Some Transformation of a Multiple Hypergeometric Series of Lauricella Function of n Variables

Nabiullah Khan* and T. Kashmin

Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh 202002, India

Abstract: This paper deals with an integral transformation involving Whittaker function $M_{k,m}(x)$ into a multiple hypergeometric series of Lauricella function $F_A^{(n)}$ of n variables. A number of known and new transformation and reduction formulae for a hypergeometric function ${}_2F_1$, Appell function F_2 , Lauricella function $F_A^{(3)}$ and a hypergeometric function of four variables $F_p^{(4)}$ are derived as special cases.

Key words: Whittaker function, Lauricella function, Laplace transform.

1. Introduction and Definition

Lauricella [1] further generalized the four Appell functions $F_1, \dots, F_4$ to functions of n -variables and defined his functions as follows [2]:

$$F_A^{(n)}[a, b_1, \dots, b_n; c_1, \dots, c_n; x_1, \dots, x_n] = \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b_1)_{m_1} \dots (b_n)_{m_n}}{(c_1)_{m_1} \dots (c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \dots \frac{x_n^{m_n}}{m_n!} \quad (1)$$

$$|x_1| + \dots + |x_n| < 1;$$

Some transformations of Lauricella function which is given by Srivastava H.M and Manocha H.L [2]:

$$F_A^{(1)} = {}_2F_1, \quad F_A^{(2)} = F_2 \quad (2)$$

where, ${}_2F_1, F_2$ are Gauss hypergeometric function and Appell function

A general triple hypergeometric series $F^{(3)}$ of Srivastava's [2] defined as:

$$F^{(3)} \left[\begin{matrix} (a):: (b); (b'); (b''); (e); (e'); (e''); \\ (d):: (g); (g'); (g''); (h); (h'); (h''); \end{matrix} \middle| x, y, z \right] = \sum_{m,n,p=0}^{\infty} \frac{[(a)]_{m+n+p} [(b)]_{m+n} [(b')]_{n+p} [(b'')]_{p+m} [(e)]_m [(e')]_n [(e'')]_p}{[(d)]_{m+n+p} [(g)]_{m+n} [(g')]_{n+p} [(g'')]_{p+m} [(h)]_m [(h')]_n [(h'')]_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \quad (3)$$

where, (a) means $a_1, a_2, \dots, a_n$ and $(a)_n$ has the interpretation

$$\prod_{j=1}^A (a_j)_n = \prod_{j=1}^A \frac{\Gamma(a_j + n)}{\Gamma(a_j)}$$

A hypergeometric function of four variables defined by Pathan [3] are given in the following form:

*Corresponding author: Nabiullah Khan, Associate Professor, research field: special function. E-mail: nabi_khan1@rediffmail.com.

$$F_p^{(4)} \left[\begin{matrix} a: : b; -; -; -; d; e; f; g; \\ c: : -; -; -; -; d'; e'; f'; \end{matrix} \quad u, x, y, z \right] = \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q} (b)_{m+n+p} (d)_m (e)_n (f)_p (g)_q}{(c)_{m+n+p+q} (d')_m (e')_n (f')_p} \frac{u^m}{m!} \frac{x^m}{n!} \frac{y^n}{p!} \frac{z^p}{q!} \quad (4)$$

2. Main Transformation

An integral is established in the following form:

$$\begin{aligned} & \int_0^{\infty} t^{\lambda-1} e^{-(z+\frac{p}{2})t} M_{k,\mu}(pt) M_{k_1, \nu_1 - \frac{1}{2}}(y_1 t) \dots M_{k_n, \nu_n - \frac{1}{2}}(y_n t) dt \\ &= \frac{p^{\mu+\frac{1}{2}} y_1^{\nu_1} \dots y_n^{\nu_n} \Gamma(b+r)}{\tau^{(b+r)}} \sum_{r=0}^{\infty} \frac{(\frac{1}{2} + \mu - k)_r p^r}{(2\mu+1)_r r!} \\ & F_A^{(n)} \left[b+r; \nu_1 - k_1, \dots, \nu_n - k_n; 2\nu_1, \dots, 2\nu_n; \frac{y_1}{\tau}, \dots, \frac{y_n}{\tau} \right] \end{aligned} \quad (5)$$

where, $b = \lambda + \mu + \nu_1 + \dots + \nu_n + \frac{1}{2}$, $Re(z + p + \frac{1}{2}Y) > 0$, $Y = y_1 + \dots + y_n$, $Re(b+r) > 0$,

$$\tau = z + p + \frac{1}{2}Y.$$

$F_A^{(n)}$ is called Lauricella function defined by Eq. (1) and $M_{k,m}(x)$ is Whittaker function [4, 5] defined as:

$$M_{k,m}(x) = x^{1/2+m} e^{-1/2x} {}_1F_1(1/2 + m - k, 2m_1 + 1; x) = x^{1/2+m} e^{-1/2x} \sum_{q=0}^{\infty} \frac{(\frac{1}{2}+m-k)_q x^q}{(2m+1)_q q!} \quad (6)$$

Eq. (5) can be obtained by expanding $M_{k,m}(x)$ in series with the help of Eq. (6) and integrating term by term with the help of the following result [6]:

$$\begin{aligned} & \int_0^{\infty} t^{\nu-1} e^{-pt} M_{k_1, \mu_1 - \frac{1}{2}}(a_1 t) \dots M_{k_n, \mu_n - \frac{1}{2}}(a_n t) dt = \frac{a_1^{\mu_1} \dots a_n^{\mu_n} \Gamma(\nu + M)}{(p+A)^{(\nu+M)}} \\ & F_A^{(n)} \left[\nu + M; \mu_1 - k_1, \dots, \mu_n - k_n; 2\mu_1, \dots, 2\mu_n; \frac{a_1}{(p+A)}, \dots, \frac{a_n}{(p+A)} \right] \end{aligned} \quad (7)$$

where, $M = \mu_1 + \dots + \mu_n$, $A = 1/2(a_1 + \dots + a_n)$, $Re(\nu + M) > 0$, $Re(p \pm 1/2a_1 + \dots \pm 1/2a_n) > 0$.

3. Special Cases

Some special cases of the main result Eq. (5) are given below:

(1) Setting $n = 1$ in Eq. (5), we get a result associated with Eq. (2):

$$\begin{aligned} & \int_0^{\infty} t^{\lambda-1} e^{-(z+\frac{p}{2})t} M_{k,\mu}(pt) M_{k_1, \nu_1 - \frac{1}{2}}(y_1 t) dt = \frac{p^{\mu+\frac{1}{2}} y_1^{\nu_1} \Gamma(\lambda + \mu + \nu_1 + \frac{1}{2} + r)}{(z + p + \frac{y_1}{2})^{(\lambda+\mu+\nu_1+\frac{1}{2}+r)}} \sum_{r=0}^{\infty} \frac{(\frac{1}{2} + \mu - k)_r p^r}{(2\mu+1)_r r!} \\ & \times {}_2F_1 \left(\begin{matrix} \lambda + \mu + \nu_1 + \frac{1}{2} + r, \nu_1 - k_1 & ; \\ & \frac{y_1}{(z+p+\frac{y_1}{2})} \\ 2\nu_1 & ; \end{matrix} \right) \end{aligned} \quad (8)$$

where, $Re(\lambda + \mu + \nu_1 + \frac{1}{2}) > 0, Re(z + p + \frac{y_1}{2}) > 0$

(2) Setting $n = 2$ in Eq. (5) and using Eq. (2), we get Appell function F_2 :

$$\int_0^\infty t^{\lambda-1} e^{-(z+\frac{p}{2})t} M_{k,\mu}(pt) M_{k_1,\nu_1-\frac{1}{2}}(y_1 t) M_{k_2,\nu_2-\frac{1}{2}}(y_2 t) dt = \frac{p^{\mu+\frac{1}{2}} y_1^{\nu_1} y_2^{\nu_2} \Gamma(b+r)}{(z+p+\frac{y_1}{2}+\frac{y_2}{2})^{b+r}} \sum_{r=0}^\infty \frac{(\frac{1}{2}+\mu-k)_r p^r}{(2\mu+1)_r r!}$$

$$\times F_2 \left[b+r; \nu_1-k_1, \nu_2-k_2; 2\nu_1, 2\nu_2; \frac{y_1}{(z+p+\frac{y_1}{2}+\frac{y_2}{2})}, \frac{y_2}{(z+p+\frac{y_1}{2}+\frac{y_2}{2})} \right] \quad (9)$$

where, $b = \lambda + \mu + \nu_1 + \nu_2 + \frac{1}{2}, Re(z + p + \frac{y_1}{2} + \frac{y_2}{2}) > 0, Re(b+r) > 0$

Appell's function F_2 [2] is defined as:

$$F_2[a, b, b'; c, c'; x, y] = \sum_{m,n=0}^\infty \frac{(a)_{m+n} (b)_m (b')_n x^m y^n}{(c)_m (c')_n m! n!} \quad (10)$$

(3) Setting $k_1 = k_2 = 0$ in Eq. (9) and using the following known transformation [7]:

$$F_2[\alpha, \beta, \beta', 2\beta, 2\beta'; 2x, y] = (1-x)^{-\alpha} H_4 \left[\alpha, \beta', \beta + \frac{1}{2}, 2\beta'; \frac{x^2}{4(1-x)^2}, \frac{y}{1-x} \right] \quad (11)$$

$$\int_0^\infty t^{\lambda-1} e^{-(z+\frac{p}{2})t} M_{k,\mu}(pt) M_{0,\nu_1-\frac{1}{2}}(y_1 t) M_{0,\nu_2-\frac{1}{2}}(y_2 t) dt = \frac{p^{\mu+\frac{1}{2}} y_1^{\nu_1} y_2^{\nu_2} \Gamma(b+r)}{(z+p+\frac{y_1}{2}+\frac{y_2}{2})^{b+r}} \sum_{r=0}^\infty \frac{(\frac{1}{2}+\mu-k)_r p^r}{(2\mu+1)_r r!}$$

$$\times F_2 \left[b+r, \nu_1, \nu_2; 2\nu_1, 2\nu_2; \frac{y_1}{(z+p+\frac{y_1}{2}+\frac{y_2}{2})}, \frac{y_2}{(z+p+\frac{y_1}{2}+\frac{y_2}{2})} \right] \quad (12)$$

We get an infinite sum of Horn function H_4 in the following form:

$$\int_0^\infty t^{\lambda-1} e^{-(z+\frac{p}{2})t} M_{k,\mu}(pt) M_{0,\nu_1-\frac{1}{2}}(y_1 t) M_{0,\nu_2-\frac{1}{2}}(y_2 t) dt = p^{\mu+\frac{1}{2}} y_1^{\nu_1} y_2^{\nu_2} \Gamma(b+r) \sum_{r=0}^\infty \frac{(\frac{1}{2}+\mu-k)_r p^r}{(2\mu+1)_r r!}$$

$$\times \left(z + p - \frac{3y_1}{2} + \frac{y_2}{2} \right)^{-b-r} H_4 \left[b+r, \nu_2, \nu_1 + \frac{1}{2}, 2\nu_2; \frac{y_1^2}{(z+p-\frac{3y_1}{2}+\frac{y_2}{2})^2}, \frac{y_2}{(z+p-\frac{3y_1}{2}+\frac{y_2}{2})} \right] \quad (13)$$

where, H_4 is the Horn function [8] defined as:

$$H_4[\alpha, \beta, \gamma, \delta; x, y] = \sum_{m,n=0}^\infty \frac{(\alpha)_{2m+n} (\beta)_n x^m y^n}{(\gamma)_m (\delta)_n m! n!} \quad (14)$$

(4) Setting $k_1 = k_2 = 0$ in Eq. (9) and using the following result of Srivastava and Karlsson [9]

$$F_2 = F_{0:1:1}^{1:1:1}, \quad (15)$$

we get a Kampé de Fériet function of two variables:

$$\int_0^\infty t^{\lambda-1} e^{-(z+\frac{p}{2})t} M_{k,\mu}(pt) M_{0,\nu_1-\frac{1}{2}}(y_1 t) M_{0,\nu_2-\frac{1}{2}}(y_2 t) dt = \frac{p^{\mu+\frac{1}{2}} y_1^{\nu_1} y_2^{\nu_2} \Gamma(b+r)}{\tau^{b+r}} \sum_{r=0}^\infty \frac{(\frac{1}{2}+\mu-k)_r p^r}{(2\mu+1)_r r!}$$

$$F_{0:1:1}^{1:1:1} \left[\begin{matrix} b+r; \nu_1, \nu_2; \\ : 2\nu_1, 2\nu_2; \end{matrix} \quad \frac{y_1}{\tau}, \frac{y_2}{\tau} \right] \quad (16)$$

where, $b = \lambda + \mu + \nu_1 + \nu_2 + \frac{1}{2}, \tau = z + p + \frac{y_1}{2} + \frac{y_2}{2}$ and $F_{l:m:n}^{p:q:s}$ is Kampé de Fériet function defined by Srivastava and Panda [10].

(5) Setting $n = 3$ in Eq. (5), we get a new result associated with Lauricella's function $F_A^{(3)}$ as:

$$\begin{aligned} & \int_0^\infty t^{\lambda-1} e^{-(z+\frac{p}{2})t} M_{k,\mu}(pt) M_{k_1, \nu_1 - \frac{1}{2}}(y_1 t) M_{k_2, \nu_2 - \frac{1}{2}}(y_2 t) M_{k_3, \nu_3 - \frac{1}{2}}(y_3 t) dt \\ &= \frac{p^{\mu+\frac{1}{2}} y_1^{\nu_1} y_2^{\nu_2} y_3^{\nu_3} \Gamma(b+r)}{\tau^{b+r}} \sum_{r=0}^\infty \frac{(\frac{1}{2} + \mu - k)_r p^r}{(2\mu + 1)_r r!} \\ & \times F_A^{(3)} \left[b+r; \nu_1 - k_1, \nu_2 - k_2, \nu_3 - k_3; 2\nu_1, 2\nu_2, 2\nu_3; \frac{y_1}{\tau}, \frac{y_2}{\tau}, \frac{y_3}{\tau} \right] \end{aligned} \quad (17)$$

where, $b = \lambda + \mu + \nu_1 + \nu_2 + \nu_3 + \frac{1}{2}$, $Re(b+r) > 0$, $\tau = z + p + \frac{y_1+y_2+y_3}{2}$, $Re(z + p + \frac{y_1+y_2+y_3}{2}) > 0$.

Using a following known result of Srivastava and Karlsson [9]:

$$\begin{aligned} & F_A^{(3)}[\alpha, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_3; x, y, z] \\ &= F^{(3)} \left[\begin{matrix} (\alpha) : _ ; _ ; _ ; \beta_1, \beta_2, \beta_3; \\ _ : _ ; _ ; _ ; \gamma_1, \gamma_2, \gamma_3; \end{matrix} \quad x, y, z \right] \end{aligned} \quad (18)$$

the Eq. (17) becomes:

$$\begin{aligned} & \int_0^\infty t^{\lambda-1} e^{-(z+\frac{p}{2})t} M_{k,\mu}(pt) M_{k_1, \nu_1 - \frac{1}{2}}(y_1 t) M_{k_2, \nu_2 - \frac{1}{2}}(y_2 t) M_{k_3, \nu_3 - \frac{1}{2}}(y_3 t) dt \\ &= \frac{p^{\mu+\frac{1}{2}} y_1^{\nu_1} y_2^{\nu_2} y_3^{\nu_3} \Gamma(b+r)}{\tau^{b+r}} \sum_{r=0}^\infty \frac{(\frac{1}{2} + \mu - k)_r p^r}{(2\mu + 1)_r r!} \\ & F^{(3)} \left[\begin{matrix} b+r : _ ; _ ; _ ; \nu_1 - k_1, \nu_2 - k_2, \nu_3 - k_3; \\ _ : _ ; _ ; _ ; 2\nu_1, 2\nu_2, 2\nu_3; \end{matrix} \quad \frac{y_1}{\tau}, \frac{y_2}{\tau}, \frac{y_3}{\tau} \right] \end{aligned} \quad (19)$$

where, $b = \lambda + \mu + \nu_1 + \nu_2 + \nu_3 + \frac{1}{2}$, $Re(b+r) > 0$, $\tau = z + p + \frac{y_1+y_2+y_3}{2}$, $Re(z + p + \frac{y_1+y_2+y_3}{2}) > 0$. $F^{(3)}$ is

the triple hypergeometric series defined by Eq. (3).

Further expanding $F^{(3)}$ into a series with the help of Eq. (3), we get a hypergeometric function $F_p^{(4)}$ of four variables defined by Pathan [7]:

$$\begin{aligned} & \int_0^\infty t^{\lambda-1} e^{-(z+\frac{p}{2})t} M_{k,\mu}(pt) M_{k_1, \nu_1 - \frac{1}{2}}(y_1 t) M_{k_2, \nu_2 - \frac{1}{2}}(y_2 t) M_{k_3, \nu_3 - \frac{1}{2}}(y_3 t) dt = \frac{p^{\mu+\frac{1}{2}} y_1^{\nu_1} y_2^{\nu_2} y_3^{\nu_3} \Gamma(b)}{\tau^b} \\ & \times F_p^{(4)} \left[\begin{matrix} b : _ ; _ ; _ ; _ ; \nu_1 - k_1, \nu_2 - k_2, \nu_3 - k_3; \mu - k + 1/2; \\ _ : _ ; _ ; _ ; _ ; 2\nu_1, 2\nu_2, 2\nu_3; 2\mu + 1; \end{matrix} \quad \frac{y_1}{\tau}, \frac{y_2}{\tau}, \frac{y_3}{\tau}, \frac{p}{\tau} \right] \end{aligned} \quad (20)$$

If we replace Whittaker function $M_{k,\mu}(pt)$ by $W_{k,\mu}(pt)$ in Eq. (20), we will get a known result of Pathan [7].

(6) Setting $k = k_1 = k_2 = k_3 = 0 = \lambda$, $z = \frac{-p}{2}$, $\nu_1 = \mu + \frac{1}{2}$, $\nu_2 = \nu_3 = \nu + \frac{1}{2}$, $p = 2\alpha$, $y_1 = 2\beta$, $y_2 = 2\gamma$, $y_3 = 2\delta$ in Eq. (20) and using the following known result [4]

$$M_{0,\mu}(x) = 2^{2\mu} \Gamma(\mu + 1) x^{\frac{1}{2}} I_\mu\left(\frac{1}{2}x\right),$$

we get

$$\begin{aligned}
& \int_0^{\infty} t I_{\nu}(\alpha t) I_{\nu}(\beta t) I_{\mu}(\gamma t) I_{\mu}(\delta t) dt \\
&= \frac{(\alpha\beta)^{\mu}(\gamma\delta)^{\nu} \rho^{-2-2\mu-2\nu} \Gamma(2+2\mu+2\nu)}{2^{2\mu+2\nu} (\Gamma(\mu+1))^2 (\Gamma(\nu+1))^2} \\
& F_p^{(4)} \left[\begin{matrix} 2+2\mu+2\nu; _; _; _; \mu+\frac{1}{2}, \nu+\frac{1}{2}, \nu+\frac{1}{2}, \mu+\frac{1}{2}; \\ _; _; _; _; _; 2\mu+1, 2\nu+1, 2\nu+1, 2\mu+1; \end{matrix} \quad \frac{2\alpha}{\rho}, \frac{2\beta}{\rho}, \frac{2\gamma}{\rho}, \frac{2\delta}{\rho} \right] \quad (21)
\end{aligned}$$

where, $\rho = \alpha + \beta + \gamma + \delta$, $Re(1 + \mu + \nu) > 0$ and $Re(\alpha + \beta + \gamma + \delta) > 0$. Eq. (21) gives an interesting new result for the hypergeometric function $F_p^{(4)}$ and $I_{\nu}(x)$ is the modified Bessel function [2; p. 209(1)] defined as

$$I_{\nu}(x) = \frac{(x/2)^{\nu}}{\Gamma(\nu+1)} {}_0F_1 \left[\begin{matrix} -; \\ \nu+1; \end{matrix} \quad \frac{x^2}{4} \right] \quad (22)$$

4. Conclusions

In the present investigation, we have established an integral transformation involving Whittaker function $M_{k,m}(x)$ into a multiple hypergeometric series of Lauricella function $F_A^{(n)}$ of n variables, and generalize a number of known and new transformation and reduction formulae for a hypergeometric function ${}_2F_1$, Appell function F_2 , Lauricella function $F_A^{(3)}$ and a hypergeometric function of four variables $F_p^{(4)}$. The results appear in this paper may be found useful in some areas of mathematical physics and Engineering.

Acknowledgements

The authors are thankful to professor Pathan M.A of Aligarh Muslim University, India for his valuable suggestions.

References

- [1] Lauricella, G. 1893. "Sulle Funzioni Ipergeometriche a Piu Variabili." *Rend. Circ. Mat. Palermo* 7: 111-158.
- [2] Srivastava, H. M., and Manocha, H. L. 1984. *A Treatise on Generating Functions*. Halsted Press (Ellis Horwood Limited, Chichester): John Wiley and sons, New York.
- [3] Pathan, M. A. 1979. "On a Transformation of a General Hypergeometric Series of Four Variables." *Nederl. Akad. Wetensch. Proc Ser. A*. 82=Indag Math 41, New York, 171-5.
- [4] Whittaker, E. T. 1903. "An Expression of Certain Known Function as Generalized Hypergeometric Functions." *Bull. Amer. Math. Soc.* 10: 125-134.
- [5] Whittaker, E. T., and Watson, G. N. 1990. *A Course of Modern Analysis*, 4th ed. Cambridge, England. Cambridge University.
- [6] Erdelyi, A., 1954. "Table of integral transforms." *Vol. I, McGraw-Hill, New York*.
- [7] Erdelyi, A. 1948. "Transformation of Hypergeometric Functions of Two Variables." *proc. Roy. Soc. Edinburgh Sect. A* 62: 378-385.
- [8] Khan, B., and Pathan, M. A. 1982. "Some Transformations of Appell's Function F_4 ." *Tamkang J. Math.* 13 (1): 25-35.
- [9] Srivastav, H. M., and Karlsson, P. W. 1985. *Multiple Gaussian Hypergeometric Series*. Halsted Press (Ellis Horwood Limited, Chichester): John Wiley and sons, New York, Chichester Brisbane, Toronto.
- [10] Srivastav, H. M., and Panda, R. 1976. "An Integral Representation for the Product of Two Jacobi Polynomials." *J. London Math. Soc.* 12 (2): 419-425.