Basic Entropies for Positive-Definite Matrices

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Abstract: We extend basic entropies in the classical information theory to matrix ones in the quantum information theory. Then we show that relations between matrix entropies similar to the classical ones hold.

Keywords: Matrix entropy, Information theory, Quantum entropy, Relative operator entropy

1. Introduction

Let a_j and b_i be elementary stochastic events for probability vectors $X = (p(a_j))$ and $Y = (p(b_i))$ for $j = 1, \dots, n$ and $i = 1, \dots, m$. Then a (classical) channel is a probability matrix $P_c = (p_{ij})$ which transforms X into Y and the entropies in the classical information theory are defined by:

compound entropy

$$H(X, Y) = - \sum_{i,j} p(a_j, b_i) \log p(a_j, b_i),$$

conditional entropies

$$H(X | Y) = - \sum_{i,j} p(a_j, b_i) \log p(a_j | b_i),$$

$$H(Y | X) = - \sum_{i,j} p(a_j, b_i) \log p(b_i | a_j),$$

mutual entropy

$$I(X; Y) = \sum_{i,j} p(a_j, b_i) \log \frac{p(a_j, b_i)}{p(a_j)p(b_i)}.$$

The relations between them are:

$$\begin{aligned} H(X, Y) &= H(X) + H(Y | X) \\ &= H(Y) + H(X | Y) = I(X; Y) \\ &\quad + H(X | Y) + H(Y | X), \end{aligned}$$

$$H(X) = I(X; Y) + H(X | Y),$$

$$H(Y) = I(X; Y) + H(Y | X).$$

Though all entropies are non-negative, we note that the some individual mutual informations

$$I(a_j; b_i) = \log \frac{p(a_j, b_i)}{p(a_j)p(b_i)}$$

may take negative values.

In this paper, we try to extend these classical information entropies to matrix ones. The key quantity is the *relative entropy* called the *Kullback-Leibler divergence*:

$$s(X | Y) = \sum_{ij} p(a_j) \log \frac{p(a_j)}{p(b_i)}.$$

The mutual entropy can be expressed by the relative entropy:

$$\begin{aligned} I(X; Y) &= I(X | P_c X) = \\ &= s((p(a_j, b_i)) | (p(a_j), p(b_i))) \end{aligned}$$

In quantum information theory, X is considered as the *density matrix* A (i.e., $A \geq 0$ and $\text{Tr} A = 1$), the channel P_c as a trace-preserving completely positive map Φ and the elementary events $\{b_i\}$ as a POVM (positive operator-valued measure) $F = \{F_i\}$ (i.e., $F_j \geq 0$ and $\sum_j F_j = I$). Note that an input event $\{a_j\}$ is considered as a PVM

(projection-valued measure) $E = \{E_j\}$, that is, a decomposition of the identity. Suppose that $A = \sum_n t_n E_n$ is a spectral decomposition and $\Phi(A) = \sum_n t_n \Phi(E_n)$ where $\{\Phi(E_i)\}$ is a POVM. Then the Ohya mutual entropy $I(A; \Phi)$ is defined by the Umegaki relative entropy

$$s_U(A|B) = \text{Tr } A(\log A - \log B)$$

as

$$I(A; \Phi) = \sup_E s_U\left(\sum_n t_n E_n \otimes \Phi(E_n) \middle| A \otimes \Phi(A)\right)$$

([16]) which is an extension of (1). Thereby, if a relative matrix entropy corresponding to $s_U(A|B)$ is realized, then we might extend information entropies to matrix ones. Simply one might think of $A^{\frac{1}{2}}(\log A - \log B)A^{\frac{1}{2}}$ as an observable, but it is probably not suitable since we think a relative entropy should be an initial vector of a geodesic for a certain geometry. In fact, the geodesic of one of the Hiai-Petz geometries [13] is

$M_t(A, B) = \exp((1-t)\log A + t\log B)$ and hence its initial tangent vector is expressed by

$$\begin{aligned} \mathfrak{S}_U(A|B) &\equiv \left. \frac{dM_t(A, B)}{dt} \right|_{t=0} \\ &= U \left(\left(\frac{1}{\log^{[1]}(d_i, d_j)} \right) \circ U^*(\log B - \log A)U \right) U^* \end{aligned}$$

where U is a unitary with $\text{diag}(d_i) = U^*AU$ and the function $f^{[1]}$ is the divided difference $f^{[1]}(x, y) = \frac{f(x) - f(y)}{x - y}$, see [6, 7]. We think it is a matrix version of the Umegaki entropy. In fact,

$$\begin{aligned} \text{Tr } \mathfrak{S}_U(A|B) \\ = \text{Tr } A(\log B - \log A) = -s_U(A|B). \end{aligned}$$

Also Petz [17] defined a *quantum conditional entropy*

$$h(\rho_{AB} | B) = s(\rho_{AB}) - s(B)$$

(where a composite matrix $\rho_{AB} = W_{AB}$ will be

discussed later) and it is related to the Umegaki entropy:

$$h(\rho_{AB} | B) = \log \dim H_A - s_U(\rho_{AB} | \tau_A \otimes \rho_B)$$

where τ_A is a tracial state. But $h(\rho_{AB} | B)$ is not always positive unfortunately.

Since the quantum conditional entropy is not positive though it is a numerical quantity and the matrix entropy $\mathfrak{S}_U(A|B)$ is a somewhat awkward tool, here we do not use $\mathfrak{S}_U(A|B)$ while we fully use the above idea, in particular, Ohya's construction. In this paper, we use the relative operator entropy

$$S(A|B) = A^{\frac{1}{2}} \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$$

introduced in [9] and try to extend information entropies to matrix ones.

2. Relative Operator Entropy

To begin with, we review the relative operator entropy for positive (bounded linear) operators on a Hilbert space, see [9, 10, 11, 6]. Let $A \#_t B$ be a weighted geometric operator mean in the sense of Kubo-Ando [15];

$$A \#_t B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^t A^{\frac{1}{2}}$$

for invertible A . If A is not invertible, we may define as $A \#_t B = \lim_{n \rightarrow \infty} (A + \frac{1}{n}) \#_t B$ in the strong operator topology. In [10, 9], we introduced the relative operator entropy $S(A|B)$ as a derivative for a differentiable path of geometric operator means $A \#_t B$ if the following strong-limit exists as a bounded operator;

$$\lim_{t \rightarrow 0} \frac{A \#_t B - A}{t}$$

If B is invertible, then

$$S(A|B) = B^{\frac{1}{2}} \eta(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) B^{\frac{1}{2}}$$

where η is the entropy function:

$$\eta(x) = -x \log x \text{ if } x > 0, \quad \eta(0) = 0.$$

In addition, if A is invertible, then

$$S(A|B) = A^{\frac{1}{2}} \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$$

So the existence condition is;

Lemma 1. If A is majorized by B , i.e., $A \leq \alpha B$ for some $\alpha > 0$, then $S(A|B)$ exists.

In fact, by Douglas' majorization theorem [5], we have $A^{\frac{1}{2}} = DB^{\frac{1}{2}}$ for some operator D and

$$S(A|B) = B^{\frac{1}{2}} \eta(D^* D) B^{\frac{1}{2}}.$$

To discuss matrix entropies based on $S(A|B)$, we summarize its properties [8, 9, 10]:

Lemma 2. The relative operator entropy has the following properties if it exists:

(1) If $B \leq B'$, then $S(A|B) \leq S(A|B')$.

(2) $T^* S(A|B) T \leq S(T^* A T | T^* B T)$

(the equality holds for invertible T).

(2') $S(\alpha A | \alpha B) = \alpha S(A|B)$ for $\alpha > 0$.

(3) $S(A_1 | B_1) + S(A_2 | B_2) \leq S(A_1 + A_2 | B_1 + B_2)$.

(3') $(1-t)S(A_1 | B_1) + tS(A_2 | B_2)$

$$\leq S((1-t)A_1 + tA_2 | (1-t)B_1 + tB_2)$$

for all $t \in [0, 1]$.

(4) $S(\bigoplus_k A_k | \bigoplus_k B_k) = \bigoplus_k S(A_k | B_k)$.

(4') $S(\sum_k E_k \otimes A_k | \sum_k E_k \otimes B_k)$
 $= \sum_k E_k \otimes S(A_k | B_k)$ for a PVM $\{E_k\}$.

(5) $S(A|B) \leq B - A$.

(6) $S(A|\alpha B) = (\log \alpha)A + S(A|B)$ for $\alpha > 0$.

The properties (2), (2'), (3), (3') and (4) are called *transformer inequality*, *homogeneity*, *subadditivity*, *joint concavity* and *orthogonality* respectively.

3. Matrix Probability Space

From now on, we assume that the space is finite dimensional, that is, operators are matrices. Assume that $A \in \mathcal{M}_n^+$, the $n \times n$ positive-definite matrices and $B \in \mathcal{M}_m^+$, the $m \times m$ ones. Let $\{E_k\}$ be the

(fixed) decomposition of the identity, that is, each E_k be a projection and $\sum_k E_k = I_n$. A set $\{E_k\}$ is considered as elementary probability events. Let $A = \sum_k t_k E_k$ be a spectral decomposition of of an invertible density matrix, that is, is positive-definite and $\text{Tr } A = 1$. Then, we can observe that the probability $p(E_k)$ is given by $\text{Tr}(t_k E_k) = t_k \text{Tr}(E_k)$.

Let Φ be a quantum channel from $\mathcal{M}_n$ to $\mathcal{M}_m$. Then each $F_k = \Phi(E_k)$ is considered as an elementary event, but it is no longer a projection. So we take a fixed POVM $\{F_\ell\}$, and consider a density matrix $B = \sum_\ell s_\ell F_\ell$. Assume $s_\ell > 0$. Then note that B is invertible since
 $B \geq \sum_\ell \min_j \{s_j\} F_\ell = \min_j \{s_j\} I_m$.

In this situation, we define a *composite matrix* W_{AB} for A and B by

$$\sum_{k,\ell} w_{k\ell} E_k \otimes F_\ell$$

where

$w_{k\ell} \geq 0$, $\sum_k w_{k\ell} \text{Tr } E_k = s_\ell$, $\sum_\ell w_{k\ell} \text{Tr } F_\ell = t_k$. In fact, for the partial traces Tr_1 and Tr_2 for each term, we have

$$\text{Tr}_2(W_{AB}) = \sum_k \text{Tr} \sum_\ell w_{k\ell} F_\ell E_k = \sum_k t_k E_k = A$$

and similarly $\text{Tr}_1(W_{AB}) = B$, so that A and B are the 'marginal distributions'. The composite probability is $p(E_k \otimes F_\ell) = w_{k\ell} \text{Tr } E_k \text{Tr } F_\ell$, and hence the conditional ones are as follows:

$$p(E_k \otimes F_\ell | F_\ell) = \frac{w_{k\ell} \text{Tr } E_k}{s_\ell},$$

$$p(E_k \otimes F_\ell | E_k) = \frac{w_{k\ell} \text{Tr } F_\ell}{t_k}.$$

A typical example for W_{AB} is $\sum_{k,\ell} t_k s_\ell E_k \otimes F_\ell$. In this case, A and B are considered *independent*. In fact, in this case, $p(E_k \otimes F_\ell) = p(E_k)p(F_\ell)$ holds for each k, ℓ . The following example is not independent:

Example 1. Let (w_{ij}) be given by a nonnegative

matrix

$$(w_{k\ell}) = \frac{1}{12} \begin{pmatrix} 6 & 1 \\ 2 & 3 \end{pmatrix}$$

and all E_k and F_k projections of rank one. Then we have the marginal probabilities

$$t_1 = \frac{7}{12}, t_2 = \frac{5}{12}, s_1 = \frac{2}{3}, s_2 = \frac{1}{3}.$$

They are not independent since, for instance, $w_{11} = \frac{1}{2} \neq \frac{14}{36} = t_1 s_1$. For positive semidefinite matrices of rank one

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$F_1 = \frac{1}{5} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}, F_2 = \frac{1}{5} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix},$$

we have the marginal density matrices

$$A = \frac{7}{12} E_1 + \frac{5}{12} E_2 = \frac{1}{12} \begin{pmatrix} 7 & 0 \\ 0 & 5 \end{pmatrix},$$

$$B = \frac{2}{3} F_1 + \frac{1}{3} F_2 = \frac{1}{15} \begin{pmatrix} 9 & 2 \\ 2 & 6 \end{pmatrix},$$

and the composite matrix

$$W_{AB} = \begin{pmatrix} \frac{6}{12} F_1 + \frac{1}{12} F_2 & 0 \\ 0 & \frac{2}{12} F_1 + \frac{3}{12} F_2 \end{pmatrix}$$

$$= \frac{1}{60} \begin{pmatrix} 25 & 10 & 0 & 0 \\ 10 & 10 & 0 & 0 \\ 0 & 0 & 11 & -2 \\ 0 & 0 & -2 & 14 \end{pmatrix}.$$

4. Matrix Entropies

The composite matrix entropy $H(W_{AB})$ is defined by $H(W_{AB}) = \eta(W_{AB})$. Also the mutual matrix entropy by $I(A; B)$ and the conditional entropies $H(W_{AB} | A)$ and $H(W_{AB} | B)$ defined by

$$I(A; B) = -S(W_{AB} | A \otimes B),$$

$$H(W_{AB} | A) = S(W_{AB} | A \otimes I),$$

$$H(W_{AB} | B) = S(W_{AB} | I \otimes B).$$

Note that these matrices can be defined since A and B are invertible (see Lemma exist). Immediately we have $H(W_{AB}) \geq 0$ and $H(W_{AB} | B) \geq 0$, while $I(A; B)$ is not always positive-semidefinite (see Example ex2 in the below) since the individual mutual information may be negative. But its trace is positive:

Lemma 3. $\text{Tr } I(A; B) \geq 0$.

By Lemma 2(5), we have

$$\text{Tr } I(A; B) \geq \text{Tr } (\text{Tr}_2 W_{AB}) - \text{Tr } A \otimes B$$

$$= \text{Tr } A - (\text{Tr } A)(\text{Tr } B) = 1 - 1 = 0.$$

Remark 1. Contrary to the classical case, another mutual matrix entropy $I(B; A)$, even if it is defined well, is not equal to $I(A; B)$ since $A \otimes B \neq B \otimes A$ and the usual product also is not commuting.

Then, by Lemma 2(4'), we express these entropies as follows:

Lemma 4. Matrix entropies have the following decompositions:

$$H(W_{AB}) = \sum_k E_k \otimes \eta(\sum_l w_{kl} F_l).$$

$$I(A; B) = -\sum_k E_k \otimes S(\sum_l w_{kl} F_l | t_k B)$$

$$= -\sum_{k,l} \log t_k E_k \otimes w_{kl} F_l - \sum_k E_k \otimes S(\sum_l w_{kl} F_l | t_k B)$$

$$H(W_{AB} | A) = \sum_k E_k \otimes S(\sum_l w_{kl} F_l | t_k I)$$

$$= \sum_k t_k E_k \otimes \eta\left(\sum_l \frac{w_{kl}}{t_k} F_l\right).$$

$$H(W_{AB} | B) = \sum_k E_k \otimes S(\sum_l w_{kl} F_l | B).$$

In addition, all F_ℓ are (mutually orthogonal) projections, then

$$H(W_{AB}) = \sum_{k,\ell} \eta(w_{k\ell}) E_k \otimes F_\ell.$$

$$I(A; B) = -\sum_{k,\ell} w_{k\ell} \log \frac{t_k s_\ell}{w_{k\ell}} E_k \otimes F_\ell.$$

$$H(W_{AB} | A) = \sum_{k,\ell} w_{k\ell} \log \frac{t_k}{w_{k\ell}} E_k \otimes F_\ell.$$

$$H(W_{AB} | B) = \sum_{k,\ell} w_{k\ell} \log \frac{s_\ell}{w_{k\ell}} E_k \otimes F_\ell.$$

Thus, the latter case, where $(\{F_\ell\})$ is a PVM,

shows the entropy values in the classical (commutative) case. So we confirm easily that $I(A; B)$ is not always positive even for diagonal matrices corresponding to the classical ones:

Example 2. We simplify Example 1; $F_1 = E_1$, $F_2 = E_2$ and $B = \frac{1}{3} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. By the above lemma

(2), we have only to show $S(\sum_{\ell} w_{1\ell} F_{\ell} | t_1 B)$ is neither positive nor negative:

$$\begin{aligned} & S\left(\frac{1}{12} \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \middle| \frac{7}{36} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \frac{1}{36} S\left(\begin{pmatrix} 18 & 0 \\ 0 & 3 \end{pmatrix} \middle| \begin{pmatrix} 14 & 0 \\ 0 & 7 \end{pmatrix}\right) \\ &= \frac{1}{36} \begin{pmatrix} 18 \log \frac{14}{18} & 0 \\ 0 & 3 \log \frac{7}{3} \end{pmatrix}. \end{aligned}$$

Since $\log \frac{14}{18} < 0$ and $\log \frac{7}{3} > 0$, we have the required result. Also (2') shows this;

$$\log \frac{w_{11}}{t_1 s_1} = \log \frac{9}{7} > 0 \text{ and } \log \frac{w_{12}}{t_1 s_2} = \log \frac{3}{7} < 0.$$

In the context for the composite elementary events $\{E_k \otimes F_{\ell}\}$, the entropy $\eta(A)$, $\eta(B)$ should be extended to

$$H_F(A) = - \sum_{k,\ell} \log(t_k) w_{k\ell} E_k \otimes F_{\ell}, \text{ and}$$

$$H_E(B) = - \sum_{k,\ell} \log(s_{\ell}) w_{k\ell} E_k \otimes F_{\ell}.$$

In fact, we obtain

$$\begin{aligned} \text{Tr}_2(H_F(A)) &= - \sum_k \text{Tr}(w_{k\ell} F_{\ell}) \log(t_k) E_k \\ &= - \sum_k t_k \log(t_k) E_k \\ &= \sum_k \eta(t_k) E_k = \eta(A) \end{aligned}$$

and similarly $\text{Tr}_1(H_E(B)) = \eta(B)$. Then we have

$$\begin{aligned} & H(W_{AB}) \\ &= \frac{1}{12} \begin{pmatrix} (6 \log 2) F_1 + (\log 12) F_2 & 0 \\ 0 & (2 \log 6) F_1 + (3 \log 4) F_2 \end{pmatrix}, \end{aligned}$$

the following relations similar to the classical cases:

Theorem 5. The following equalities hold:

$$(1) H(W_{AB} | B) + I(A; B) = H_F(A)$$

$$(2) H_F(A) + H(W_{AB} | A) = H(W_{AB}).$$

Proof. By Lemma 4 and Lemma 2 (6), we have (1) as follows:

$$\begin{aligned} & H(W_{AB} | B) + I(A; B) \\ &= \sum_k E_k \otimes S\left(\sum_l w_{kl} F_{\ell} | B\right) \\ &\quad - \sum_k E_k \otimes S\left(\sum_l w_{kl} F_{\ell} | t_k B\right) \\ &= \sum_k E_k \otimes \left[S\left(\sum_l w_{kl} F_{\ell} | B\right) \right. \\ &\quad \left. - (\log t_k) \left(\sum_l w_{kl} F_{\ell} - S\left(\sum_l w_{kl} F_{\ell} | t_k B\right)\right) \right] \\ &= - \sum_{k,l} \log(t_k) w_{kl} E_k \otimes F_{\ell} = H_F(A). \end{aligned}$$

Similarly (2) follows from

$$\begin{aligned} & H_F(A) + H(W_{AB} | B) = \\ & - \sum_{k,l} \log(t_k) w_{kl} E_k \otimes F_{\ell} + \sum_k E_k \otimes S\left(\sum_l w_{kl} F_{\ell} | t_k I\right) \\ &= \sum_k E_k \otimes \eta\left(\sum_l w_{kl} F_{\ell}\right) \\ &= \eta\left(\sum_k E_k \otimes \sum_l w_{kl} F_{\ell}\right) \\ &= \eta\left(\sum_{k,l} w_{kl} E_k \otimes F_{\ell}\right) = \eta(W_{AB}) = H(W_{AB}). \end{aligned}$$

Remark 2. As in the above,

$H(W_{AB} | A) + I(A; B)$ is not equal to $H_E(B)$ in general.

Example 3. In Example 1, we exchange $\{f_{\ell}\}$ to some POVM. Then the following matrix entropies similar to the classical ones as in Example 2:

$$\begin{aligned}
I(A; B) &= \frac{1}{12} \begin{pmatrix} (6 \log \frac{9}{7})F_1 + (\log \frac{3}{7})F_2 & 0 \\ 0 & (2 \log \frac{3}{5})F_1 + (3 \log \frac{9}{5})F_2 \end{pmatrix}, \\
H(W_{AB} | A) &= \frac{1}{12} \begin{pmatrix} (6 \log \frac{7}{6})F_1 + (\log 7)F_2 & 0 \\ 0 & (2 \log \frac{5}{2})F_1 + (3 \log \frac{5}{3})F_2 \end{pmatrix}, \\
H(W_{AB} | B) &= \frac{1}{12} \begin{pmatrix} (6 \log \frac{4}{3})F_1 + (\log 4)F_2 & 0 \\ 0 & (2 \log 4)F_1 + (3 \log 2)F_2 \end{pmatrix}, \\
H_F(A) &= \frac{1}{12} \begin{pmatrix} (6 \log \frac{12}{7})F_1 + (\log \frac{12}{7})F_2 & 0 \\ 0 & (2 \log \frac{12}{5})F_1 + (3 \log \frac{12}{5})F_2 \end{pmatrix}.
\end{aligned}$$

Similar identities hold like classical cases for a PVM $\{F_\ell\}$:

Corollary 6. If $\{F_\ell\}$ is a PVM, then

- (1) $H(W_{AB} | A) + I(A; B) = H_E(B)$, and
- (2) $H_E(B) + H(W_{AB} | B) = H(W_{AB})$.

In the above corollary, the assumption are necessary as in the following example:

Example 4. In the situation of Example 3, consider the case that each F_k is not a projection: Let

$$P_1 = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad P_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$F_1 = \frac{1}{4}P_1 + \frac{3}{4}P_2 = \frac{1}{4} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ and}$$

$$F_2 = \frac{3}{4}P_1 + \frac{1}{4}P_2 = \frac{1}{4} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

$$\text{Then } B = \frac{2}{3}F_1 + \frac{1}{3}F_2 = \frac{1}{12} \begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix} = \frac{5P_1}{12} + \frac{7P_2}{12}$$

and

$$\begin{aligned}
H_E(B) &= \frac{1}{12} \begin{pmatrix} 6 \log \frac{3}{2}F_1 + \log 3F_2 & \\ & 2 \log \frac{3}{2}F_1 + 3 \log 3F_2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{16} \log \frac{3^3}{2^2}P_1 + \frac{1}{48} \log \frac{3^{19}}{2^{18}}P_2 & \\ & \frac{1}{48} \log \frac{3^{11}}{2^2}P_1 + \frac{1}{16} \log \frac{3^3}{2^2}P_2 \end{pmatrix}.
\end{aligned}$$

By

$$\sum_\ell w_{1\ell} F_\ell = \frac{6F_1}{12} + \frac{F_2}{12} = \frac{9}{48}P_1 + \frac{19}{48}P_2,$$

$$\sum_\ell w_{2\ell} F_\ell = \frac{2F_1}{12} + \frac{3F_2}{12} = \frac{11}{48}P_1 + \frac{9}{48}P_2,$$

we have

$$S\left(\sum_\ell w_{1\ell} F_\ell \mid t_1 I\right) = \frac{9}{48} \log \frac{28}{9}P_1 + \frac{19}{48} \log \frac{28}{19}P_2,$$

$$S\left(\sum_\ell w_{2\ell} F_\ell \mid t_2 I\right) = \frac{11}{48} \log \frac{20}{11}P_1 + \frac{9}{48} \log \frac{20}{9}P_2$$

and hence $H(W_{AB} | A)$

$$= \begin{pmatrix} \frac{9}{48} \log \frac{28}{9}P_1 + \frac{19}{48} \log \frac{28}{19}P_2 & \\ & \frac{11}{48} \log \frac{20}{11}P_1 + \frac{9}{48} \log \frac{20}{9}P_2 \end{pmatrix}.$$

Since $t_1 B = \frac{35P_1 + 49P_2}{144}$ and $t_2 B = \frac{25P_1 + 35P_2}{144}$, we

have $I(A; B)$

$$= - \begin{pmatrix} \frac{9}{48} \log \frac{35}{27}P_1 + \frac{19}{48} \log \frac{49}{57}P_2 & \\ & \frac{11}{48} \log \frac{25}{33}P_1 + \frac{9}{48} \log \frac{35}{27}P_2 \end{pmatrix},$$

we have $H(W_{AB} | A) + I(A; B)$

$$= \begin{pmatrix} \frac{9}{48} \log \frac{12}{5}P_1 + \frac{19}{48} \log \frac{12}{7}P_2 & \\ & \frac{11}{48} \log \frac{12}{5}P_1 + \frac{9}{48} \log \frac{12}{7}P_2 \end{pmatrix},$$

which does not coincides with $H_E(B)$.

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