Existence of Positive Solutions to Semilinear Elliptic Systems Involving Concave and Convex Nonlinearities

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Abstract: In this paper, we studied the combined effect of concave and convex nonlinearities on the number of positive solutions for a semilinear elliptic system. We prove the existence of at least four positive solutions for a semilinear elliptic system involving concave and convex nonlinearities by using the Nehari manifold and the center mass function.

Key words: Elliptic systems, concave and convex nonlinearities, Nehari manifold, maximum principle.

1. Introduction

This paper deals with the existence and nonexistence of positive solutions to the following problem:

$$\begin{aligned} -\Delta u + u &= (\alpha + 1)u^\alpha v^{\beta+1} + \mu(\alpha' + 1)u^{\alpha'} v^{\beta'+1} \text{ in } \Omega \\ -\Delta v + v &= (\beta + 1)u^{\alpha+1} v^\beta + \mu(\beta' + 1)u^{\alpha'+1} v^{\beta'} \text{ in } \Omega \\ u > 0, v > 0 & \text{ in } \Omega \\ u = v = 0 & \text{ on } \partial\Omega \end{aligned} \quad (1)$$

where, Ω is a bounded regular domain in $\mathcal{R}^N$ ($N \geq 3$) containing 0 in its interior, $1 < p = \alpha + \beta + 1 < 2^* - 1$, $2^* = 2N/(N-2)$ is the Sobolev critical exponent, $0 \leq q = \alpha' + \beta' + 1 < 1$ and μ is a real parameter.

Semilinear scalar elliptic equations with concave and convex nonlinearities are widely studied; we refer the readers to [2, 3, 7, 13, 14, 21] etc.. For the semilinear elliptic systems, we refer to Ahammou [1, 5, 6, 9, 12, 15, 17, 19]. The model type is written as follows:

$$\begin{cases} -\Delta u + u = hu^s + \mu gu^t & \text{in } \Omega \\ 0 < u \in \mathcal{H}_0^1(\Omega) \end{cases} \quad (2)$$

When $h \equiv g \equiv 1$, Eq. (1) can be regarded as a

perturbation problem of the following equation:

$$\begin{cases} -\Delta u + u = u^s & \text{in } \Omega \\ 0 < u \in \mathcal{H}_0^1(\Omega). \end{cases} \quad (3)$$

The number of positive solutions of Eq. (3) is affected by the shape of the domain Ω , which has been the focus of a great deal of research in recent years. For example, we quote the works of Byeon [8], Dancer [10], Damascelli, L. et al [11] and Wu [22].

Considering the multiplicity of positive solutions to problem Eq. (1), we extended this method in this paper. We consider the Sobolev spaces $\mathcal{H}(\Omega) = \mathcal{H}_0^1(\Omega) \times \mathcal{H}_0^1(\Omega)$ with respect to the norm

$$\|(u, v)\| = (\|u\|^2 + \|v\|^2)^{1/2};$$

where,

$$\|u\| = \|u\|_{\mathcal{H}_0^1} = \left(\int_{\Omega} |\nabla u|^2 + u^2 \right)^{1/2}$$

is a standard norm in $\mathcal{H}_0^1(\Omega)$.

Since our approach is variational, we define the functional I_μ on $\mathcal{H}(\Omega)$ by

$$I_\mu(u, v) := (1/2)\|(u, v)\|^2 - \int_{\Omega} u^{\alpha+1} v^{\beta+1} dx - \mu \int_{\Omega} u^{\alpha'+1} v^{\beta'+1} dx$$

Let

$$C_{(\alpha, \beta)} := \inf_{u \in \mathcal{H}_0^1 \setminus \{0\}} \frac{\|u\|^2}{\left(\int_{\Omega} |u|^{(p+1)} dx \right)^{2/(p+1)}}$$

and

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$$\tilde{C}_{(\alpha,\beta)} := \inf_{(u,v) \in \mathcal{H} \setminus \{(0,0)\}} \frac{\|(u,v)\|^2}{\left(\int_{\mathbb{R}^N} |u|^{\alpha+1} |v|^{\beta+1} dx \right)^{2/(p+1)}}$$

From [16], $C_{(\alpha,\beta)}$ is achieved.

Lemma 1 Let Ω be a domain (not necessarily bounded) and $1 < p = \alpha + \beta + 1 < 2^* - 1$, we have

$$\tilde{C}_{(\alpha,\beta)} :=$$

$$\left[\left(\frac{\alpha+1}{\beta+1} \right)^{(\beta+1)/(p+1)} + \left(\frac{\alpha+1}{\beta+1} \right)^{-(\alpha+1)/(p+1)} \right] C_{(\alpha,\beta)}$$

with

$$\left[\left(\frac{\alpha+1}{\beta+1} \right)^{(\beta+1)/(p+1)} + \left(\frac{\alpha+1}{\beta+1} \right)^{-(\alpha+1)/(p+1)} \right]$$

simply written as $K(\alpha, \beta)$.

Proof The proof is essentially the same as in Ref. [5], but with minor modifications. Under the assumption $\mu \neq 0$, our problem Eq. (1) can be regarded as a perturbation of the system when $\mu = 0$; let $z = (x, y) \in \mathcal{R}^{N-1} \times \mathcal{R}$ and let O be a bounded smooth domain in $\mathcal{R}^{N-1}$; denote the N -ball B^N , the infinite strip S and the finite strip $S_{l,t}$ as follows:

$$B^N(z_0, r) = \{z \in \mathcal{R}^N / |z - z_0| < r\};$$

$$S = \{(x, y) \in \mathcal{R}^N / x \in O\};$$

$$S_{l,t} = \{(x, y) \in \mathcal{R}^N / l < y < t\}.$$

Wu [21] considered a perturbation of the finite strip $S_{-t,t}$, that is, a finite strip with a hole:

$$\Theta_t = S_{-t,t} \setminus \bar{\omega},$$

where ω is a bounded domain in $\mathcal{R}^N$ with $\omega \subset \subset S_{-t',t'}$ for some $t' > 0$.

In our work, we research the critical points as the minimizers of the energy functional associated with the problem Eq. (1) on the constraint defined by the Nehari manifold, which are solutions of our system.

Theorem 1 Assume that $N \geq 3$, $1 < p = \alpha + \beta + 1 < 2^* - 1$ and $0 \leq q = \alpha' + \beta' + 1 < 1$. For each $t > t_0$, there exists $\Lambda > 0$, so the problem Eq. (1) has at least four positive solutions for all $\mu \in (0, \Lambda)$.

This paper is organized as follows: in the Section 2, we give some preliminaries; while Section 3 is devoted to the proof of **theorem 1**.

2. Preliminaries

2.1 Nehari Manifold

Considering the Nehari manifold:

$$\mathcal{M}_\mu(\Omega) =$$

$$\{(u, v) \in \mathcal{H}(\Omega) \setminus \{(0, 0)\} / \langle I'_\mu(u, v), (u, v) \rangle = 0\}$$

Define

$$\begin{aligned} \phi_\mu(u, v) &= \langle I'_\mu(u, v), (u, v) \rangle \\ &= \|(u, v)\|^2 - (p+1) \int_\Omega u^{\alpha+1} v^{\beta+1} dx - \end{aligned}$$

$$\mu(q+1) \int_\Omega u^{\alpha'+1} v^{\beta'+1} dx$$

Then, for $(u, v) \in \mathcal{M}_\mu(\Omega)$

$$\begin{aligned} \langle \phi'_\mu(u, v), (u, v) \rangle &= 2\|(u, v)\|^2 - \\ &= (p+1)^2 \int_\Omega u^{\alpha+1} v^{\beta+1} dx - \mu(q+1)^2 \int_\Omega u^{\alpha'+1} v^{\beta'+1} dx \\ &= (1-p)\|(u, v)\|^2 - \mu(q-p)(1+q) \int_\Omega u^{\alpha'+1} v^{\beta'+1} dx \\ &= (1-q)\|(u, v)\|^2 - (p+1)(p-q) \int_\Omega u^{\alpha+1} v^{\beta+1} dx. \end{aligned} \quad (4)$$

Now, as in Tarantello [18], we split $\mathcal{M}_\mu(\Omega)$ in three parts:

$$\mathcal{M}_\mu^+(\Omega) = \{(u, v) \in \mathcal{M}_\mu(\Omega) : \langle \phi'_\mu(u, v), (u, v) \rangle > 0\}$$

$$\mathcal{M}_\mu^0(\Omega) = \{(u, v) \in \mathcal{M}_\mu(\Omega) : \langle \phi'_\mu(u, v), (u, v) \rangle = 0\}$$

$$\mathcal{M}_\mu^-(\Omega) = \{(u, v) \in \mathcal{M}_\mu(\Omega) : \langle \phi'_\mu(u, v), (u, v) \rangle < 0\}$$

It is well known that I_μ is of class C^1 in $\mathcal{H}(\Omega)$, and the solutions of Eq. (1) are the critical points of I_μ which is not bounded below on $\mathcal{H}(\Omega)$. Note that $\mathcal{M}_\mu(\Omega)$ contains every nontrivial solution of the problem Eq. (1). Moreover, we have the following results:

Let

$$\begin{aligned} \mu_0 &:= \\ &= \left[\frac{(p-1)}{(p-q)(q+1)} \right] \left[\frac{(1-q)}{(p-q)(p+1)} \right]^{\frac{(1-q)}{(p-1)}} \\ &= (K(\alpha, \beta) C_{(\alpha,\beta)})^{\frac{(1-q)(p+1)}{2(p-1)}} |\Omega|^{\frac{(p+1)}{2(p-1)} + \frac{(1+q)}{(1-q)}} \end{aligned} \quad (5)$$

where, $|\Omega|$ is the Lebesgue measure of domain Ω .

Lemma 1 We have $\mathcal{M}_\mu^0(\Omega) = \emptyset$ for all $\mu \in (0, \mu_0)$.

Proof Let us reason by contradiction. Suppose $\mathcal{M}_\mu^0(\Omega) \neq \emptyset$, let $u \in \mathcal{M}_\mu^0(\Omega)$, by Eq. (5), we have

$$\begin{aligned} \|(u, v)\|^2 &= [\mu(p-q)(q+1)/(p-1)] \int_{\Omega} u^{\alpha'+1} v^{\beta'+1} dx \\ &= [(p+1)(p-q)/(1-q)] \int_{\Omega} u^{\alpha+1} v^{\beta+1} dx \end{aligned} \quad (6)$$

by Eq. (6) and the Sobolev embedding, we have

$$\|(u, v)\| \leq \left[\frac{(1-q)}{(p-q)(p+1)} \right]^{1/(p-1)} (K(\alpha, \beta) C_{(\alpha, \beta)})^{(p+1)/2(p-1)} |\Omega|^{\frac{(p+1)}{2(p-1)}} \quad (7)$$

by Eq.(6) and the Holder inequality, we obtain

$$\|(u, v)\| \geq \mu^{1/(1-q)} \left[\frac{(p-1)}{(p-q)(q+1)} \right]^{1/(q-1)} |\Omega|^{\frac{(1-q)}{(1+q)}} \quad (8)$$

From Eqs. (7) and (8), we get $\mu \geq \mu_0$ which contradicts the fact that $\mu \in (0, \mu_0)$.

By **Lemma 1**, we can write $\mathcal{M}_{\mu}(\Omega) = \mathcal{M}_{\mu}^+(\Omega) \cup \mathcal{M}_{\mu}^-(\Omega)$, for $\mu \in (0, \mu_0)$. Define

$$\alpha_{\mu} = \inf_{(u,v) \in \mathcal{M}_{\mu}^+(\Omega)} I_{\mu}(\Omega), \alpha_{\mu}^+ = \inf_{(u,v) \in \mathcal{M}_{\mu}^+(\Omega)} I_{\mu}(\Omega) \text{ and } \alpha_{\mu}^- = \inf_{(u,v) \in \mathcal{M}_{\mu}^-(\Omega)} I_{\mu}(\Omega).$$

The following Lemma shows that the minimizers on $\mathcal{M}_{\mu}(\Omega)$ are usually critically points for I_{μ} .

Lemma 2 [22] Suppose that (u_0, v_0) is a local minimizer for I_{μ} on $\mathcal{M}_{\mu}(\Omega)$. Then for $\mu \in (0, \mu_0)$, (u_0, v_0) is a critical point of I_{μ} .

For each $(u, v) \in \mathcal{H}(\Omega) \setminus \{(0, 0)\}$, we write

$$t_{\max}(u, v) = \left[\frac{(1-q)\|(u, v)\|^{1-q}}{(p+1)(p-q) \int_{\Omega} u^{\alpha+1} v^{\beta+1} dx} \right]^{1/(p-1)}$$

Let

$$\mu_1 := \left[\frac{2aq}{b(q-p)} \right]^{\frac{-(q+1)(2q+1)}{(p+q)}} - b \left[\frac{2aq}{b(q-p)} \right]^{\frac{(q+1)(p-q-1)}{(p+q)}} |\Omega|^{\frac{(p-q)(p+1)}{(1+q)}}$$

where

$$a := (t_m)^{(1-q)}$$

and

$$b := (p+1)(t_m)^{(p-q)} [K(\alpha, \beta) C_{(\alpha, \beta)}]^{-(p+1)/2}$$

Lemma 3 For each $\mu \in (0, \mu_1)$ and $(u, v) \in \mathcal{H}(\Omega) \setminus \{(0, 0)\}$,

(i) there is an unique $t^- = (t^-u, t^-v) > t_m > 0$ such that

$$(t^-u, t^-v) \in \mathcal{M}_{\mu}^-(\Omega)$$

and

$$I_{\mu}(t^-u, t^-v) = \max_{t \geq t_m} I_{\mu}(tu, tv)$$

(ii) (t^-u, t^-v) is a continuous function for nonzero (u, v) ;

(iii) $\mathcal{M}_{\mu}^-(\Omega) =$

$$\left\{ (u, v) \in \mathcal{H}(\Omega) \setminus \{(0, 0)\} / \frac{1}{\|(u, v)\|} t^- \left(\frac{(u, v)}{\|(u, v)\|} \right) = 1 \right\}$$

(iv) there is an unique $0 < t^+ = (t^+u, t^+v) < t_m$ such that

$$(t^+u, t^+v) \in \mathcal{M}_{\mu}^+(\Omega)$$

and

$$I_{\mu}(t^+u, t^+v) = \min_{0 \leq t \leq t^-} I_{\mu}(tu, tv)$$

Proof Similar to the proof of Lemma 5 in Wu [22].

For $c > 0$, we define

$$I_0^c(u, v) = (1/2)\|(u, v)\|^2 - \int_{\Omega} cu^{\alpha+1} v^{\beta+1} dx$$

$$\mathcal{M}_0^c(\Omega) =$$

$$\left\{ (u, v) \in \mathcal{H}(\Omega) \setminus \{(0, 0)\} / \langle (I_0^c)'(u, v), (u, v) \rangle = 0 \right\}$$

$$\mathcal{M}_0(\Omega) =$$

$$\left\{ (u, v) \in \mathcal{H}(\Omega) \setminus \{(0, 0)\} / \langle (I_0)'(u, v), (u, v) \rangle = 0 \right\}$$

Note that $I_0 = I_0^c$ for $c = 1$ and for each $(u, v) \in \mathcal{M}_{\mu}^-(\Omega)$, there is an unique $S_{(u,v)} > 0$ such that $S_{(u,v)}(u, v) \in \mathcal{M}_0(\Omega)$. Furthermore, we have the following result.

Lemma 4 For each $(u, v) \in \mathcal{M}_{\mu}^-(\Omega)$,

(i) there is an unique $S^c(u, v) > 0$ such that $S^c((u, v))(u, v) \in \mathcal{M}_0^c(\Omega)$

and

$$\max_{t \geq 0} I_0^c(tu, tv) = I_0^c(S^c((u, v))(u, v))$$

$$= \left[(1/2)\|(u, v)\|^2 - \right.$$

$$\left. \|(u, v)\|^{(2-p)} \right] \left[\frac{\|(u, v)\|^{p+1}}{\int_{\Omega} cu^{\alpha+1} v^{\beta+1} dx} \right]^{2/p-1};$$

(ii) for $\theta \in (0, 1)$,

$$\begin{aligned} I_{\mu}(u, v) &\geq (1 - \mu\theta)^{\frac{p+1}{p-1}} I_0(S_{(u,v)}(u, v)) \\ &\quad - \left[\mu \left(\frac{1-q}{2(q+1)} \right) \right] \theta^{\frac{q+1}{q-1}} \end{aligned}$$

$$(K(\alpha, \beta)C_{(\alpha, \beta)})^{\frac{2(q+1)}{1-q}} |\Omega|^{\frac{2(p-q)(p+1)}{(p+1)(1-q)}}$$

and

$$\begin{aligned} I_\mu(u, v) &\leq (1 + \mu\theta)^{\frac{p+1}{p-1}} I_0(S_{(u, v)}(u, v)) \\ &\quad + \left[\mu \left(\frac{1-q}{2(q+1)} \right) \right] \theta^{\frac{q+1}{q-1}} \\ &\quad (K(\alpha, \beta)C_{(\alpha, \beta)})^{\frac{2(q+1)}{1-q}} |\Omega|^{\frac{2(p-q)(p+1)}{(p+1)(1-q)}} \end{aligned}$$

Proof (i) we have that

$$I_0^c(tu, tv) =$$

$$(t^2/2) \|(u, v)\|^2 - t^{(q+1)} \int_{\Omega} cu^{\alpha'+1} v^{\beta'+1} dx$$

Let $g(t) = at^p - bt^{(q+1)}$ with $a = (1/2$

$\|(u, v)\|^2$ and $b = \int_{\Omega} cu^{\alpha'+1} v^{\beta'+1} dx$. Furthermore,

we have that g attains its maximum at $t_m = (ap/b(q+1))^{1/(q-p)} = S^c(u, v) > 0$. Thus,

$$I_0^c(S^c((u, v))(u, v)) = g(t_m)$$

$$= \left[(1/2) \|(u, v)\|^2 - \right.$$

$$\left. \|(u, v)\|^{(2-p)} \right] \left[\frac{\|(u, v)\|^{p+1}}{\int_{\Omega} cu^{\alpha'+1} v^{\beta'+1} dx} \right]^{2/p-1}$$

(ii) for each $(u, v) \in \mathcal{M}_\mu^-(\Omega)$, let $c = 1/(1 - \mu\theta)$, $S^c = S^c(u, v) > 0$ and $S_u > 0$ such that $S^c(u, v) \in \mathcal{M}_0^c(\Omega)$ and $S_u(u, v) \in \mathcal{M}_0(\Omega)$. Then, for each $\theta \in (0, 1)$, we have

$$\begin{aligned} &\int_{\Omega} S^c u^{\alpha'+1} v^{\beta'+1} dx \leq \\ &(K(\alpha, \beta)C_{(\alpha, \beta)})^{(q+1)} |\Omega|^{\frac{(p-q)(q+1)}{(p+1)}} \|S^c(u, v)\|^{q+1} \\ &\leq [(1-q)/2] \theta^{\frac{q+1}{q-1}} |\Omega|^{\frac{(p-q)(q+1)}{(p+1)}} (K(\alpha, \beta)C_{(\alpha, \beta)})^{\frac{2(q+1)}{1-q}} \\ &\quad + \left(\frac{(1+q)}{2} \theta \right) \|S^c(u, v)\|^2 \end{aligned}$$

Thus, by (i) and Lemma 3 (i), we obtain (ii).

3. Existence Result

3.1 Existence of a Local Minimum in $\mathcal{M}_\lambda^+(\Omega)$

Let $\mu_2 = (p-1)(q+1)/2(p-q)$.

Lemma 5 For $\mu \in (0, \mu_2]$, we have

(i) for each $(u, v) \in \mathcal{M}_\mu^+(\Omega)$, $I_\mu(u, v) < 0$. In

particular

$$\alpha_\mu(\Omega) \leq \alpha_\mu^+(\Omega) < 0;$$

(ii) I_μ is coercive and bounded from below on $\mathcal{M}_\mu(\Omega)$.

Proof (i) for each $(u, v) \in \mathcal{M}_\mu^+(\Omega)$, we have

$$\begin{aligned} &\mu \int_{\Omega} u^{\alpha'+1} v^{\beta'+1} dx > \\ &[(p-1)/(1-q)] \int_{\Omega} u^{\alpha'+1} v^{\beta'+1} dx \end{aligned}$$

and

$$I_\mu(u, v) <$$

$$[(p-1)(p-q)/2(q+1)] \int_{\Omega} u^{\alpha'+1} v^{\beta'+1} dx.$$

Thus one from of deduces the result (i).

(ii) for $(u, v) \in \mathcal{M}_\mu(\Omega)$, by the Holder and Young inequalities:

$$\begin{aligned} I_\mu(u, v) &= \left(\frac{p-1}{2(p+1)} \right) \|(u, v)\|^2 - \\ &\mu \left(\frac{p-q}{(p+1)(q+1)} \right) \int_{\Omega} u^{\alpha'+1} v^{\beta'+1} dx \\ &\geq \left(\frac{1}{(p+1)} \right) \|(u, v)\|^2 \left[\left(\frac{p-1}{2} \right) - \right. \\ &\left. \left(\frac{p-1}{2} \right) - \mu \frac{(p-q)}{(q+1)} (K(\alpha, \beta)C_{(\alpha, \beta)})^{q+1} \right] \\ &\quad - \mu \left[\frac{(p-1)(p-q)}{2(q+1)} \right] (K(\alpha, \beta)C_{(\alpha, \beta)})^{q+1} \quad (9) \end{aligned}$$

Thus, I_μ is coercive and bounded from below on $\mathcal{M}_\mu(\Omega)$ for all $\mu \in (0, \mu_2]$.

Now, we establish the existence of a local minimum.

Theorem 2 Let $\mu_3 = \min\{\mu_0, \mu_1, \mu_2\}$, for $\mu \in (0, \mu_3)$, the functional I_μ has an unique minimizer $(u_{\min}, v_{\min})$ in $\mathcal{M}_\mu^+(\Omega)$ which satisfies

(i) $I_\mu((u_{\min}, v_{\min})) = \alpha_\mu(\Omega) = \alpha_\mu^+(\Omega)$;

(ii) $(u_{\min}, v_{\min})$ is positive solution of Eq. (1);

(iii) $I_\mu((u_{\min}, v_{\min}))$ goes to 0 as μ tends to 0.

Proof there exists a minimizing sequence (u_n, v_n) for I_μ on $\mathcal{M}_\mu(\Omega)$ such that

$$\begin{aligned} I_\mu((u_n, v_n)) &= \alpha_\mu(\Omega) + o(1) \text{ and } I'_\mu((u_n, v_n)) = \\ &o(1) \text{ in } \mathcal{H}'(\Omega) \text{ (dual of } \mathcal{H}(\Omega)) \end{aligned}$$

as n tends to ∞ .

By Lemma 5 and the compact imbedding theorem, there exists a subsequence still denoted by

(u_n, v_n) and $(u_{\min}, v_{\min})$ in $\mathcal{M}_\mu(\Omega)$ such that

$$(u_n, v_n) \rightharpoonup (u_{\min}, v_{\min}) \text{ weakly in } \mathcal{H}(\Omega);$$

$$(u_n, v_n) \rightarrow (u_{\min}, v_{\min}) \text{ strongly in } L^{p+1}(\Omega);$$

$$(u_n, v_n) \rightarrow (u_{\min}, v_{\min}) \text{ strongly in } L^{q+1}(\Omega).$$

Now, we show that (u_n, v_n) converges to $(u_{\min}, v_{\min})$ strongly in $\mathcal{H}(\Omega)$. Suppose otherwise, by the lower semi-continuity of the norm, then $\|(u_{\min}, v_{\min})\| < \liminf_{n \rightarrow \infty} \|(u_n, v_n)\|$ and so

$$\|(u_{\min}, v_{\min})\|^2 - \int_{\Omega} u_{\min}^{\alpha+1} v_{\min}^{\beta+1} dx - \mu \int_{\Omega} u_{\min}^{\alpha'+1} v_{\min}^{\beta'+1} dx < 0$$

We get a contradiction with the fact that $(u_{\min}, v_{\min}) \in \mathcal{M}_\mu(\Omega)$. Hence, (u_n, v_n) converge to $(u_{\min}, v_{\min})$ strongly in $\mathcal{H}(\Omega)$. This implies

$$I_\mu((u_n, v_n)) \rightarrow I_\mu((u_{\min}, v_{\min})) = \alpha_\mu(\Omega),$$

as $n \rightarrow \infty$.

Moreover, we have $(u_{\min}, v_{\min}) \in \mathcal{M}_\mu^+(\Omega)$. If not, then by Lemma 3, there are two numbers t_0^+ and t_0^- uniquely defined so that $t_0^+(u_{\min}, v_{\min}) \in \mathcal{M}_\mu^+(\Omega)$ and $t_0^-(u_{\min}, v_{\min}) \in \mathcal{M}_\mu^-(\Omega)$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt} I_\mu(t_0^+(u_{\min}, v_{\min}))|_{t=t_0^+} = 0$$

and

$$\frac{d^2}{dt^2} I_\mu(t_0^+(u_{\min}, v_{\min}))|_{t=t_0^+} > 0,$$

there exists $t_0^+ < t \leq t_0^-$ such that $I_\mu(t_0^+(u_{\min}, v_{\min})) < I_\mu(t^-(u_{\min}, v_{\min}))$. By Lemma 3, we get

$$\begin{aligned} I_\mu(t_0^+(u_{\min}, v_{\min})) &< \\ I_\mu(t^-(u_{\min}, v_{\min})) &\leq I_\mu(t_0^-(u_{\min}, v_{\min})) \\ &= I_\mu((u_{\min}, v_{\min})) \end{aligned}$$

which contradicts the fact that $I_\mu((u_{\min}, v_{\min})) = \alpha_\mu^+(\Omega)$. Since

$$I_\mu((u, v)_{\min}) = I_\mu((|u|, |v|)_{\min})$$

and

$$(|u|, |v|)_{\min} \in \mathcal{M}_\mu^+(\Omega)$$

then by Lemma 2 and the maximum principle, we

may assume that $(u_{\min}, v_{\min})$ is a positive solution of our problem Eq. (1).

By Lemma 5, we have

$$\begin{aligned} 0 > I_\mu((u_{\min}, v_{\min})) &\geq \\ -\mu[(p-1)(p-q)/2(q+1)](K(\alpha, \beta)C_{(\alpha, \beta)})^{q+1} \end{aligned}$$

Thus, we obtain that

$$I_\mu((u_{\min}, v_{\min})) \rightarrow 0 \text{ as } \mu \rightarrow 0.$$

3.2 Existence of Two Positive Solutions in $\mathcal{M}_\lambda^-(\Theta_t)$

In this section, we consider the filtration of $\mathcal{M}_\mu^-(\Theta_t)$ and we will prove that Eq. (1) has two positive solutions for μ sufficiently small in Θ_t . For that, we need the following notations.

$$S_l^+ = \{(x, y) \in S / y > l\}$$

and

$$S_l^- = \{(x, y) \in S / y < l\}.$$

For positive number δ , let

$$\begin{aligned} \mathcal{M}_0(\delta, \Theta_t) &= \\ \{(u, v) \in \mathcal{M}_0(\Theta_t) / I_0(u, v) &\leq \alpha_0(S) + \delta\} \\ \mathcal{M}_0^+(\delta, \Theta_t) &= \\ \{(u, v) \in \mathcal{M}_0(\Theta_t) / \int_{S_0^+ \cap \Theta_t} u^{\alpha+1} v^{\beta+1} dx < \\ ((p+1)/(p-1))\alpha_0(S)\} & \\ \mathcal{M}_0^-(\delta, \Theta_t) &= \\ \{(u, v) \in \mathcal{M}_0(\Theta_t) / \int_{S_0^- \cap \Theta_t} u^{\alpha+1} v^{\beta+1} dx < \\ ((p+1)/(p-1))\alpha_0(S)\} & \end{aligned}$$

Lemma 6 There exists $\delta_0, t_0 > 0$ such that for $t > t_0$, we have

$$(i) \mathcal{M}_0^\pm(\delta_0, \Theta_t) \neq \emptyset;$$

$$(ii) \mathcal{M}_0^+(\delta_0, \Theta_t) \cap \mathcal{M}_0^-(\delta_0, \Theta_t) = \emptyset;$$

$$(iii) \mathcal{M}_0(\delta_0, \Theta_t) = \mathcal{M}_0^+(\delta_0, \Theta_t) \cup \mathcal{M}_0^-(\delta_0, \Theta_t)$$

Proof Similar as in [21].

Furthermore, Eq. (1) has two positive solutions $v_0^\pm$ such that $v_0^\pm \in \mathcal{M}_0^\pm(\delta_0, \Theta_t)$

and

$$I_0(v_0^\pm) = \inf_{v \in \mathcal{M}_0^\pm(\delta_0, \Theta_t)} I_0(v) < \\ \alpha_0(S_0^\pm \cap \Theta_t) < \alpha_0(S) + \delta_0.$$

Since $\alpha_0(S_0^\pm \cap \Theta_t) < \alpha_0(S) + \delta_0$, we can choose a positive number $\bar{\delta} < \delta_0$ such that $\alpha_0(S_0^\pm \cap \Theta_t) < \alpha_0(S) + \bar{\delta}$. Moreover, we consider the filtration of the manifold $\mathcal{M}_\mu^-(\Theta_t)$ as follows:

$$\mathcal{N}_\mu(\bar{\delta}, \Theta_t) =$$

$$\{(u, v) \in \mathcal{M}_\mu^-(\Theta_t) / I_\lambda(u) \leq \alpha_0(S) + \bar{\delta}\}$$

$$\mathcal{N}_\mu^+(\bar{\delta}, \Theta_t) =$$

$$\left\{ (u, v) \in \mathcal{N}_\mu(\bar{\delta}, \Theta_t) / \int_{S_0^+ \cap \Theta_t} u^{\alpha+1} v^{\beta+1} dx < \left(\frac{p+1}{p-1} \right) \alpha_0(S) \right\}$$

$$\mathcal{N}_\mu^-(\bar{\delta}, \Theta_t) =$$

$$\left\{ (u, v) \in \mathcal{N}_\mu(\bar{\delta}, \Theta_t) / \int_{S_0^- \cap \Theta_t} u^{\alpha+1} v^{\beta+1} dx < \left(\frac{p+1}{p-1} \right) \alpha_0(S) \right\}$$

Then we have the following result.

Lemma 7 Let μ_3 as in Theorem 3, then there exists $\mu_4 < \mu_3$ such that for $\mu \in (0, \mu_4)$, $\mathcal{N}_\mu^\pm(\bar{\delta}, \Theta_t)$ are nonempty sets. Furthermore,

$$\inf \{ I_\mu(u, v) / (u, v) \in$$

$$\mathcal{N}_\mu^\pm(\bar{\delta}, \Theta_t) \} < \alpha_\mu^+(\Theta_t) + \alpha_0(S) + \bar{\delta}.$$

Proof We only need to prove the case “+”. Since $S_0^+ \cap \Theta_t$ is bounded domain. Thus, Eq. (1) in $S_0^+ \cap \Theta_t$ has a positive solution (u^+, v^+) such that $I_0(u^+, v^+) = \alpha_0(S_0^+ \cap \Theta_t)$ and $(u^+, v^+) \equiv (0, 0)$ in $S_0^- \cap \Theta_t$. Let $(u_{\min}, v_{\min})$ be a positive solution of Eq. (1) in Θ_t as in Theorem 3. Then, For $v, w > 0$, we have

$$I_\mu((u_{\min}, u_{\min}) + l(u^+, v^+)) < I_\mu(u_{\min}, u_{\min}) + I_0(l(u^+, v^+)) \\ - \int_{\Theta_t} \left\{ \int_0^{l\|(u^+, v^+)\|} [(\|(u_{\min}, v_{\min})\| + s)^{p+1} - s^{p+1} - (\|(u_{\min}, v_{\min})\|)^{p-1}] ds \right\} \quad (10)$$

$$(v + w)^k =$$

$$(v + w)^{k-1} v + (v + w)^{k-1} w > v^k + w^k \quad (11)$$

Thus,

$$I_\mu((u_{\min}, v_{\min}) + l(u^+, v^+)) < \\ I_\mu((u_{\min}, v_{\min})) + I_0(l(u^+, v^+))$$

Since $I_0(l(u^+, v^+)) \rightarrow -\infty$ as $l \rightarrow +\infty$, there exists $l_0 > 0$ such that

$$\sup_{l \geq 0} I_\mu((u_{\min}, v_{\min}) + l(u^+, v^+)) =$$

$$\sup_{0 \leq l \leq l_0} I_\mu((u_{\min}, v_{\min}) + l(u^+, v^+)).$$

Let $g_1(l) = I_\mu((u_{\min}, v_{\min}) + l(u^+, v^+))$ for $l \geq 0$. For the continuity of g_1 , given $\varepsilon > 0$, there exists $0 < l_1 < l_0$ such that

$$g_1(l) < g_1(0) + I_0((u^+, v^+)) \text{ for } 0 \leq l < l_1.$$

Then,

$$\sup_{0 \leq l < l_1} I_\lambda((u_{\min}, v_{\min}) + l(u^+, v^+)) <$$

$$I_\mu((u_{\min}, v_{\min})) + I_0((u^+, v^+)).$$

Now, we only need to show that

$$\sup_{l_1 \leq l < l_0} I_\mu((u_{\min}, v_{\min}) + l(u^+, v^+)) <$$

$$I_\mu((u_{\min}, v_{\min})) + I_0((u^+, v^+)).$$

Let $g_2(l) = I_0((u_{\min}, v_{\min}) + l(u^+, v^+))$ for $l \geq 0$. Then

$$g_2'(l) = l \|(u^+, v^+)\|^2 -$$

$$l^{p+1} \int_{\Theta_t} (u^+)^{p+1} (v^+)^{p+1} dx$$

$$g_2''(l) = (p-1)l^{p-2} \|(u^+, v^+)\| -$$

$$(p+1)l^p \int_{\Theta_t} (u^+)^{p+1} (v^+)^{p+1} dx$$

and there is an unique $\bar{l} = 1$ such that $g_2'(\bar{l}) = 0$ and $g_2''(\bar{l}) < 0$. Thus, g_2 has an absolutely maximum at $\bar{l} = 1$. Therefore,

$$\sup_{l \geq 0} I_0(l(u^+, v^+)) =$$

$$I_0((u^+, v^+)) = \alpha_0(S_0^+ \cap \Theta_t).$$

By Eqs. (8) and (9) we obtain

$$\sup_{l_1 \leq l < l_0} I_\mu((u_{\min}, v_{\min}) + l(u^+, v^+)) <$$

$$I_\mu(u_{\min}, v_{\min}) + \alpha_0(S_0^+ \cap \Theta_t).$$

Thus,

$$\sup_{l \geq 0} I_\mu((u_{\min}, v_{\min}) + l(u^+, v^+))$$

$$< I_\mu(u_{\min}, v_{\min}) + \alpha_0(S_0^+ \cap \Theta_t)$$

$$< I_\mu(u_{\min}, v_{\min}) + \alpha_0(S) + \bar{\delta}. \quad (12)$$

Next, we prove that there exists two positive numbers l_0 and μ_4 such that for $\mu \in (0, \mu_4)$,

$$\int_{S_0^\pm \cap \Theta_t} |u_{\min} + l u^+|^{\alpha+1} |v_{\min} + l v^+|^{\beta+1} dx < \\ ((p+1)/(p-1)) \alpha_0(S).$$

Let

$$A_1 = \left\{ (u, v) \in \mathcal{H}(\Omega) \setminus \{0\} / \right. \\ \left. \frac{1}{\|(u, v)\|} t^- \left(\frac{(u, v)}{\|(u, v)\|} \right) > 1 \right\} \cup \{0\} \\ A_2 = \left\{ (u, v) \in \mathcal{H}(\Omega) \setminus \{0\} / \right. \\ \left. \frac{1}{\|(u, v)\|} t^- \left(\frac{(u, v)}{\|(u, v)\|} \right) < 1 \right\}$$

Then $\mathcal{M}_\mu^-(\Theta_t)$ disconnects $\mathcal{H}(\Theta_t)$ in two connected components A_1, A_2 and

$$\mathcal{H}(\Theta_t) \setminus \mathcal{M}_\mu^-(\Theta_t) = A_1 \cup A_2.$$

For each $(u, v) \in \mathcal{M}_\mu^+(\Theta_t)$, we have $1 < t_m t((u, v))$. Since $t^-((u, v)) = \frac{1}{\|(u, v)\|} t^- \left(\frac{(u, v)}{\|(u, v)\|} \right)$ then $\mathcal{M}_\mu^+(\Theta_t) \subset A_1$. In particular, $(u_0, v_0) \in A_1$. We claim that there exists $l_0 > 0$ such that $((u_{\min}, u_{\min}) + l(u^+, v^+)) \in A_2$. Firstly, we find a constant $c > 0$ such that

$$0 < t^- \left(\frac{(u_{\min}, v_{\min}) + l(u^+, v^+)}{\|(u_{\min}, v_{\min}) + l(u^+, v^+)\|} \right) < c$$

for all $l \geq 0$.

Otherwise, there exists a sequence (l_n) such that

$$l_n \rightarrow \infty \text{ and } t^- \left(\frac{(u_{\min}, v_{\min}) + l(u^+, v^+)}{\|(u_{\min}, v_{\min}) + l(u^+, v^+)\|} \right) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

$$\text{Let } (u_n, v_n) = \frac{(u_{\min}, v_{\min}) + l(u^+, v^+)}{\|(u_{\min}, v_{\min}) + l(u^+, v^+)\|}, \text{ since}$$

$$t^-((u_n, v_n))(u_n, v_n) \in \mathcal{M}_\mu^-(\Theta_t) \subset \mathcal{M}_\mu(\Theta_t)$$

by the Lebesgue dominated convergence theorem,

$$\int_{\Theta_t} (u_n)^{p+1} (v_n)^{p+1} dx \rightarrow \int_{\Theta_t} (u^+)^{p+1} (v^+)^{p+1} dx \text{ as } n \rightarrow \infty$$

Hence, we have

$$I_\mu(t^-((u_n, v_n))(u_n, v_n)) \rightarrow \infty \text{ as } n \rightarrow \infty,$$

this contradicts to the fact that I_μ is bounded below on $\mathcal{M}_\mu(\Theta_t)$. Let

$$l_0^+ = \|(u^+, v^+)\|^{-1} \left| c^2 - \|(u^+, v^+)\|^{-2} \right|^{1/2} + 1.$$

Then,

$$\|(u_{\min}, v_{\min}) + l(u^+, v^+)\|^2 > c^2 >$$

$$\left[t^- \left(\frac{(u_{\min}, v_{\min}) + l(u^+, v^+)}{\|(u_{\min}, v_{\min}) + l(u^+, v^+)\|} \right) \right]^2,$$

that is $((u_{\min}, v_{\min}) + l(u^+, v^+)) \in A_2$. Define a path $\gamma(s) = (u_{\min}, v_{\min}) + sl_0(u^+, v^+)$ for $s \in [0, 1]$, then $\gamma(0) = (u_{\min}, v_{\min}) \in A_1$, $\gamma(1) = (u_{\min}, v_{\min}) + l_0(u^+, v^+) \in A_2$ and there exists $s_0 \in (0, 1)$ such that $((u_{\min}, v_{\min}) + sl_0(u^+, v^+)) \in \mathcal{M}_\mu^-(\Theta_t)$. By (12), we obtain that:

$$I_\mu((u_{\min}, v_{\min}) + s_0 l_0(u^+, v^+)) < I_\mu(u_{\min}, v_{\min}) + \alpha_0(S) + \bar{\delta}.$$

Thus,

$$((u_{\min}, v_{\min}) + sl_0(u^+, v^+)) \in \mathcal{N}_\mu^+(\bar{\delta}, \Theta_t).$$

Moreover,

$$\int_{S_0^\pm \cap \Theta_t} (u^+)^{p+1} (v^+)^{p+1} dx \equiv 0$$

and

$$\|(u_{\min}, v_{\min})\| \rightarrow 0 \text{ as } \mu \rightarrow 0.$$

Thus, there exists $\mu_4 < \mu_3$ such that for $\mu \in (0, \mu_4)$

$$\int_{S_0^\pm \cap \Theta_t} |u_{\min} + s_0 l_0 u^+|^{\alpha+1} |v_{\min} + s_0 l_0 v^+|^{\beta+1} dx \\ = \int_{S_0^\pm \cap \Theta_t} |u_{\min}|^{\alpha+1} |v_{\min}|^{\beta+1} dx < \left(\frac{p+1}{p-1} \right) \alpha_0(S)$$

This implies that $\mathcal{N}_\mu^\pm(\bar{\delta}, \Theta_t)$ are nonempty sets for all $\mu \in (0, \mu_4)$.

Note that $\alpha_0(S) < \alpha_0(\Theta_t)$ for all $t > 0$ and for each $u \in \mathcal{M}_\mu^-(\Theta_t)$, there is a unique $S_{(u,v)} > 0$ such that $S_{(u,v)}(u, v) \in \mathcal{M}_0(\Theta_t)$. Moreover, we have the following result.

Lemma 8 There exists $\mu_5 < \mu_4$ such that for $\mu \in (0, \mu_5)$, we have

$$(i) 1 < S_{(u,v)}^{(p+1)} < \frac{\alpha_0(\Theta_t)}{\alpha_0(S)} \text{ for all } (u, v) \in \mathcal{M}_\mu^-(\Theta_t)$$

$$(ii) \int_{\Theta_t} |u|^{\alpha+1} |v|^{\beta+1} dx >$$

$$\left(\frac{p+1}{p-1} \right) \alpha_0(S) \text{ for all } (u, v) \in \mathcal{M}_\mu^-(\Theta_t)$$

Proof (i) for $(u, v) \in \mathcal{M}_\mu^-(\Theta_t)$, we have

$$\|(u, v)\|^2 - \int_{\Theta_t} u^{\alpha+1} v^{\beta+1} dx -$$

$$\mu \int_{\Omega} u^{\alpha'+1} v^{\beta'+1} dx = 0 \quad (13)$$

and

$$(1-p)\|(u,v)\|^2 < (p-q)\int_{\Theta_t} u^{\alpha+1} v^{\beta+1} dx \quad (14)$$

Thus, there is an unique $S_{(u,v)} > 0$ such that $S_{(u,v)}(u,v) \in \mathcal{M}_0(\Theta_t)$ and so

$$S_{(u,v)}^2 \|(u,v)\|^2 = S_{(u,v)}^{(p+1)} \int_{\Theta_t} u^{\alpha+1} v^{\beta+1} dx.$$

Then by Eq. (13) and the Holder inequality we get

$$1 < S_{(u,v)}^{(p-1)} = \frac{\|(u,v)\|^2}{\int_{\Theta_t} u^{\alpha+1} v^{\beta+1} dx} \leq 1 + \mu \left(\int_{\Theta_t} u^{\alpha+1} v^{\beta+1} dx \right)^{(q-p+1)/(p+1)}.$$

Moreover, by Eq. (14) and the Sobolev inequality,

$$\int_{\Theta_t} u^{\alpha+1} v^{\beta+1} dx > \left(\frac{1-q}{p-q-1} \right)^{(p+1)/(p-1)} S^{-(p+1)/(p-1)}$$

Therefore,

$$1 < S_{(u,v)} \leq \left[1 + \mu \left(\frac{p-q-1}{1-q} \right)^{(p+1)/(p-1)} S^{-(p+1)/(p-1)} \right]^{1/(p-1)} \quad (15)$$

Then there exists $\mu_5 < \mu_4$ such that for $\mu \in (0, \mu_5)$, we have

$$1 < S_{(u,v)}^{(p+1)} < \frac{\alpha_0(\Theta_t)}{\alpha_0(S)} \text{ for all } (u,v) \in \mathcal{M}_\mu^-(\Theta_t).$$

(ii) Since

$$\int_{\Theta_t} u^{\alpha+1} v^{\beta+1} dx > \left(\frac{2(p+1)}{p-1} \right)^{(p+1)/(p-1)} S_{(u,v)}^{-(p+1)} \alpha_0(\Theta_t).$$

By (i), we can conclude that

$$\int_{\Theta_t} u^{\alpha+1} v^{\beta+1} dx > \left(\frac{2(p+1)}{p-1} \right) \alpha_0(S) \text{ for all } (u,v) \in \mathcal{M}_\mu^-(\Theta_t).$$

This completes the proof.

Lemma 9 There exists $\mu_* \leq \mu_5$ such that for $\mu \in (0, \mu_*)$, we have

- (i) $\mathcal{N}_\mu^\pm(\bar{\delta}, \Theta_t) \neq \emptyset$;
- (ii) $\mathcal{N}_\mu^+(\bar{\delta}, \Theta_t) \cap \mathcal{N}_\mu^-(\bar{\delta}, \Theta_t) = \emptyset$;
- (iii) $\mathcal{N}_\mu(\bar{\delta}, \Theta_t) = \mathcal{N}_\mu^+(\bar{\delta}, \Theta_t) \cup \mathcal{N}_\mu^-(\bar{\delta}, \Theta_t)$.

Proof Let $(u,v) \in \mathcal{N}_\mu(\bar{\delta}, \Theta_t)$, then by Lemma 4, there is an unique $S_{(u,v)} > 0$ such that $S_{(u,v)}(u,v) \in \mathcal{M}_0$ and

$$I_0(S_{(u,v)}(u,v)) \leq (1-\mu\theta)^{\frac{p+1}{p-1}} \left[\alpha_0(S) + \bar{\delta} + \mu \left(\frac{1-q}{2(q+1)} \right) \theta^{\frac{q+1}{q-1}} S^{\frac{2(q+1)}{1-q}} \right]$$

Since $\bar{\delta} < \delta_0$, we can conclude that for each $\theta \in (0,1)$ there exists $\mu_* \leq \mu_5$ so that for $\mu \in (0, \mu_*)$, we obtain

$$I_0(S_{(u,v)}(u,v)) \leq \alpha_0(S) + \delta_0 \text{ for all } (u,v) \in \mathcal{N}_\mu(\bar{\delta}, \Theta_t) \quad (16)$$

By Eq.(16) and Lemma 6, for each $(u,v) \in \mathcal{N}_\mu(\bar{\delta}, \Theta_t)$ there is either

$$S_{(u,v)}(u,v) \in \mathcal{M}_0^+(\delta_0, \Theta_t) \quad \text{or}$$

$S_{(u,v)}(u,v) \in \mathcal{M}_0^-(\delta_0, \Theta_t)$. Without loss generality, we may assume that $S_{(u,v)}(u,v) \in \mathcal{M}_0^+(\delta_0, \Theta_t)$. Then by Lemma 8

$$\int_{\Theta_t} u^{\alpha+1} v^{\beta+1} dx <$$

$$\left(\frac{p+1}{p-1} \right) S_{(u,v)}^{-(p+1)} \alpha_0(S) < \left(\frac{p+1}{p-1} \right) \alpha_0(S)$$

Thus, $(u,v) \in \mathcal{N}_\mu^+(\bar{\delta}, \Theta_t)$.

To complete the proof of Lemma 9, it remains to be shown that

$$\mathcal{N}_\mu^+(\bar{\delta}, \Theta_t) \cap \mathcal{N}_\mu^-(\bar{\delta}, \Theta_t) = \emptyset.$$

Suppose that there exists $(u_0, v_0) \in \mathcal{N}_\mu(\bar{\delta}, \Theta_t)$ such that

$$\int_{S_0^\pm \cap \Theta_t} u_0^{\alpha+1} v_0^{\beta+1} dx < \left(\frac{p+1}{p-1} \right) \alpha_0(S).$$

It implies

$$2 \left(\frac{p+1}{p-1} \right) \alpha(\Theta_t) \leq \int_{\Theta_t} u_0^{\alpha+1} v_0^{\beta+1} dx < \left(\frac{p+1}{p-1} \right) \alpha_0(S)$$

which is a contradiction.

Now, we have the following result

Lemma 10 $\mathcal{N}_\mu^\pm(\bar{\delta}, \Theta_t)$ are closed.

Proof We only need to prove the case “+”. Supposing that (u_0, v_0) is a limit point of $\mathcal{N}_\mu^+(\bar{\delta}, \Theta_t)$. Then

$$\int_{S_0^\pm \cap \Theta_t} u_0^{\alpha+1} v_0^{\beta+1} dx \leq \left(\frac{p+1}{p-1} \right) \alpha_0(S)$$

and

$$I_\mu(u_0, v_0) \leq \alpha_0(S) + \bar{\delta}.$$

That implies $(u_0, v_0) \in \mathcal{N}_\mu^+(\bar{\delta}, \Theta_t)$. Since

$$\mathcal{N}_\mu(\bar{\delta}, \Theta_t) = \mathcal{N}_\mu^+(\bar{\delta}, \Theta_t) \cup \mathcal{N}_\mu^-(\bar{\delta}, \Theta_t)$$

and

$$\mathcal{N}_\mu^+(\bar{\delta}, \Theta_t) \cap \mathcal{N}_\mu^-(\bar{\delta}, \Theta_t) = \emptyset.$$

Hence, if $\int_{S_0^+ \cap \Theta_t} u^{\alpha+1} v^{\beta+1} dx = \left(\frac{p+1}{p-1}\right) \alpha_0(S)$,

then $(u_0, v_0) \in \mathcal{N}_\mu^-(\bar{\delta}, \Theta_t)$. By Lemma 8,

$$\begin{aligned} 2 \left(\frac{p+1}{p-1} \right) \alpha_0(S) &< \int_{\Theta_t} u_0^{\alpha+1} v_0^{\beta+1} dx \\ &\leq \int_{S_0^+ \cap \Theta_t} u_0^{\alpha+1} v_0^{\beta+1} dx + \int_{S_0^- \cap \Theta_t} u_0^{\alpha+1} v_0^{\beta+1} dx \\ &< 2 \left(\frac{p+1}{p-1} \right) \alpha_0(S), \end{aligned}$$

which is a contradiction. Thus, $\mathcal{N}_\mu^\pm(\bar{\delta}, \Theta_t)$ are closed.

Now, we consider the minimization problems in $\mathcal{N}_\mu^\pm(\bar{\delta}, \Theta_t)$ for I_μ ,

$$\sigma^\pm(\bar{\delta}, \Theta_t) = \inf_{(u,v) \in \mathcal{N}_\mu^\pm(\bar{\delta}, \Theta_t)} I_\mu(u, v).$$

Then we have the following result.

Lemma 11 For each $\mu \in (0, \mu_*)$, Eq. (1) has two positive solutions $(u_0^\pm, v_0^\pm)$ in Θ_t such that $(u_0^\pm, v_0^\pm) \in \mathcal{N}_\mu^\pm(\bar{\delta}, \Theta_t)$ and $I_\mu((u_0^\pm, v_0^\pm)) = \sigma^\pm(\bar{\delta}, \Theta_t)$.

Proof Similar to the proof in Wu [21], there exists minimizing sequences

$(u_n^\pm, v_n^\pm)$ for I_μ on $\mathcal{N}_\mu^\pm(\bar{\delta}, \Theta_t)$ such that

$$I_\mu((u_n^\pm, v_n^\pm)) = \sigma^\pm(\bar{\delta}, \Theta_t) + o(1)$$

and

$$I'_\mu((u_n^\pm, v_n^\pm)) = o(1) \text{ in } \mathcal{H}(\Theta_t), \text{ as } n \rightarrow \infty.$$

By Lemma 5 and the compact imbedding theorem, there exists subsequences still denoted by $(u_n^\pm, v_n^\pm)$ and $u_0^\pm \in \mathcal{N}_\mu^\pm(\bar{\delta}, \Theta_t)$ such that

$$(u_n^\pm, v_n^\pm) \rightharpoonup (u_0^\pm, v_0^\pm) \text{ weakly in } \mathcal{H}(\Theta_t);$$

$$(u_n^\pm, v_n^\pm) \rightarrow (u_0^\pm, v_0^\pm) \text{ strongly in } L^{p+1}(\Theta_t);$$

$$(u_n^\pm, v_n^\pm) \rightarrow (u_0^\pm, v_0^\pm) \text{ strongly in } L^{q+1}(\Theta_t).$$

Now, we show that $(u_n^\pm, v_n^\pm)$ converges to $(u_0^\pm, v_0^\pm)$ strongly in $\mathcal{H}(\Theta_t)$. Suppose otherwise.

By the lower semi-continuity of the norm, then either

$$\|(u_0^\pm, v_0^\pm)\| < \liminf_{n \rightarrow \infty} \|(u_n^\pm, v_n^\pm)\|$$

and so

$$\begin{aligned} \|(u_0^\pm, v_0^\pm)\|^2 - \int_\Omega (u_0^\pm)^{\alpha+1} (v_0^\pm)^{\beta+1} dx - \\ \mu \int_\Omega (u_0^\pm)^{\alpha'+1} (v_0^\pm)^{\beta'+1} dx < 0 \end{aligned}$$

We get a contradiction with the fact that $(u_0^\pm, v_0^\pm) \in \mathcal{N}_\mu^\pm(\bar{\delta}, \Theta_t)$. Hence, $(u_n^\pm, v_n^\pm)$ converge to $(u_0^\pm, v_0^\pm)$ strongly in $\mathcal{H}(\Theta_t)$. This implies

$$\begin{aligned} I_\mu((u_n^\pm, v_n^\pm)) &\rightarrow I_\mu((u_0^\pm, v_0^\pm)) = \\ &\sigma^\pm(\bar{\delta}, \Theta_t) \text{ as } n \rightarrow \infty. \end{aligned}$$

Since $I_\mu((u_0^\pm, v_0^\pm)) = I_\mu(|u_0^\pm|, |v_0^\pm|)$ and $(|u_0^\pm|, |v_0^\pm|) \in \mathcal{N}_\mu^\pm(\bar{\delta}, \Theta_t)$, then by Lemma 2 and the maximum principle, we may assume that $(u_0^\pm, v_0^\pm)$ are positive solutions of our problem Eq. (1).

Proof of Theorem 1

Let (u, v) be the ground state solution of Eq. (1) in the infinite strip S , then $I_\mu((\bar{u}, \bar{v})) = \alpha_0(S)$. Without loss of generality, we shall assume that $0 \in \omega$ and there exists $r_0 > 0$ such that $B^{N-1}(0, r_0) \subset \omega$ and $\omega \subset B^{N-1}(0, r_0/2)$. Let $\bar{h} = (0, h) \in \mathbb{R}^{N-1} \times \mathbb{R}$ with $|h| = 1/2$ and let $\varphi : \mathcal{R}^N \rightarrow [0, 1]$ be a C^∞ cut-off function such that $0 \leq \varphi \leq 1$ and

$$\varphi(z) = \begin{cases} 0, & z \in B^N(0, r_0/2) \cup S_t^+ \cup S_{-t}^-, \\ 1, & z \in S_{-t+1, t-1} \setminus B^N(0, r_0) \end{cases}$$

and $v_t(z) = \varphi(z) \bar{u}(z - \bar{h})$ for $z \in S$. Then $v_t \in \mathcal{H}(\Theta_t)$ for all $t > t_0$. Moreover, for each $t > t_0$, there exists $S_{t,\mu} > 0$ such that $(u, v)_{t,\mu} = S_{t,\mu} v_t$ in $\mathcal{M}_\mu^-(\Theta_t)$. For $\mu \geq 0$, defining the following

$$\begin{aligned} I_t &= [-t, t] \\ V_\mu &= \{(u, v) \in \mathcal{M}_\mu^-(\Theta_t) / (u, v) \geq 0\} \\ \Gamma_\mu(\Theta_t) &= \left\{ \gamma \in C[I_t, V_\mu] / \gamma(s) = \right. \\ &\quad \left. (u, v)_{t,\mu} \text{ for } s \in \partial I_t \right\} \\ \beta_\mu(\Theta_t) &= \inf_{\gamma \in \Gamma_\mu(\Theta_t)} \max_{s \in I_t} I_\mu(\gamma(s)) \end{aligned}$$

By Lemma 4.6 in Wu [21], there exists $d_0 > 0$ such that

$$\inf_{(u,v) \in \mathcal{M}_\mu^\pm(\delta_0, \Theta_t)} I_0((u,v)) < d_0 \leq \beta_0(\Theta_t) \quad (17)$$

Similar to the argument of Theorem 4.1 in Wu [21] and by Lemmas 4 and 8, for any $\theta \in (0, 1)$, we have

$$\begin{aligned} \beta_\mu(\Theta_t) &\geq (1 - \mu\theta)^{\frac{p+1}{p-1}} \beta_0(\Theta_t) - \\ &\left[\mu \frac{(1-q)}{2(q+1)} \right] |\Theta_t|^{\frac{(q+1)}{q-1}} (K_{(\alpha,\beta)} C_{(\alpha,\beta)})^{\frac{2(q+1)}{1-q}} \end{aligned} \quad (18)$$

and

$$\begin{aligned} \beta_\mu(\Theta_t) &\leq (1 + \mu\theta)^{\frac{p+1}{p-1}} \beta_0(\Theta_t) + \\ &\left[\mu \frac{(1-q)}{2(q+1)} \right] |\Theta_t|^{\frac{(q+1)}{q-1}} (K_{(\alpha,\beta)} C_{(\alpha,\beta)})^{\frac{2(q+1)}{1-q}}. \end{aligned} \quad (19)$$

Furthermore, we have the following result.

Lemma 12 For each $\delta > 0$, there exists $\Lambda \leq \mu^*$ so that for $\mu \in (0, \Lambda)$, we have

$$\begin{aligned} \sigma^\pm(\bar{\delta}, \Theta_t) &< \inf_{(u,v) \in \mathcal{M}_\mu^\pm(\delta_0, \Theta_t)} I_0((u,v)) < \\ &\beta_0(\Theta_t) - \delta < \beta_\mu(\Theta_t). \end{aligned}$$

Proof By Eqs. (17)-(19), we have

$$\inf_{(u,v) \in \mathcal{M}_\mu^\pm(\delta_0, \Theta_t)} I_0((u,v)) < \beta_0(\Theta_t) - \delta < \beta_\mu(\Theta_t)$$

Similar to the argument in Lemma 7,

$$\sigma^\pm(\bar{\delta}, \Theta_t) < \alpha_\mu^+(\Theta_t) + \inf_{(u,v) \in \mathcal{M}_\mu^\pm(\delta_0, \Theta_t)} I_0((u,v))$$

This completes the proof.

Lemma 13 As in Lemma 12, for $\mu \in (0, \Lambda)$, Eq. (1) has a positive solution $(\hat{u}, \hat{v})$ in Θ_t such that $I_\mu((\hat{u}, \hat{v})) = \beta_\mu(\Theta_t)$.

Proof By Lemma 12 and the minmax principle (2) there exists a sequence $((u_n, v_n))$ in $\mathcal{H}(\Theta_t)$ with $(u_n, v_n) \geq 0$ such that

$$\begin{cases} I_\mu((u_n, v_n)) = \beta_\mu(\Theta_t) + o(1) \\ I'_\mu((u_n, v_n)) = o(1) \end{cases} \quad \text{in } \mathcal{H}'(\Theta_t) \text{ as } n \rightarrow \infty.$$

By the compact imbedding theorem, the Holder inequality and the maximum principle, we can conclude that Eq. (1) has a positive solution $(\hat{u}, \hat{v}) \in \mathcal{M}_\mu^-(\Theta_t)$ such that $I_\mu((\hat{u}, \hat{v})) = \beta_\mu(\Theta_t)$.

By Lemmas 3, 11 and 13 we obtain that Eq. (1) has four positive solutions $(u_{\min}, v_{\min})$, (u_0^+, v_0^+) , (u_0^-, v_0^-) and $(\hat{u}, \hat{v})$. Since

$$\begin{aligned} \mathcal{M}_\mu^+(\Theta_t) \cap \mathcal{N}_\mu^+(\bar{\delta}, \Theta_t) &= \\ \emptyset, \mathcal{N}_\mu^+(\bar{\delta}, \Theta_t) \cap \mathcal{N}_\mu^-(\bar{\delta}, \Theta_t) &= \emptyset \end{aligned}$$

and

$$I_\mu((\hat{u}, \hat{v})) = \beta_\mu(\Theta_t) > \sigma^\pm(\bar{\delta}, \Theta_t),$$

this

implies

that

$(u_{\min}, v_{\min})$, (u_0^+, v_0^+) , (u_0^-, v_0^-) and $(\hat{u}, \hat{v})$ are distinct.

4. Conclusions

Drawing on the work of Wu (22), this work shows that there exists at least four positive solutions to our system, and by splitting twice in the Nehari manifold: $\mathcal{M}_\mu(\Theta_t)$ in which $\mathcal{M}_\mu^+(\Theta_t)$ and $\mathcal{M}_\mu^-(\Theta_t)$ and which provides a positive solution first in $\mathcal{M}_\mu^+(\Theta_t)$ and $\mathcal{M}_\mu^-(\Theta_t)$ in which $\mathcal{N}_\mu^+(\bar{\delta}, \Theta_t)$ and $\mathcal{N}_\mu^-(\bar{\delta}, \Theta_t)$ and provides two positive solutions. In the end, using the compact imbedding theorem, the Holder inequality and the maximum principle, we prove the existence of a fourth positive solution in $\mathcal{M}_\mu^-(\Theta_t)$, so we can continue this subdivision, for example $\mathcal{N}_\mu^-(\bar{\delta}, \Theta_t)$, to find other positive solutions.

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