Calculating Binding Energy for Odd Isotopes of Beryllium ($7 \leq A \leq 13$)

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Abstract: In this article, we present exact solution of the Schrödinger equation (for an N-identical body-force) for odd-A isotopes of Beryllium in the presence of Yukawa potential by Nikiforov-Uvarov (NU) method. The NU method can be used to solve second order differential equation. By this method, we find the wave equation and binding energy. Numerical results of binding energy are presented and show that these results are in good agreement with experimental values.

Key words: Schrödinger equation, yukawa potential, NU method, odd isotopes of beryllium, binding energy, wave equation.

1. Introduction

The exact solution of the problem has always played an important role in quantum mechanics, because the important information is hidden in solutions of questions [1]. The Schrodinger equation is one of the fundamental wave equation in physics. Its solutions for some certain potential have important applications in atomic, nuclear, condensed and high energy physics and particle physics [2-14]. So far, several approaches have presented to solve Schrödinger equation two-particle systems and three-particle systems in the presence of different potentials [15]. In this paper, we solve the N-particle Schrödinger equation for odd isotopes of Beryllium. Schrodinger equation for the N-particles which interact with each other is [16, 17]:

$$\frac{-\hbar^2}{2m} \left(\frac{d^2\psi(x)}{dx^2} + \frac{D-1}{x} \frac{d\psi(x)}{dx} - \frac{\gamma(\gamma+D-2)}{x^2} \psi(x) \right) + V(x)\psi(x) = E\psi(x) \quad (1)$$

And $D = 3N - 3$

In this equation, D , γ and N denote respectively the space dimension, total angular momentum and the number of particles, m is one of particle masses, and x is the hyper radius. If we write the equation for Beryllium 7 (7Be),

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$$\text{with substitution } \rightarrow \psi(x) = x^{-(D-1)} \varphi(x) \\ \frac{d\varphi^2(x)}{dx^2} + \frac{2m}{\hbar^2} \left[E - V(x) - \hbar^2 \frac{(2\gamma+15)(2\gamma+17)}{8mx^2} \right] \times \varphi(x) = 0 \quad (2)$$

$D = 3 \times 7 - 3$

where the potential $v(r)$ is taken as Yukawa potential in Eq. (1) [16]:

$$V(x) = -\frac{v_0}{x} e^{-ax}$$

Since the Schrodinger with Yukawa potential has no solution [18], we use an approximation for the centrifugal terms as [19]:

$$\frac{1}{x} \simeq 2a \frac{e^{-ax}}{(1 - e^{-2ax})}$$

Or equivalently:

$$\frac{1}{x^2} \simeq 4a^2 \frac{e^{-2ax}}{(1 - e^{-2ax})^2}$$

The radial Schrödinger equation will be as follows:

$$\frac{d\varphi^2(x)}{dx^2} + \frac{2m}{\hbar^2} \times [E + 2av_0 \frac{e^{-2ax}}{(1 - e^{-2ax})} - \hbar^2 \times \frac{(2\gamma+15)(2\gamma+17)}{8} 4a^2 \frac{e^{-2ax}}{(1 - e^{-2ax})^2}] \varphi(x) = 0 \quad (3)$$

This paper is arranged as follows. In Section 2, we solved the Schrodinger equation for odd isotopes of Beryllium in the presence of the Yukawa potential, and then the energy spectrum and the wave equation of the system are obtained by using the NU method. The details of the NU method are given in Appendix A.

Finally, our conclusion is given in Section 3.

2. Solution of the Schrödinger Equation by Nikiforov-Uvarov Method

Several methods are used in the solution of the Schrodinger equation, one of them is the Nikiforov-Uvarov method [19]. We use NU generalized method to solve the Schrödinger equation (see the Appendix A) [20]:

$$\frac{d\varphi^2(x)}{dx^2} + \frac{2m}{\hbar^2} \times [E + 2av_0 \frac{e^{-2ax}}{(1 - e^{-2ax})} - \hbar^2 \times \frac{(2\gamma + 15)(2\gamma + 17)}{8} 4a^2 \frac{e^{-2ax}}{(1 - e^{-2ax})^2}] \varphi(x) = 0 \quad (3)$$

Using variables the form below:

$$\epsilon = \frac{-2mE}{\hbar^2}, s = e^{-2ax}$$

Substituting into Eq. (3) we have:

$$\frac{d\varphi^2(s)}{ds^2} + \frac{(1-s)}{s(1-s)} \frac{d\varphi(s)}{ds} + \frac{1}{s^2(1-s)^2} \left[-\frac{\epsilon(1-s)^2}{4a^2} + \frac{mv_0}{\hbar^2 a} s(1-s) - (2\gamma + 15)(2\gamma + 17) \frac{s}{4} \right] \varphi(s) = 0 \quad (4)$$

Comparing Eq. (4) and Eq. (A11), we can easily obtain the coefficients α_i ($i = 1, 2, 3$) and analytical expression for ξ_i as follows:

$$\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1 \quad (5)$$

And

$$\xi_1 = \frac{\epsilon}{4a^2} + \frac{mv_0}{\hbar^2 a},$$

$$\xi_2 = \frac{\epsilon}{2a^2} + \frac{mv_0}{\hbar^2 a} - (2\gamma + 15)(2\gamma + 17) \frac{1}{4} \quad (6)$$

$$\xi_3 = \frac{\epsilon}{4a^2}$$

The values of the coefficients α_i ($i = 4, 5, \dots, 9$) are found from Eqs. (A16)-(A18), the specific values of the coefficients α_i ($i = 4, 5, \dots, 9$) necessary for the energy eigenvalues and eigenfunctions are displayed as follows:

$$\begin{aligned} \alpha_4 &= 0, \alpha_5 = \frac{1}{2}, \alpha_6 = \frac{1}{4} + \frac{\epsilon}{4a^2} + \frac{mv_0}{\hbar^2 a} \\ \alpha_7 &= \frac{-\epsilon}{2a^2} + \frac{mv_0}{\hbar^2 a} = (2\gamma + 15)(2\gamma + 17) \frac{1}{4} \quad (7) \\ \alpha_8 &= \frac{\epsilon}{4a^2}, \alpha_9 = \frac{1}{4} + \frac{1}{4} (2\gamma + 15)(2\gamma + 17) \end{aligned}$$

By using Eq. (A22), we obtained the energy eigenvalues of the Yukawa potential as:

$$E_{7Be} = \frac{-a^2}{2m} \times \frac{1}{\left[(2n+1) + \left(2\sqrt{\frac{1}{4} (1 + (2\gamma + 15)(2\gamma + 17))} \right) \right]^2} \times \left[n^2 + n + \frac{1}{2} - \frac{mv_0}{\hbar^2 a} + \frac{1}{4} (2\gamma + 15)(2\gamma + 17) + (2n+1) \left(\sqrt{\frac{1}{4} (1 + (2\gamma + 15)(2\gamma + 17))} \right) \right]^2 \quad (8)$$

To find the corresponding wave function, referring to the above parametric constant and Eqs. (A18) and (A23), we find the wave function:

$$\begin{aligned} \varphi(s)_{7Be} &= e^{-x\sqrt{\frac{-2mE}{\hbar^2}}} (1 - e^{-2ax})^{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{4}(2\gamma+15)(2\gamma+17)}} \\ &\times P_n \left[\sqrt{\frac{-2mE}{a^2\hbar^2}}, -\frac{1}{2} + \sqrt{\frac{-2mE}{a^2\hbar^2}} \left(2\sqrt{\frac{1}{4} + \frac{1}{4}(2\gamma+15)(2\gamma+17)} + 1 \right) \right] \\ &\times (1 - e^{-2ax}) \quad (9) \end{aligned}$$

Repeat this procedure for the other isotopes we will have:

$$\begin{aligned} \varphi(s)_{9Be} &= e^{-x\sqrt{\frac{-2mE}{\hbar^2}}} (1 - e^{-2ax})^{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{4}(2\gamma+21)(2\gamma+23)}} \\ &\times P_n \left[\sqrt{\frac{-2mE}{a^2\hbar^2}}, -\frac{1}{2} + \sqrt{\frac{-2mE}{a^2\hbar^2}} \left(2\sqrt{\frac{1}{4} + \frac{1}{4}(2\gamma+21)(2\gamma+23)} + 1 \right) \right] \\ &\times (1 - e^{-2ax}) \quad (10) \end{aligned}$$

and

$$E_{9Be} = \frac{-a^2}{2m} \times \frac{1}{\left[(2n+1) + \left(2\sqrt{\frac{1}{4} + \frac{1}{4} (2\gamma + 21)(2\gamma + 23)} \right) \right]^2} \times \left[n^2 + n + \frac{1}{2} - \frac{mv_0}{\hbar^2 a} + \frac{1}{4} (2\gamma + 21)(2\gamma + 23) + (2n+1) \left(\sqrt{\frac{1}{4} + \frac{1}{4} (2\gamma + 21)(2\gamma + 23)} \right) \right]^2 \quad (11)$$

For $11Be$, we will have:

$$\begin{aligned} \varphi(s)_{11Be} &= e^{-x\sqrt{\frac{-2mE}{\hbar^2}}} (1 - e^{-2ax})^{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{4}(2\gamma+27)(2\gamma+29)}} \end{aligned}$$

$$P_n \left[\sqrt{\frac{-2mE}{a^2 \hbar^2}}, -\frac{1}{2} + \sqrt{\frac{-2mE}{a^2 \hbar^2} \left(2\sqrt{\frac{1}{4} + \frac{1}{4}(2\gamma+27)(2\gamma+29)} \right) + 1} \right] \\ (1 - e^{-2ax}) \quad (12)$$

and

$$E_{11Be} = \frac{-a^2}{2m} \times \frac{1}{\left[(2n+1) + \left(2\sqrt{\frac{1}{4} + \frac{1}{4}(2\gamma+27)(2\gamma+29)} \right) \right]^2} \times \\ \left[n^2 + n + \frac{1}{2} - \frac{mv_0}{\hbar^2 a} + \frac{1}{4}(2\gamma+27)(2\gamma+29) \right] \\ + (2n+1) \left(\sqrt{\frac{1}{4} + \frac{1}{4}(2\gamma+27)(2\gamma+29)} \right)^2 \quad (13)$$

The wave function and energy eigenvalue for ^{13}Be become:

$$\varphi(s)_{13Be} = e^{-x\sqrt{\frac{-2mE}{a^2 \hbar^2}}} (1 - e^{-2ax})^{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{4}(2\gamma+33)(2\gamma+35)}} \\ \times P_n \left[\sqrt{\frac{-2mE}{a^2 \hbar^2}}, -\frac{1}{2} + \sqrt{\frac{-2mE}{a^2 \hbar^2} \left(2\sqrt{\frac{1}{4} + \frac{1}{4}(2\gamma+33)(2\gamma+35)} \right) + 1} \right] \\ \times (1 - e^{-2ax}) \quad (14)$$

and

$$E_{13Be} = \frac{-a^2}{2m} \times \frac{1}{\left[(2n+1) + \left(2\sqrt{\frac{1}{4} + \frac{1}{4}(2\gamma+33)(2\gamma+35)} \right) \right]^2} \times \\ \left[n^2 + n + \frac{1}{2} - \frac{mv_0}{\hbar^2 a} + \frac{1}{4}(2\gamma+33)(2\gamma+35) \right] \\ + (2n+1) \left(\sqrt{\frac{1}{4} + \frac{1}{4}(2\gamma+33)(2\gamma+35)} \right)^2 \quad (15)$$

3. Conclusions

In this paper, we solve the Schrödinger equation for odd isotopes Beryllium $7 \leq A \leq 13$ by using generalized NU method and choosing Yukawa potential and we obtain the wave function and binding energy of these isotopes. Comparing the obtained binding energy with the experimental results, we can

see that a good agreement has been achieved. By putting the values of n , γ , v_0 , $\hbar = 1$, considering $a = 0.1 \text{ fm}^{-1}$ and finally, by comparison with experimental values (Table 1), we will see that a good agreement is obtained [21-23].

Table 1 The comparison of the binding energy with the experimental results.

Odd isotopes	n	γ	V_0	E (Present calculation)	E (Experimental value)
^7Be	3	1	6	36.5979	37.600
^9Be	1	1	9	58.5015	58.165
^{11}Be	1	0	1	65.2684	65.481
^{13}Be	2	0	5	75.9345	68.550

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Appendix A: Nikiforov-Uvarov Method

The Schrodinger equation is transformed into a second order differential equation of the form with an appropriate coordinate transformation [22]. The Nikiforov-Uvarov method [21] can be used to solve second order differential equations with an appropriate coordinate transformation $s = s(r)$:

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)} \psi_n' + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \psi_n(s) = 0 \quad (A1)$$

where, $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second-degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To find particular solution Eq. (A1) by separating of variables, one deals with the transformation as follows:

$$\psi_n(s) = \phi(s) y_n(s) \quad (A2)$$

And it reduces to an equation of hyper geometric type:

$$\sigma(s)y_n''(s) + \tau(s)y_n'(s) + \lambda y_n(s) = 0 \quad (A3)$$

$\phi(s)$ is defined as logarithmic derivative:

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \quad (A4)$$

$y_n(s)$ is the hyper geometric-type function whose polynomials solutions are given by Rodriguez relation:

$$y_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} [\sigma^n(s) \rho(s)] \quad (A5)$$

where, B_n is the normalization constant and the weight function $\rho(s)$ must satisfy the condition:

$$(\sigma(s)\rho(s))' = \tau(s)\rho(s), \sigma'(s)\rho(s) + \sigma(s)\frac{\rho'(s)}{\rho(s)} = \tau(s) \quad (A6)$$

The function $\pi(s)$ and the parameter λ required for this method are defined as follows:

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' + \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma} \quad (A7)$$

and

$$\lambda = k + \pi'(s) \quad (A8)$$

and

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \tau'(s) < 0 \quad (A9)$$

Since $\rho(s) > 0$ and $\sigma(s) > 0$, hence the derivative of τ should be negative [10]. This leads to the choice of the solution. If λ in Eq. (A8) is:

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \quad n = 0, 1, 2, \dots \quad (\text{A10})$$

The following equation is a general form of the Schrodinger equation written for any potential [22]:

$$\left[\frac{d^2}{ds^2} + \frac{(\alpha_1 - \alpha_2 s)}{s(1 - \alpha_3 s)} \frac{d}{ds} + \frac{-\xi_1 s^2 + -\xi_2 s - \xi_3}{[s(1 - \alpha_3 s)]^2} \right] \times \psi(s) = 0 \quad (\text{A11})$$

We may solve this as follows. When Eq. (A11) is compared with Eq. (A1), we get

$$\tilde{\tau} = \alpha_1 - \alpha_1 s \quad (\text{A12})$$

and

$$\sigma = s(1 - \alpha_3 s) \quad (\text{A13})$$

also

$$\tilde{\sigma} = -\xi_1 s^2 - \xi_2 s - \xi_3 \quad (\text{A14})$$

Substituting these into Eq. (A7):

$$\pi(s) = \alpha_4 + \alpha_5 s \pm \sqrt{(\alpha_6 - k\alpha_3)^2 s^2 + (\alpha_7 + k)s + \alpha_8} \quad (\text{A15})$$

where

$$\alpha_4 = \frac{1}{2}(1 - \alpha_1), \alpha_5 = \frac{1}{2}(\alpha_2 - 2\alpha_3), \alpha_6 = \alpha_5^2 + \xi_1 \quad (\text{A16})$$

and

$$\alpha_7 = 2\alpha_4\alpha_5 - \xi_2, \alpha_8 = \alpha_4^2 + \xi_3 \quad (\text{A17})$$

and

$$\begin{aligned} \alpha_9 &= \alpha_3\alpha_7 + \alpha_3^2\alpha_8 + \alpha_6, \\ \alpha_{10} &= \alpha_1\alpha_7 + 2\alpha_4 + 2\sqrt{\alpha_8} \\ \alpha_{11} &= \alpha_2 - 2\alpha_5 + 2\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8} \\ \alpha_{12} &= \alpha_4 + \sqrt{\alpha_8} \\ \alpha_{13} &= \alpha_5 - (\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) \end{aligned} \quad (\text{A18})$$

in Eq. (A15), the function under square root must be the square of polynomial according to the NU method [22], so that π and τ becomes:

$$\pi(s) = \alpha_4 + \alpha_5 s - [(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})s - \sqrt{\alpha_8}] \quad (\text{A19})$$

$$\tau = \alpha_1 + 2\alpha_4 - (\alpha_2 - 2\alpha_5)s - 2[(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})s - \sqrt{\alpha_8}] \quad (\text{A20})$$

and

$$\dot{t} = -(\alpha_2 - 2\alpha_5) - 2[(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) - \sqrt{\alpha_8}] \rightarrow \tau < 0 \quad (\text{A21})$$

is obtained. When Eq. (A8) is used with Eqs. (A20) and (A21) the following equation is derived:

$$\alpha_2 n - (2n + 1)\alpha_5 + (2n + 1)(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) + n(n - 1)\alpha_3 + \alpha_7 + 2\alpha_3\alpha_8 + 2\sqrt{\alpha_8\alpha_9} = 0 \quad (\text{A22})$$

This Eq. (A21) gives the energy spectrum. And the wave function will be as follows:

$$\psi(s) = s^{\alpha_{12}} (1 - \alpha_3 s)^{-\alpha_{12} \frac{\alpha_{13}}{\alpha_3}} P_n^{\left(\alpha_{10}-1, \frac{\alpha_{11}}{\alpha_3} \alpha_{10}-1\right)} \times (1 - 2\alpha_3 s) \quad (\text{A23})$$

$P_n^{\left(\alpha_{10}-1, \frac{\alpha_{11}}{\alpha_3} \alpha_{10}-1\right)}$ are Jacobi polynomials.