The First – Passage Time Moment of the GNP Diffusion Process to a Determined Value

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This paper describes an accurate method of approximating the moments of the first-passage time for the birth and death Gross National Product GNP diffusion process when the GNP is a determined value or constant absorbing barrier. This was done by approximating the differential equations by equivalent difference equations.

Keywords: First-passage time, Birth and death GNP diffusion process, Constant absorbing barrier, Difference equations.

Introduction

First-passage time plays an important role in the area of applied probability theory especially in stochastic modeling. Several examples of such problems are the extinction time of absorbing process, the exogenous factors in economics, or the cycle lengths of a certain vehicle actuated traffic signals. More specifically, the first-passage time to a moving barrier for diffusion and other Markov processes arises in economics modeling (cf. Gutierrez et.al.(1997), (1999)), in biological modeling (cf. Ewens (1979)), in statistics (cf. Darling and Siegert (1953) and Durbin (1971)) and in engineering (cf. Blake and Lindsey (1973)).

Many important results related to the first-passage time have been studied from different points of view by different authors. For example, McNeil (1970) has derived the distribution of the integral functional $W_x = \int_0^{T_x} \{X(t)\} dt$ where T_x is the first-passage time to the origin in a general birth and death process with $X(0) = x$ and $g(i)$ is an arbitrary function. And, Iglehart (1965), McNeil and Schach (1973) have shown a number of classical birth and death processes upon taking diffusion limits to asymptotically approach the Ornstein-Uhlenbeck (O. U.).

Many properties such as a first-passage time to a barrier, absorbing or reflecting, located some distance from an initial starting point of the O. U. process and the related diffusion process such as the case of the first passage time of a Wiener process to a linear barrier is a closed form expression for the density available is discussed in Cox and Miller (1965). Gutierrez et. al. (1999) presented a methodology to build a log normal

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diffusion process with exogenous factors that models economic variables. They studied the problem of forecasting the first passage time and applied to the GNP of Spain. Thomas (1975) describes some mean first-passage time approximation for the Orenstein-Uhlenbeck process. Tuckwell and Wan (1984) have studied the first-passage time of a Markov process to moving barriers as a first-exit time for a vector whose components include the process and the barrier. Also, others such as, Karlin and Taylor (1981), Ferebee (1982), Alawneh and Aleideh (2002), etc have discussed the first passage time from different points of view. In particular, Alawneh and Aleideh (2002) have discussed the problem of finding the moments of the first-passage time distribution for the Orenstein-Uhlenbeck process with a single absorbing barrier using the method of approximating the differential equations by difference equations.

In this paper, we consider the birth and death GNP diffusion process and study the first-passage time for such a process to a fixed determined value single constant absorbing barrier. More specifically, the moment approximations are derived using the method of difference equations.

First-Passage Moment Approximation to the GNP Diffusion Process

Let $\{X(t) : t \geq 0\}$ be the birth and death diffusion process with infinitesimal mean bx and variance ax starting at some $x_0 > 0$, where b and a are the drift and the diffusion coefficients respectively and satisfies the Ito stochastic differential equation

$$dX(t) = bX(t)dt + \sqrt{2aX(t)} dW(t) \quad (1)$$

where $\{W(t) : t \geq 0\}$ is a standard Wiener process with zero mean and variance $\sigma^2 t$. Assume that the existence and uniqueness conditions are satisfied (cf. Gihman and Skorohod (1972)).

Let $\{Y(t) : t \geq 0\}$ be the single constant absorbing barrier (fixed determined value of the GNP) $Y(t) = k$, with $Y(0) = k$, or equivalently;

$$\frac{dY(t)}{dt} = 0$$

Thus the first-passage time of the GNP diffusion process $X(t)$ to a fixed determined value or a constant single absorbing barrier $Y(t) = k$ by the random variables

$$T_x = \inf \{t \geq 0 : x(t) \geq k\} \quad (2)$$

with probability density function $g(t; x_0) = -\frac{d}{dt} \int_{-\infty}^k P(x_0, x; t) dt$. Here $p(x_0, x; t)$ is the probability density function of $X(t)$. Letting $M_n(x_0, k; t) ; n = 1, 2, 3, \dots$, be the n -th moment of the first-passage time T_x to a constant barrier k , i.e.

$$M_n(x_0, k; t) = E(T_x^n) ; n = 1, 2, 3, \dots \quad (3)$$

It follows from the forward Kolmogorov equation that the n^{th} moment of T_y must satisfy the ordinary differential equation

$$ax M_n''(x_0, k; t) + bx M_n'(x_0, k; t) = -n M_{n-1}(x_0, k; t) \quad (4)$$

or equivalently;

$$M_n''(x_o, k; t) + \frac{b}{a} M_n'(x_o, k; t) = -\frac{n}{ax} M_{n-1}(x_o, k; t) \quad (5)$$

where $M_n'(x_o, k; t)$ and $M_n''(x_o, k; t)$ are the first and second derivatives of $M_n(x_o, k; t)$ with respect to x ($x_o \leq x \leq k$), with appropriate boundary conditions for $n = 1, 2, 3 \dots$. Note that $M_0(x_o, k; t) = 1$.

Now, rewrite Eq. (5), as

$$M_n''(x_o, k; t) = -\frac{n}{ax} M_{n-1}(x_o, k; t) - \frac{b}{a} M_n'(x_o, k; t). \quad (6)$$

Let Δ be the difference operator. Then we defined the first order difference of $M_n(x_o, k, t)$ as follows:

$$\Delta M_n(x_o, k; t) = M_{n+1}(x_o, k; t) - M_n(x_o, k; t) \quad (7)$$

(cf. Kelley and Peterson (1991)).

Note that Eq. (6) can be approximated by

$$M_n''(x_o, k; t) = -\frac{n}{ax} M_{n-1}(x_o, k; t) + \frac{b}{a} \Delta M_n(x_o, k; t) \quad (8)$$

By substituting from Eq. (7) in Eq. (8) we get:

$$M_n''(x_o, k; t) = -\frac{n}{ax} M_{n-1}(x_o, k; t) + \frac{b}{a} M_n(x_o, k; t) - \frac{b}{a} M_{n+1}(x_o, k; t) \quad (9)$$

Now, we will use the matrix theory to solve the differential equation defined in Eq. (9). If we let

$$\vec{M}(x_o, k; t) = [M_1(x_o, k; t), M_2(x_o, k; t), \dots]$$

Then we get

$$\frac{d^2 \vec{M}(x_o, k; t)}{dx^2} = A \vec{M}(x_o, k; t) \quad (10)$$

where

$$A = \begin{bmatrix} \frac{b}{a} & -\frac{b}{a} & 0 & 0 & \dots \\ -\frac{2}{ax} & \frac{b}{a} & -\frac{b}{a} & 0 & \dots \\ 0 & -\frac{3}{ax} & \frac{b}{a} & -\frac{b}{a} & \dots \\ 0 & 0 & -\frac{4}{ax} & \frac{b}{a} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Now let

$$\frac{d \vec{M}(x_o, k; t)}{dx} = \vec{R}(x_o, k; t). \quad (11)$$

This implies

$$\frac{d^2 \vec{M}(x_o, k; t)}{dx^2} = \frac{d \vec{R}(x_o, k; t)}{dx}. \quad (12)$$

Apply to Eq. (10), we get

$$\frac{d}{dx} \begin{bmatrix} \vec{R}(x_o, k; t) \\ \vec{M}(x_o, k; t) \end{bmatrix} = \begin{bmatrix} O & A \\ I & O \end{bmatrix} \cdot \begin{bmatrix} \vec{R}(x_o, k; t) \\ \vec{M}(x_o, k; t) \end{bmatrix} \quad (13)$$

where I is the identity matrix and O is the zero matrix.

Thus the solution of the system of equations in Eq. (13) is given by

$$\begin{bmatrix} \vec{R}(x_o, k; t) \\ \vec{M}(x_o, k; t) \end{bmatrix} = e^{\begin{bmatrix} O & A^* \\ D & O \end{bmatrix}} \cdot \begin{bmatrix} \vec{R}(x_o, k; t) \\ \vec{M}(x_o, k; t) \end{bmatrix} \quad (14)$$

where $D = [d_{ij}] ; i, j \geq 1$ is the diagonal matrix with entries

$$d_{ij} = \begin{cases} K - n_o ; & j = i \\ o & ; \text{ otherwise} \end{cases} \quad (15)$$

and $A^* = [\alpha_{ij}] ; i, j \geq 1$ is the matrix with entries

$$a_{ij} = \begin{cases} -\frac{i}{\alpha} \ln\left(\frac{k}{x_o}\right) ; & j = i - 1 \\ \frac{b}{a} (k - x_o) ; & j = i \\ -\frac{b}{a} (k - x_o) ; & j = i + 1 \\ o ; & \text{ otherwise.} \end{cases} \quad (16)$$

Note that the matrix e^B where $B = \begin{bmatrix} O & A^* \\ D & O \end{bmatrix}$ is defined by

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \dots$$

This series is convergent since it is a Cauchy operator of Eq. (2.6) (cf. Zeifman (1991)).

Conclusion

In conclusion the advantage of this technique is to use the difference equation to approximate the ordinary differential equation since it is the discretization of the ODE. Also, the system of the solutions in equations (14) gives an explicit solution to the first-passage time moments for the birth and death GNP diffusion process to a fixed determined value of GNP (constant single absorbing barrier). This increases the applicability of the diffusion process in stochastic modeling in the area of economics.

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