Lacunary Invariant Statistical Convergence Of Sequences Of Sets With Respect To A Modulus Function

Nimet Pancaroglu and Fatih Nuray

Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey.

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Abstract: In this paper, we introduce and study the concept of lacunary invariant convergence for sequences of sets with respect to modulus function f and give some inclusion relations.

Keywords: Statistical convergence, lacunary invariant convergence, sequences of sets, modulus function

1. Introduction

The concept of statistical convergence for sequences of real numbers was introduced Fast [5], Salat [15] and others. Let $K \subseteq N$ and $K_n = \{k \leq n : k \in K\}$. Then the natural density of K is defined by $\delta(K) = \lim_n n^{-1} |K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of K_n .

A sequence $x = (x_k)$ complex numbers is said to be statistically convergent to L if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$$

Convergence concept for sequences of set had been studied by Beer [3], Aubin and Frankowska [1], Baronti and Papini [2]. The concept of statistical convergence of sequences of set was introduced by Nuray and Rhoades [11] in 2012. Ulusu and Nuray [19] introduced the concept of Wijsman lacunary statistical convergence of sequences of set. Similarly, the concepts of Wijsman invariant statistical and Wijsman lacunary invariant statistical convergence were introduced by Pancaroglu and Nuray [13] in 2013.

A function $f : [0, \infty) \rightarrow [0, \infty)$ is called a

modulus if

- (1) $f(x) = 0$ if and only if $x = 0$,
- (2) $f(x + y) \leq f(x) + f(y)$
- (3) f is increasing
- (4) f is continuous from the right at 0.

A modulus may be unbounded (for example $f(x) = x^p$, $0 < p < 1$) or bounded (for example $f(x) = \frac{x}{x+1}$).

Modulus function was introduced by Nakano [10] in 1953. Ruckle [14] used in idea of modulus function f to construct a class of FK spaces

$$L(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

The space $L(f)$ is closely related to the space ℓ_1 which is a $L(f)$ space with $f(x) = x$ for all real $x \geq 0$.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ will be denoted by such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$.

Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. Freedman, Sember and Raphael [6] defined the space N_θ in the following way. For any lacunary sequence $\theta = (k_r)$,

$$N_\theta = \{x = (x_k) :$$

$$\text{for some } L, \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0\}$$

Corresponding author: Nimet Pancaroglu, Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey. E-mail: npancaroglu@aku.edu.tr.

Fridy and Orhan [7] introduce lacunary statistically convergent sequence as follows:

Let θ be a lacunary sequence; the number sequence (x_k) is lacunary statistically convergent to L provided that for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0.$$

2. Definitions and Notations

Let σ be a mapping of the positive integers into itself. A continuous linear functional ϕ on ℓ_∞ , the space of real bounded sequences, is said to be an invariant mean or a σ mean, if and only if,

- (1) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ is such that $x_n \geq 0$ for all n ,
- (2) $\phi(e) = 1$, where $e = (1, 1, 1, \dots)$,
- (3) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

The mappings ϕ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m , where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n . Thus ϕ extends the limit functional on c , the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In case σ is translation mapping $\sigma(n) = n + 1$, the σ mean is often called a Banach limit and V_σ , the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences.

It can be shown that

$$V_\sigma = \{x = (x_n) :$$

$$\lim_n t_{mn}(x) = L \text{ uniformly in } m, L = \sigma - \lim x\}$$

where,

$$t_{mn}(x) = \frac{x_m + x_{\sigma(m)} + x_{\sigma^2(m)} + \dots + x_{\sigma^{n-1}(m)}}{n}.$$

Definition 1. [12] A set E of positive integers said to have uniform lacunary invariant density of zero if

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n :$$

$$E \cap \{\sigma(m), \sigma^2(m), \dots, \sigma^k(m)\}| = 0$$

uniformly in m .

By using uniform invariant density, the following definition was given.

Definition 2. [16] A complex number sequence $x = (x_k)$ is said to be lacunary σ -statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} |\{k \in I_r :$$

$$|x_{\sigma^k(m)} - L| \geq \varepsilon\}| = 0 \text{ uniformly in } m = 1, 2, \dots$$

This will be denoted by $S_{\sigma\theta} - \lim x = L$ or $x_k \rightarrow L(S_{\sigma\theta})$.

Let (X, ρ) be a metric space. For any point $x \in X$ and non-empty subset A of X we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

The concept of Wijsman convergence was introduced by Wijsman [20] as follow:

Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k \rightarrow \infty} d(x, A_k) = d(x, A).$$

for each $x \in X$. This will be denoted by $W - \lim A_k = A$.

Convergence concept for sequences of set had been studied by Beer [3], Aubin and Frankowska [1], Baronti and Papini [2]. The concepts of Wijsman statistical convergence and Wijsman strong Cesaro summability were introduced by Nuray and Rhoades [11] as follows:

Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, the sequence $\{A_k\}$ is said to be Wijsman strongly Cesaro summable to A if for each $x \in X$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |d(x, A_k) - d(x, A)| = 0.$$

Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, the sequence $\{A_k\}$ is said to be Wijsman statistically convergent to A if for $\varepsilon > 0$ and each $x \in X$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case we write $st\text{-}\lim_W A_k = A$ or $A_k \rightarrow A(WS)$.

3. Main Result

The purpose of this paper is, by using modulus function, to introduce and study new sequence spaces of sequences of sets. The following three definitions were given [13].

Definition 3. Let (X, ρ) be metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary invariant convergent to A , if for each $x \in X$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} d(x, A_{\sigma^k(m)}) = d(x, A)$$

uniformly in m .

In this case we write $A_k \rightarrow A(WN_{\sigma\theta})$ and the set of all Wijsman lacunary invariant convergent sequences of sets will be denoted $WN_{\sigma\theta}$.

Definition 4. Let (X, ρ) be metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman strongly lacunary invariant convergent to A , if for each $x \in X$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_{\sigma^k(m)}) - d(x, A)| = 0$$

uniformly in m .

In this case we write $A_k \rightarrow A([WN_{\sigma\theta}])$ and the set of all Wijsman strongly lacunary invariant convergent sequences of sets will be denoted $[WN_{\sigma\theta}]$.

Definition 5. Let (X, ρ) be metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary invariant statistically convergent to A , for each $\varepsilon > 0$ and if for each $x \in X$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\}| = 0$$

uniformly in m .

In this case we write $A_k \rightarrow A(WS_{\sigma\theta})$ and the set of all Wijsman lacunary invariant statistically convergent sequences of sets will be denoted $WS_{\sigma\theta}$.

Let (X, ρ) be metric space. For any non-empty closed subsets $A, A_k \subseteq X$, $x \in X$, we define the sequences of sets space $[WN_{\sigma\theta}]_\infty$ as follows:

$$[WN_{\sigma\theta}]_\infty = \{\{A_k\} : \sup_{m,r} \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_{\sigma^k(m)})| < \infty\}.$$

Now, by using a modulus function, we introduce following new sequence spaces of sequences of sets.

Definition 6. Let (X, ρ) be metric space and f be a modulus function. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman strongly lacunary invariant convergent to A with respect to the modulus f , if for each $x \in X$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} f(|d(x, A_{\sigma^k(m)}) - d(x, A)|) = 0$$

uniformly in m .

In this case we write $A_k \rightarrow A([WN_{\sigma\theta}(f)])$ and the set of all Wijsman strongly lacunary invariant convergent sequences of sets with respect to the modulus f will be denoted $[WN_{\sigma\theta}(f)]$.

Let (X, ρ) be metric space and f be a modulus function. For any non-empty closed subsets $A, A_k \subseteq X$, $x \in X$, we define the sequences of sets space $[WN_{\sigma\theta}(f)]_\infty$ as follows:

$$[WN_{\sigma\theta}(f)]_\infty = \{\{A_k\} : \sup_{m,n} \frac{1}{h_r} \sum_{k \in I_r} f(|d(x, A_{\sigma^k(m)})|) < \infty\}.$$

If $f(x) = x$, then the spaces $[WN_{\sigma\theta}(f)]$, $[WN_{\sigma\theta}(f)]_\infty$ reduce to $[WN_{\sigma\theta}]$, $[WN_{\sigma\theta}]_\infty$ respectively.

Now we study the relation between $WS_{\sigma\theta}$ and $[WN_{\sigma\theta}(f)]$ convergence.

Theorem 1. Let (X, ρ) be a metric space then for any non empty closed subsets A , $A_k \subseteq X$, for each m ,

i $A_k \rightarrow A([WN_{\sigma\theta}(f)])$ implies $A_k \rightarrow A(WS_{\sigma\theta})$.

ii f is bounded and $A_k \rightarrow A(WS_{\sigma\theta})$ implies $A_k \rightarrow A([WN_{\sigma\theta}(f)])$.

iii $WS_{\sigma\theta} = [WN_{\sigma\theta}(f)]$ if f is bounded.

Proof. (i) If $\varepsilon > 0$ and $A_k \rightarrow A([WN_{\sigma\theta}(f)])$, we can write,

$$\begin{aligned} & \sum_{k \in I_r} f(|d(x, A_{\sigma^k(m)}) - d(x, A)|) \\ & \geq |d(x, A_{\sigma^k(m)}) - d(x, A)| \\ & \geq \varepsilon \sum_{k \in I_r} f(|d(x, A_{\sigma^k(m)}) - d(x, A)|) \\ & \geq \varepsilon \cdot |\{k \in I_r : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\}| \end{aligned}$$

which yields result.

(ii) Suppose that $A_k \rightarrow A(WS_{\sigma\theta})$ and f is bounded, for each $m \geq 1$, set

$$G = \sup_{k, m} f(|d(x, A_{\sigma^k(m)})| + |d(x, A)|)$$

Let $\varepsilon > 0$ and select N_ε such that

$$\frac{1}{h_r} |\{k \in I_r : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \frac{\varepsilon}{2}\}| < \varepsilon 2G$$

for all m and $r > N_\varepsilon$ and set $L(r, m, x) = \{k \in I_r : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon 2\}$.

Now for all m and $r > N_\varepsilon$ we have that

$$\begin{aligned} & \sum_{k \in I_r} f(|d(x, A_{\sigma^k(m)}) - d(x, A)|) \\ & = \sum_{k \in L(r, m, x)} f(|d(x, A_{\sigma^k(m)}) - d(x, A)|) \end{aligned}$$

$$\begin{aligned} & + \sum_{k \notin L(r, m, x)} f(|d(x, A_{\sigma^k(m)}) - d(x, A)|) \\ & < \frac{1}{h_r} (h_r \frac{\varepsilon}{2G}) G + \frac{1}{h_r} (\frac{\varepsilon}{2}) h_r = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $\{A_k\}$ is strongly lacunary invariant convergent to A with respect to the modulus function f .

(iii) This is an immediate consequence of (i) and (ii).

This completes proof of theorem.

Theorem 2. Let f be a modulus. Then $[WN_{\sigma\theta}(f)] \subset [WN_{\sigma\theta}(f)]_\infty$.

Proof. Suppose that, $\{A_k\} \in [WN_{\sigma}(f)]$.

$$\begin{aligned} & 1h_r \sum_{k \in I_r} f(|d(x, A_{\sigma^k(m)})|) \\ & = \frac{1}{h_r} \sum_{k \in I_r} f(|d(x, A_{\sigma^k(m)}) - d(x, A) + d(x, A)|) \\ & \leq \frac{1}{h_r} \sum_{k \in I_r} f(|d(x, A_{\sigma^k(m)}) - d(x, A)| + \frac{1}{h_r} \sum_{k \in I_r} f(|d(x, A)|) \\ & \leq \frac{1}{h_r} \sum_{k \in I_r} f(|d(x, A_{\sigma^k(m)}) - d(x, A)|) + M \frac{1}{h_r} \sum_{k \in I_r} f(1) \end{aligned}$$

where M is an integer such that $d(x, A) < M$. Therefore $\{A_k\} \in [WN_{\sigma\theta}(f)]_\infty$.

Theorem 3. If f is a modulus and $\{A_k\}$ is strongly lacunary invariant convergent to A , then $\{A_k\}$ is strongly lacunary invariant convergent to A with respect to the modulus v , that is, $[WN_{\sigma\theta}] \subset [WN_{\sigma\theta}(f)]$.

Proof. Let $\{A_k\} \in [WN_{\sigma\theta}]$, then,

$$\begin{aligned} S_{nm} & = \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_{\sigma^k(m)}) - d(x, A)| \rightarrow 0 \\ (r \rightarrow \infty) \end{aligned}$$

uniformly in m .

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$. Write $a_{km} = |d(x, A_{\sigma^k(m)}) - d(x, A)|$

$$\sum_{k \in I_r} f(a_{km}) = \Sigma_1 + \Sigma_2$$

where the first summation is over $a_{km} \leq \delta$ and second over $a_{km} > \delta$. Then $\Sigma_1 \leq \varepsilon h_r$ and for $a_{km} > \delta$,

$$f(a_{km}) < a_{km} \delta < 1 + \left\lceil \frac{a_{km}}{\delta} \right\rceil$$

where $[z]$ denotes the integer part of z . By definition modulus function, we have for $a_{km} > \delta$,

$$f(a_{km}) \leq (1 + \left\lceil \frac{a_{km}}{\delta} \right\rceil) f(1) \leq 2f(1)a_{km}\delta$$

Hence $\Sigma_2 \leq 2f(1)\frac{a_{km}}{\delta}$, which together with $\Sigma_1 \leq \varepsilon h_r$ yields $[WN_{\sigma\theta}] \subset [WN_{\sigma\theta}(f)]$.

Lemma 1. [16] Let f be a modulus. Let $\alpha > 0$ be given constant. Then there is a constant $c > 0$ such that $f(x) > cx$ ($0 < x < \alpha$).

Theorem 4. Let $\{A_k\}$ be a bounded sequence, f be a modulus function. Then $\{A_k\}$ is Wisjman strongly lacunary invariant convergent to A with respect to modulus f if and only if $\{A_k\}$ is Wisjman strongly lacunary invariant convergent to A , i.e.,

$$[WN_{\sigma\theta}(f)] \cap L_\infty \Leftrightarrow [WN_{\sigma\theta}]$$

where L_∞ denotes the set of bounded sequences of sets.

Proof. The proof of the theorem follows from Theorem 3 and Lemma 1.

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