{t,r} B = B \#_{1-t,r} A$ holds for $t \in [0,1]$ and $r \in [-1,1]$, we have

$$T_{1-t,r}(A \#_{s,r} B|A) = \frac{(A \#_{s,r} B) \#_{1-t,r} A - A \#_{s,r} B}{1-t}$$

$$= \frac{A \#_{t,r} (A \#_{s,r} B) - A \#_{s,r} B}{1-t}$$

$$= \frac{A \#_{s,r} B - A \#_{s,r} B}{1-t}$$

$$= \frac{(A \#_{s,r} B - A) - (A \#_{s,r} B - A)}{1-t}$$

$$= \frac{tT_{st,r}(A|B) - T_{s,r}(A|B)}{1-t}.$$  

\[\square\]

4. Expanded Operator Valued $\alpha$-Divergence

In this section, we denote $D_t(\cdot \mid \cdot)$ instead of $D_\alpha(\cdot \mid \cdot)$. In [12], we had shown that for strictly positive operators $A$, $B$, and $t \in [0,1]$,

$$D_t(A|B) = -T_{1-t}(B|A) - T_t(A|B).$$

This relation gives a geometrical interpretation for operator valued $\alpha$-divergence. Tsallis relative operator entropy $T_t(A|B)$ can be regarded as the slope of the line passing through points $A$ and $A \#_t B$. Since

$$-T_{1-t}(B|A) = -\frac{B \#_{1-t} A - B}{1-t} = \frac{B - A \#_t B}{1-t},$$

we can regard this operator value as the slope of the line passing through points $A \#_t B$ and $B$. Therefore, $D_t(A|B)$ gives the difference between the slopes of these two lines. We can illustrate the quantity corresponding to $D_t(A|B)$ by bold straight line in Fig. 6.

We obtain the following results of operator valued $\alpha$-divergence for fixed point $A$ and any point on the path.

**Proposition 4.1.** Let $A$ and $B$ be strictly positive operators. Then,

- (1) $D_t(A\mid A\#_s B) = s \frac{T_s(A|B) - T_{st}(A|B)}{1-t},$  
  $t \in [0,1),$

- (2) $D_0(A\mid A\#_s B) = s\{T_s(A|B) - S(A|B)\},$

- (3) $D_1(A\mid A\#_s B) = s\{sS_s(A|B) - T_s(A|B)\}$

hold for $s \in \mathbb{R}$.

**Proof.** (1) By Lemma 2.1, we have

$$D_t(A\mid A\#_s B) = -\frac{T_s(A|B) - T_{st}(A|B)}{1-t},$$

$$ \Rightarrow A \#_{1-t} B = A \#_s B - A \#_{s,t} B, \quad A \#_{s,t} B = \frac{A \#_s B - A \#_{s,t} B}{1-t},$$

$$= \frac{A \#_{s,t} B - A}{1-t} = \frac{T_s(A|B) - T_{st}(A|B)}{1-t}.$$  

(2) We can get this result by putting $t = 0$ in (1).

(3) By Theorem 2.3 and $T_0(X|Y) = S(X|Y) = -S_1(Y|X)$, we have
\[ D_t(A, B) = -T_0(A, B) - T_1(A, B) = S_1(A, B) - s(T_s(A, B)) = s(S_s(A, B) - T_s(A, B)). \]

By applying the relation in Proposition 4.1, we can get the results of operator valued \( \alpha \)-divergence for any two points on the path as follows:

**Theorem 4.2.** Let \( A \) and \( B \) be strictly positive operators. Then,

1. \( D_t(A, B) = (s-r)(A, B)A^{-1}T_{s-r}(A, B) - T_{s-r}(A, B), \)
   \[ t \in [0,1], \]
   \[ (1) \]

2. \( D_0(A, B) = (s-r)(A, B)A^{-1}(T_{s-r}(A, B) - S(A, B)), \)
   \[ (2) \]

3. \( D_1(A, B) = (s-r)(A, B)A^{-1}(S_{s-r}(A, B) - T_{s-r}(A, B)) \)
   \[ \text{hold for } r, s \in \mathbb{R}. \]
   \[ (3) \]

**Proof.** (1) (1) in Lemma 2.2, (1) in Proposition 4.1, and (2) in Theorem 2.6, we have

\[ D_t(A, B) = D_t(A, B)^{-1}T_{(s-r+1)}(A, B) \]
\[ = D_t(A, B) - (s-r)(A, B)A^{-1}(T_{s-r}(A, B) - S(A, B)). \]
\[ (2) \]

(2) We can get this result by putting \( t = 0 \) in (1).

(3) Since \( S(B|A) = -S_1(A|B) \) hold for \( A, B > 0 \) (cf. [10]), we have

\[ D_t(A|B) = D_{t,0}(B|A) = A - B + S_1(A|B). \]

Therefore, by (1) and (2) in Theorem 2.6,

\[ D_t(A, B) = A - B - T_1(A, B) \]
\[ = (A, B)A^{-1}T_{s-r}(A, B) - T_{s-r}(A, B). \]

\[ (3) \]

We define expanded operator valued \( \alpha \)-divergence.

**Definition 4.3.** For strictly positive operators \( A \) and \( B \), and for \( t \in [0,1], r \in [-1,1] \), expanded operator valued \( \alpha \)-divergence is defined as follows:

\[ D_{t,r}(A|B) = -T_{t-r}(A|B) \]

We get the following relations for expanded operator valued \( \alpha \)-divergence immediately.

**Proposition 4.4.** Let \( A \) and \( B \) be strictly positive operators. Then,

1. \( D_{t,0}(A|B) = D_t(A|B), \)
   \[ (1) \]

2. \( D_{t,1}(A|B) = 0, \)
   \[ (2) \]

3. \( D_{0,r}(A|B) = B - A - T_r(A|B), \)
   \[ (3) \]

4. \( D_{t,r}(A|B) = A - B - T_r(A|B) \)
   \[ (4) \]

**hold for \( t \in [0,1] \) and \( r \in [-1,1] \).**

We can illustrate the relations in Proposition 4.4 in Fig. 7.

**Remark 3.** It is obvious that

\[ D_{t-r}(A|B) = D_{t,r}(A|B) \]

holds for \( t \in [0,1] \) and \( r \in [-1,1] \).

We can rewrite \( D_{t,r}(A|B) \) as the difference between weighted arithmetic mean and operator power mean as follows.

**Theorem 4.5.** Let \( A \) and \( B \) be strictly positive operators. Then,

\[ A - B - T_1(A|B) \]
\[ D_t(A|B) \]
\[ \uparrow_{t=1} \]
\[ r \rightarrow 0 \]
\[ D_{t,r}(A|B) \]
\[ \downarrow_{r=1} \]
\[ t = 0 \]
\[ B - A - T_r(A|B) \]
\[ \text{Fig. 7 Expanded operator valued } \alpha-\text{divergence.} \]
\[ D_{t,r}(A|B) = \frac{A \nabla_t B - A \#_{t,r} B}{t(1-t)} \]

holds for \( t \in (0,1) \) and \( r \in [-1,1] \).

**Proof.** We can get this relation as follows:

\[ D_{t,r}(A|B) = -T_{1-t,r}(B|A) - T_{t,r}(A|B) \]

\[ = - \frac{B \#_{1-t,r} A - B - A \#_{t,r} B - A}{t(1-t)} \]

\[ = -tA \#_{t,r} B + tB - (1-t)A \#_{t,r} B + (1-t)A \]

\[ = \frac{A \nabla_t B - A \#_{t,r} B}{t(1-t)}. \]

We get the following relations corresponding to Proposition 4.1.

**Theorem 4.6.** Let \( A \) and \( B \) be strictly positive operators. Then,

1. \[ D_{t,r}(A|A \#_{s,t} B) = s \frac{T_{s,t}(A|B) - T_{st,r}(A|B)}{1-t}, \]
   \( t \in (0,1) \),

2. \[ D_{0,r}(A|A \#_{s,r} B) = s \{T_{s,r}(A|B) - T_{r}(A|B)\}, \]

3. \[ D_{1,r}(A|A \#_{s,r} B) = s \{S_{s,r}(A|B) - T_{s,r}(A|B)\} \]
   hold for \( s \in (0,1) \) and \( r \in [-1,1] \).

**Proof.** (1) By Theorem 3.5, we have

\[ D_{t,r}(A|A \#_{s,t} B) = -T_{1-t,r}(A \#_{s,t} B|A) - T_{t,r}(A|A \#_{s,t} B) \]

\[ = s \frac{T_{s,t}(A|B) - tT_{st,r}(A|B)}{1-t} - sT_{st,r}(A|B) \]

\[ = s \frac{T_{s,t}(A|B) - T_{st,r}(A|B)}{1-t}. \]

(2) We can get this result by putting \( t = 0 \) in (1).

(3) By Theorem 3.2 and Theorem 3.5, we have

\[ D_{t,r}(A|A \#_{s,r} B) = -T_{r}(A \#_{s,r} B|A) - T_{1-r}(A|A \#_{s,r} B) \]

\[ = s1_{t,r}(A|A \#_{s,r} B) - T_{1-r}(A|A \#_{s,r} B) \]

\[ = s \{S_{s,r}(A|B) - T_{s,r}(A|B)\}. \]

**References**


