Buckling Load Evaluation Method for Single Layer Cylindrical Lattice Shells

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Abstract: A rational design evaluation procedure is investigated for the elastic overall buckling load carrying capacity of single layer cylindrical lattice shell roof structures. The nature of the imperfection sensitivity of these structures is for the first time reviewed in this paper. This allows the development of the reduced stiffness buckling analytical concept for the lattice shells based upon the introduction of a simple lower bound estimation equation through the use of the so-called continuum shell analogy theory. The linear and nonlinear buckling loads found from conventional finite element analyses are compared with the present estimations. Finally, the elastic-plastic load carrying capacity estimation method through the use of the present elastic lower bound criteria is also proposed.

Key words: Metal lattice shell, Buckling, design formula, cylindrical barrel vault, roof structure.

1. Introduction

It is well-known that metal shells and shell-like lattice frame structures have buckling behavior which is very sensitive to initial geometric imperfections [1]. In exquisite contrast to these imperfection-sensitive shell-like structures as shown in Fig. 1, the elastic buckling of columns and plates is not affected by any imperfection. In this paper, a single layer cylindrical lattice roof with central angle ϕ , radius of curvature R and longitudinal length in direction L , has first been modeled to be the analogical continuum cylindrical shell having an equivalent shell thickness t as shown in Fig. 1. This continuum shell analogical theory can contribute to give the simple lower bound criteria to initial imperfections based upon the reduced stiffness buckling theory through reviewing the previous nonlinear finite element buckling analytical results. Then an alternative simple design formula included the buckling mode index has been proposed considering the buckling load reduction due to imperfections on the basis of the mixed continuum shell analogy and reduced stiffness theory. The

present evaluation method has been confirmed by many nonlinear finite element numerical experiments for various imperfection modes and amplitudes, different support conditions and varied geometric parameters.

2. Buckling and Reduced Stiffness Analyses for Cylindrical Lattice Shells Using the Continuum Theory

Let us consider a cylindrical lattice shell panel as shown in Fig. 2, and then the lattice shell can be approximately replaced by an orthotropic shell panel through the application of the well-known continuum analogy. The initial stresses as membrane fundamental solution under pressure loading p are approximately given as

$$n_x^E = 0, \quad n_y^E = -pR, \quad n_{xy}^E = 0 \quad (1)$$

For infinitesimally small buckling displacements (u^d, v^d, w^d), the associated buckling strain components are

$$\kappa_x^d = -\frac{\partial^2 w^d}{\partial x^2}, \quad \kappa_y^d = -\frac{\partial^2 w^d}{\partial y^2}, \quad \kappa_{xy}^d = -\frac{\partial^2 w^d}{\partial x \partial y} \quad (2a)$$

$$\varepsilon_x^d = \frac{\partial u^d}{\partial x}, \quad \varepsilon_y^d = \frac{\partial v^d}{\partial y} - \frac{w^d}{R}, \quad \varepsilon_{xy}^d = \frac{1}{2} \left\{ \frac{\partial u^d}{\partial y} + \frac{\partial v^d}{\partial x} \right\} \quad (2b)$$

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Fig. 1 Buckling characteristics.

Fig. 2 An analogical continuum shell roof.

$$\varepsilon_x^{dd} = \frac{1}{2} \left\{ \frac{\partial w^d}{\partial x} \right\}^2, \quad \varepsilon_y^{dd} = \frac{1}{2} \left\{ \frac{\partial w^d}{\partial y} \right\}^2, \quad \varepsilon_{xy}^{dd} = \frac{1}{2} \frac{\partial w^d}{\partial x} \frac{\partial w^d}{\partial y} \quad (2c)$$

where $(\kappa_x^d, \kappa_y^d, \kappa_{xy}^d)$ are the corresponding bending strain components, $(\varepsilon_x^d, \varepsilon_y^d, \varepsilon_{xy}^d)$ the corresponding linear membrane strain components, and $(\varepsilon_x^{dd}, \varepsilon_y^{dd}, \varepsilon_{xy}^{dd})$ the corresponding nonlinear membrane strain components.

Bending and membrane stress resultants are related to strains through the orthotropic constitutive equations

$$\begin{aligned} m_x^d &= D_{11}\kappa_x^d + D_{12}\kappa_y^d, \quad m_y^d = D_{12}\kappa_x^d + D_{22}\kappa_y^d, \quad m_{xy}^d = 2D_{66}\kappa_{xy}^d \\ n_x^d &= A_{11}\varepsilon_x^d + A_{12}\varepsilon_y^d, \quad n_y^d = A_{12}\varepsilon_x^d + A_{22}\varepsilon_y^d, \quad n_{xy}^d = 2A_{66}\varepsilon_{xy}^d \\ n_x^{dd} &= A_{11}\varepsilon_x^{dd} + A_{12}\varepsilon_y^{dd}, \quad n_y^{dd} = A_{12}\varepsilon_x^{dd} + A_{22}\varepsilon_y^{dd}, \quad n_{xy}^{dd} = 2A_{66}\varepsilon_{xy}^{dd} \end{aligned} \quad (3)$$

where $D_{11}, D_{12}, D_{22}, D_{66}, A_{11}, A_{12}, A_{22}, A_{66}$ are bending and membrane stiffness coefficients, and for an equilateral triangle unit the followings were obtained in reference [2],

$$\begin{aligned} D_{11} &= D_{22} = (9+12d)\sqrt{3EI/(12l)}, \quad D_{12} = (3-12d)\sqrt{3EI/(12l)}, \\ D_{66} &= (3+12d)\sqrt{3EI/(12l)}, \\ A_{11} &= A_{22} = 9\sqrt{3EA/(12l)}, \quad A_{12} = A_{66} = 3\sqrt{3EA/(12l)} \end{aligned} \quad (4)$$

where $d = GJ/(4EI)$, $t (= 2\sqrt{3D_{22}/A_{22}}) = \sqrt{12I/A}$, l = the length of the lattice members, GJ = torsion rigidity, E = Young's modulus, I = the centroidal moment of inertia and A = the cross sectional area.

The buckling condition using the variational principal [1] is given as

$$\delta(U_{2b} + U_{2m} + V_{2m}) = 0 \quad (5)$$

where the first, second and third terms are respectively the quadratic components on the bending strain energy, the membrane strain energy and the quadratic part of the interactions between the idealised prebuckling state and the nonlinear membrane stresses and strains associated with buckling mode. Considering Eq.(1), these energy components are

$$U_{2b} = \frac{1}{2} \iint (m_x^d \kappa_x^d + m_y^d \kappa_y^d + 2m_{xy}^d \kappa_{xy}^d) dx dy \quad (6)$$

$$U_{2m} = \frac{1}{2} \iint (n_x^d \varepsilon_x^d + n_y^d \varepsilon_y^d + 2n_{xy}^d \varepsilon_{xy}^d) dx dy \quad (7)$$

$$V_{2m} = \iint n_y^E \varepsilon_y^{dd} dx dy \quad (8)$$

The classical simple supported boundary condition SS3 is exactly satisfied by the individual components of the form

$$\begin{aligned} u^d &= u_{mn} \cos \frac{m\pi x}{L} \sin \frac{n\pi y}{R\phi}, \quad v^d = v_{mn} \sin \frac{m\pi x}{L} \cos \frac{n\pi y}{R\phi}, \\ w^d &= w_{mn} \sin \frac{m\pi x}{L} \sin \frac{n\pi y}{R\phi} \end{aligned} \quad (9)$$

Consequently the critical load spectrum resulting the solution of eigenvalue equations takes the form

$$p^C = p^b + p^m \quad (10)$$

where

$$p^b = \frac{\pi^2}{RL^2} \frac{\{D_{11}m^4 + 2(D_{12} + 2D_{66})m^2a^2n^2 + D_{22}a^4n^4\}}{a^2n^2} \quad (11)$$

$$p^m = \frac{L^2m^4\{A_{11}A_{22} - (A_{12})^2\}A_{66}}{\pi^2R^3a^2n^2[A_{11}A_{66}m^4 + \{A_{11}A_{22} - (A_{12})^2 - 2A_{12}A_{66}\}m^2a^2n^2 + A_{22}A_{66}a^4n^4]} \quad (12)$$

In these equations $\alpha = L/(R\phi)$ is the aspect ratio, n the half-wave buckling number in arch direction, and m the half-wave number in longitudinal direction: in this buckling problem it is easy to understand from Eqs. (11) and (12) that the minimum critical load for m is obtained to be associated with $m = 1$. In Eq. (10), p^b is the bending stiffness component for buckling and is not insensitive to imperfections, however, it is well-known that the membrane stiffness component p^m of the second term is completely lost due to existing moderately large imperfections as shown in Yamada and Croll [3]. Then the bending component p^b is called to be the reduced stiffness critical load p^* . Application of this reduced stiffness method to the pressure loaded cylindrical shell buckling problem requires the concept of an idealised model as shown in Fig. 3. The actual prebuckling deflection of the geometrically perfect cylindrical shell panel may be considered to comprise two parts: an idealised uniform radial deflection w^F due to the pressure loading; and, a loading imperfection resulting from the distortions at the edge restraint.

From large number of shell buckling experiments [4, 5] it has been understood that small changes in imperfections, combined with highly unstable forms of postbuckling behaviour, were largely responsible for the immense scatter of buckling loads and at times severe reductions from the linear buckling theory. Then if an analyst were to limit the investigation of the shell-like lattice structure buckling, it is likely that a designer would be left in a state of some confusion. It would for example, be difficult to understand why the deflection mode undergoes such large change, and why at a certain level of imperfection the system

undergoes a seemingly discontinuous change in qualitative behaviour. Even if a lot of nonlinear FEM analyses (the flow shown in Fig. 4a) were carried out varying the imperfection modes and levels, the results would be of little direct benefit if another geometry or member arrangement was to be considered. An alternative is to examine the relationship between the linear and nonlinear analyses. Using the conventional linear buckling procedure shown in Fig. 4c, the innovative reduced stiffness analysis shown in Fig. 4d has been performed for continuum shells and single layer lattice roofs by the previous studies [6–10].

In addition, in the case of not only shallow spherical shells [11] but also cylindrical roof shells [12], partly as a consequence of the high levels of loading imperfection, resulting from the effects of edge constraints, a shell under pressure loading has been shown to frequently buckle into modes having a contribution not just from associated with minimum critical load but also from the adjacent longer for roller support or shorter for pinned support. Even though these modes may not correspond with integer value, the development of buckling lobes (half-wave length in arch direction) has been shown to allow an essentially non-integer form of buckling mode [3]. Extensive numerical experiments have shown that upper bounds of the sensitivity are to geometric imperfection provided by the linear critical load predicted from the idealised bifurcation analytical load by Eq. (10). As lower bound to the imperfection sensitivity, the reduced stiffness critical load is given to be $p^* = p^b$ (Eq. (11)) for the pressurised cylindrical shells.

Fig. 3 Idealised model with uniform fundamental states.

Fig. 4 Various buckling analytical procedures using FEM.

3. Nonlinear Buckling Analysis of Cylindrical Lattice Frames Using the Finite Element Method

Yamada [7, 10] adopted a rigidly jointed single layer lattice frame roof shown in Fig. 5. In this section, the lattice roof has roller supports in which the joints on rigid boundary beams can only move horizontal direction to the beams except for fixed four corner joints. The network pattern has six spans in ridge direction and ten spans in arch direction as shown in Fig. 5. All members are straight and their joints are on the cylindrical surface, in which $R = 21$ m. The base angle of triangle unit θ is 60° (1.047 rad), and the

length of all the members is the same as 3.5 m. The rise of the roof is 5.24 m, the span 27.77 m and the central angle ϕ 82.77° (1.445 rad). Each constitutive lattice member is of steel tube with $E = 205$ N/mm², a 165.2 mm outside diameter and a 7.1 mm thickness.

Fig. 6 represents the relations between the pressure and deflection at sampling point No.10 (superimposed in Fig. 5). The lattice frame would be considered as a quasi-isotropic shell and the nonlinear FEM results can be compared with results of the nonlinear Ritz method using the analogically equivalent shell model. The good agreements are obtained even these two analytical procedures are quite different: here, the geometric imperfections illustrated in Fig. 6c are

Fig. 5 A lattice vault roof.

Fig. 6 Nonlinear for different imperfections translated from Yamada [7].

chosen to be the same normalised from as the linear bending solution. Examples of the total deflection modes and the incremental displacement modes at the buckling points along the center arc are shown in Fig. 7.

Because of the very large bending distortions induced by the edge restraints, the geometrically perfect model denoted by curve A displays a nonlinear behaviour which is clearly a highly imperfect form of the underlying linear bifurcation model. The deflection response can be observed to have a dramatically form depending on the imperfection amplitude w_s^0 . A small change in w_s^0 at around -20 cm (indicated by curve E) can result in a violent change in the nature of the response as indicated by for example curve F. As imperfections tend to either large positive or large negative values, the buckling

loads approach asymptotically what appears to be a characteristic lower bound.

Fig. 8 represents the effects of initial imperfection amplitude (normalised by the equivalent shell thickness $t = 194\text{ mm}$) on the buckling load and on the membrane stiffness ratios at the buckling point. These incremental membrane strain energy ratios associated with the loss of stiffness at the buckling points are shown to fall from around 0.4 on curve D to around 0.1 on B' in proportion to the reduction of buckling load. Accepting that the loss of stiffness in large deflection and the related imperfection induced reduction in buckling load, are consequences of the loss of membrane energy it is of considerable practical interest to study whether the lower bounds to imperfection sensitivity can be explained by reference to a linear reduced stiffness lower bound analysis.

Fig. 7 Variation of selected modes at the buckling points translated from Yamada [7].

Fig. 8 Imperfection sensitivity of buckling pressure and incremental membrane strain energies for behaviour of latticed roofs translated from Yamada [7].

4. Application of the reduced stiffness concept to the design evaluation of overall buckling load

For design of light-weight cylindrical lattice roofs, it would be necessary to know adequate resistance to buckling under vertical, self-weight and snow, loading. A large number of studies in Japan [13–17] have shown that the nonlinear snap buckling exhibited by even geometrically perfect cylindrical lattice roofs is very complicated. As already mentioned in section 3, the good agreement between the critical load carrying

capacity by using the reduced stiffness method and the lower bound to imperfection-sensitive, non-linear numerical experiments is obtained. It has been also suggested that the reduced stiffness model can be applied to obtain the overall buckling load criteria at the stage for checking the overall (global) instability of lattice roofs in structural design flow shown in Fig. 9. In the stage of the check of lattice member section in Fig. 9, the check target would be the individual member buckling type whose elastic buckling loads are insensitive to any imperfection. (But the elastic-plastic interaction effect in the member buckling reduces the critical load.) However, even for the elastic buckling, the overall buckling loads of shell-like lattice frames are much affected by imperfections. In the present design procedure, the overall buckling type includes any buckling mode type having longer buckling half-wave length rather than the length of the lattice member.

The present evaluation procedure of overall buckling load carrying capacity summarized as shown in Fig. 10, needs first to set the intrinsic buckling mode which is the most important factor in buckling design and is defined as the inverse number of a buckling length (or lobe). The buckling length of lattice shell can be considered to be the same as the

Euler buckling length of a single column under compression. For increasing the buckling load carrying capacity, the structural designer can adopt how to increase the buckling mode index (how to

increase the buckling length) with slightly varying the one of some geometrical dimension, instead of changing all the lattice members to be a costly bigger size.

Fig. 9 An appropriate design procedure on lattice roof structures.

Fig. 10 Design evaluation procedure for the overall buckling load carrying capacity of the lattice roof structures.

4.1 Fundamental Buckling Mode Index

The buckling half-wave number in longitudinal direction in pressurised cylindrical lattice roofs can be easily known to be $m = 1$ from Eqs. (11) and (12). For buckling half-wave number in arch direction n , the fundamental buckling mode index $B_0 \equiv n_0 L / (R\phi)$ has been adopted to be that of SS3 boundary condition plotted in Fig. 11 originally analysed obtained by Yamada [2]. The plotted curve may be approximately written as the following regression equation,

$$B_0 = K_0 - K_1 \times \log_{10} \beta + K_2 \times (\log_{10} \beta)^2 \quad (13)$$

where $\beta \equiv \sqrt{(A_{11} / A_{22}) - (A_{12} / A_{22})^2 L^2 / (Rt)}$ is the geometric parameter. The coefficients K_0 , K_1 and K_2 in Eq.(13) are

$$K_0 = 1.029, K_1 = 0.056, K_2 = 0.444 \text{ for } 1 \leq \beta < 10^3$$

$$K_0 = 10.371, K_1 = 6.131, K_2 = 1.431 \text{ for } 10^3 \leq \beta < 10^5$$

4.2 Equivalent Buckling Mode Index

The fundamental buckling mode index B_0 has been from that of idealised prebuckling uniform stress state model shown in Fig. 3, however, the actual buckling mode is known from the previous studies [3, 10, 17] to be slightly varied in dependence on the boundary conditions. In this study, the modifier index b is adopted to be -0.8 for roller supports and 0.3 for pinned supports referring to lots of results from the preliminary nonlinear finite element analytical

Fig. 11 Effects of geometric parameter on buckling modes.

experiments considering various imperfection amplitudes [17]. Then the equivalent buckling mode index B_D is defined using B_0 and b as

$$B_D = \max \left(B_0 + b, \frac{L}{R\phi} \right) \quad (14)$$

4.3 Computation of the Reduced Stiffness Load p^*

As before mentioned in section 2, the lower bound analysis for imperfection-sensitive cylindrical roof shells gives the reduced stiffness critical load $p^* = p^b$ (Eq. (11)) for the normalised circumferential buckling mode index $B=an$ as

$$p^* = \frac{\pi^2}{RL^2} \frac{\{ D_{11} + 2(D_{12} + 2D_{66})B^2 + D_{22}B^4 \}}{B^2} \quad (15)$$

where D_{11} , D_{12} , D_{22} and D_{66} are the bending stiffness components of Eq.(4) for the continuum shell analogy.

Closed dots in Fig. 12 show the nonlinear FEM buckling loads and the associated incremental modes at buckling points computed using the inverse number of half-wave length for the same lattice roofs having the aspect ratio $\alpha = 0.69$ and the equivalent shell thickness $t = 194$ mm in section 3, however the open angle are varied to be 30 degrees ($\phi = \pi/6$), 60 degrees ($\phi = \pi/3$), and various imperfection amplitudes of dimple type imperfection modes in reference [17]. The two boundary conditions, pinned as well as roller supports, have been adopted. In the figure, the broken lines are p^* from Eq. (15) and the completely same in the case of pinned and roller support conditions. The open dots represent the buckling loads by FEM based linear analysis which are computed using the two procedures of upper bound p^c and lower bound p^* .

From Fig. 12, the lower bound of the spectrum plots of the buckling loads about the each remarkable mode at buckling point for imperfect lattice shells is showing to be limited by the present reduced stiffness load spectra equation Eq. (15). More detail confirmation is done in Fig. 13; the horizontal axis is adopted to be the imperfection amplitude. In this lattice roof model, since $L = 21$ m, the imperfection

Fig. 12 Comparison between the present simplified evaluation and the nonlinear finite element analytical results for imperfect latticed roofs by Ref. [18].

amplitude $w_0 = L/1000 = 21$ mm is around $w_0/t = 0.1$. Included the relatively large imperfection, $-1 < w_0/t < 1$, the present evaluation Eq. (15) using $B = B_D$ is shown to give rational design evaluation as shown in Fig. 13b. As the results, the elastic overall buckling load carrying capacity p^e is defined as p^* given in Eq. (15) for $B = B_D$ in this paper.

Next, in the specific case for the cylindrical lattice roof having equilateral triangle unit ($\theta = \pi/3$), the member buckling load associated with Euler buckling

for a pinned end column may be given by considering a pre-buckling uniform axial stress state [2] as

$$p^{mem} = \sqrt{3} \frac{\pi^2 EI}{Rl^3} \quad (16)$$

In the case, the elastic overall buckling load p^e is written with the member buckling load p^{mem} as Eq. (17) from Eqs (15) and (16) and $B = B_D$.

$$p^e = \frac{49}{52} \frac{l^2}{L^2} \left(B_D + \frac{1}{B_D} \right)^2 p^{mem} \quad (17)$$

If $p^e > p^{mem}$, the collapse of lattice roofs would be

Fig. 13 Good evaluation of the present overall buckling load $p^e = p^*$ obtained by $B=B_D$ compared with nonlinear FEM models having imperfections.

induced by the start of the member buckling before the occurrence of the overall buckling; for the member instability, the elastic-plastic interactive load-carrying capacity would be generally checked at the stage(4) in Fig. 9.

4.4 Evaluation of Elastic-Plastic Buckling Load Carrying Capacity for Design

If $p^e < p^{mem}$, it would be at times needed to check on the reduction of the overall buckling load due to the effects of the interaction between the elastic buckling and the steel yielding. But one of the most difficult design aspects in shells and shell-like lattice structures has been the prediction of an appropriate allowance for additional reductions arising from interaction between elastic and plastic nonlinearities. The simplicity of the reduced stiffness method makes it a particularly convenient basis for predict of plastic collapse design estimates by Croll [18]. On the basis of the reduced stiffness buckling model, any imperfection introduction into the lattice shell will, at a prescribed pressure level, provide upper bounds of

the incremental deformation compared with that predicted for the exact lattice shell behaviour. This means that incremental stress components found using the reduced stiffness model will, at this prescribed pressure level, be upper bounds of those occurring in the exact behaviour. Consequently, the load for yielding using reduced stiffness method has been proposed to be a lower bound of the exact yield occurrence by Yamada and Croll [12].

The plastic squash load for equilateral triangle unit, associated with the idealised compressed force of diagonal members ($-qRl/\sqrt{3}$) by assuming uniform membrane stress state) would be obtained as

$$p^{sqh} = \sqrt{3} \frac{A}{Rl} \sigma_Y \quad (18)$$

where $A \equiv$ the cross sectional area of the member and $\sigma_Y \equiv$ the yielding stress.

The simplest design evaluation of elastic-plastic buckling loads may be to use the following empirical formulation.

$$\left(\frac{p^u}{p^e} \right)^{\gamma_1} + \left(\frac{p^u}{p^{sqh}} \right)^{\gamma_2} = 1 \quad \text{or} \quad \left(\frac{\lambda^2 p^u}{p^{sqh}} \right)^{\gamma_1} + \left(\frac{p^u}{p^{sqh}} \right)^{\gamma_2} = 1 \quad (19)$$

In the equation, p^u is the elastic-plastic buckling load carrying capacity for design, $\lambda = (p^{sqh}/p^e)^{1/2}$ the so-called generalized slenderness ratio. Dulacska [19] adopted $\gamma_1 = \gamma_2 = 2$ for the buckling of concrete shells and called the above empirical formula “Dunkerley equation”. But Dunkerley [20] wrote the empirical equation for the vibration frequency of shafts and did not discuss on buckling or on material yielding at all. Similarly, Kato [13] recommended ($\gamma_1 = 1$, $\gamma_2 = 2$) and called Eq. (19) “modified Dunkerley equation” for the cylindrical lattice roof shells using not the load factor but the stress of a selected representative lattice member. On the other hand, Kawamoto and Yuhara [21] for complete steel cylindrical shells proposes to express the equation for computing (γ_1, γ_2) in terms of the geometric parameter of cylinders $L/\sqrt{Rt}$ and the maximum imperfection amplitude w_0/t .

5. Conclusions

It has been recognized that a single layer cylindrical lattice shell roof structure displays a complex form of imperfection-sensitive nonlinear buckling behavior. This prebuckling nonlinearity has been understood to be induced by the total equivalent imperfections involving the potential-load-induced imperfection and actual-geometric imperfection. Based upon the reduced stiffness buckling analysis associated with the lower bound for imperfection-sensitive buckling load carrying capacity, an alternative design evaluation procedure of overall buckling loads for the single layer cylindrical lattice shell roof structures has been proposed. The previous fruits using the parametric nonlinear buckling analysis for the FEM modelling have been used in this paper to confirm the quality of the results by the present evaluation method. The present design concept provides a simple, safe, and rational basis and would also provide an understanding of greater significance for many other single layer lattice shell buckling problems.

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References

- [1] S. Yamada, Buckling Theory: Handbook of the Vibration and Buckling of Shells, edited by the Japanese Society of Mechanical Engineers, Gihodo Publications, Tokyo, 2003, pp. 29–42. (in Japanese)
- [2] S. Yamada, Analysis and Design for Cylindrical Latticed Roof Shells, Architectural Institute of Japan, 1995. (in Japanese)
- [3] S. Yamada and J. G. A. Croll, Buckling behavior of pressure loaded cylindrical panels, *Journal of Engineering Mechanics* 115 (2) (1989) 327–344.
- [4] J. Singer, J. Arbocz and T. Weller, Buckling Experiments, Vol. 2, Wiley, 2002.
- [5] S. Yamada and M. Uchiyama, Imperfection-sensitive buckling and postbuckling of spherical shell caps, buckling and postbuckling structures, edited by B. G. Falzon and M. H. Aliabadi, Imperial College Press, London, 2008, pp. 309–334.
- [6] J. G. A. Croll, Towards simple estimates of shell buckling loads, *Der Stahlbau*: Part I, Heft 8, 1975, p. 8; Part II, Heft 9, 1975.
- [7] S. Yamada, Relationship between non-linear numerical experiments and a linear lower bound analysis using finite element method on the overall buckling of reticular partial cylindrical space frames, *Computer Applications in Civil and Building Engineering*, Kozo System Inc, 1991, pp. 259–266.
- [8] S. Yamada, Buckling Analysis: Handbook of Computational Mechanics, edited by Yagawa and Miyazaki, Asakura Publication, Tokyo, 2007, pp. 252–261. (in Japanese)
- [9] S. Yamada, A. Takeuchi, Y. Tada and K. Tsutsumi, Imperfection-sensitive overall buckling of single-layer lattice domes, *Journal of Engineering Mechanics* 127 (4) (2001) 382–386.
- [10] S. Yamada and T. Taguchi, Snow Load Carrying Capacity of Single-Layer Latticed Cylindrical Roof Structures, *Snow Engineering Recent Advances*, Balkema Publishers, 1997, pp. 235–240.
- [11] S. Yamada, K. Uchiyama and M. Yamada, Experimental investigation of the buckling of shallow spherical shells, *International Journal of Non-Linear Mechanics* 18 (1) (1983) 37–54.

- [12] S. Yamada and J. G. A. Croll, Elasto-plastic buckling design for shallow cylindrical shells under external pressure, in: Proceedings of IASS-ASCE International Symposium, ASCE, 1994, pp. 936–945.
- [13] S. Kato, M. Iida and T. Yamashita, Buckling loads of simply supported single layer latticed roofs under uniform vertical loading, Journal of Structural Engineering 41B (1995) 247–257. (in Japanese)
- [14] M. Murakami, K. Heki and H. Ohnishi, Examples of numerical analytical solutions for the elastic buckling of single layer latticed cylindrical shells having equilateral triangle unites, in: Proceedings of Annual Meeting of the Architectural Institute of Japan, 1989, pp. 1259–1260. (in Japanese)
- [15] M. Konakawa, Overall buckling analysis of cylindrical latticed shell roofs having arbitrary supports on straight beam edges in ridge direction, Journal of Structural and Construction Engineering 283 (1979) 76–86. (in Japanese)
- [16] T. Suzuki and T. Ogawa, Buckling analysis of cylindrical truss roof shells, Journal of Structural and Construction Engineering 288 (1980) 29–37. (in Japanese)
- [17] S. Yamada, T. Kikuchi and Y. Matsumoto, Effects of localized single dimple imperfections on the buckling of latticed cylindrical shells, in: Proceedings of 9th Asian Pacific Conference on Shell and Spatial Structures, Nagoya, 2009.
- [18] J. G. A. Croll, Lower bound elasto-plastic buckling of cylinders, in: Proceedings of Instn Civil Engrg, 1981, Part 2, Vol. 71, pp. 235–261.
- [19] E. Dulacska, Buckling of elastic-plastic shells, IASS Bulletin, 1979, Vol. 68, pp. 15–20.
- [20] S. Dunkerley, On the whirling and vibration of shafts, Philosophical Transactions of the Royal Society of London, Ser. A 185, 1894, pp. 279–360.
- [21] Y. Kawamoto and T. Yuhara, Buckling of fabricated ring-stiffened steel cylinders under axial compression, in: Proceedings of International Conference on Advances in Marine Structures, Dunfermline, 1986, pp. 262–280.