Stationary and Non-stationary Self-Induced Vibrations in Waveguiding Systems

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Abstract: With the use of a wave model, the non-linear problem about realization of the Poincare-Hopf bifurcations in waveguiding systems is stated. The constitutive non-linear differential equations are deduced, the methods for their solution are elaborated. The example of torsion wave propagation in an elongated drill string is considered. Computer simulation of auto-oscillation generation in the examined system is performed for the cases of stationary and non-stationary variations of the perturbation parameter. The diapason of the drilling rotation velocity values corresponding to regimes of stable self-excited periodic motions of the system is found. This domain is shown to be limited by the states of the Poincare-Hopf bifurcations. Owing to the feature that the stated problem is singularly perturbed, the autovibrations are of relaxation type with fast and slow motions. Influence of the length of the uniform and articulated drill strings on the bifurcation values of their angular velocities of generation and accomplishment of the auto-oscillation processes in the drill strings is discussed.

Key words: Waveguiding systems, singularly perturbed problem, self-induced vibrations, Hopf’s bifurcation, relaxation vibrations.

1. Introduction

Autovibration of dynamical systems is one of the most widespread self-organization phenomena in nature. It can play both the positive and negative roles in many devices, beginning, for example, from bow and wind musical instruments and to complex objects of modern industry and electronic technical equipment. The simplest and clearest model, illustrating the process of mechanical autovibration generating, is the 1 DOF oscillator including a conveyor belt with a load on it, restrained by elastic weightless spring (Fig. 1) [1]. Between the belt and load, the conditions of nonlinear frictional interaction are realized, which at certain constant values of the belt velocity v cause self-excitation of periodic reciprocating motions of the weight. Owing to its simplicity, this system was exhaustively investigated for different laws of nonlinear friction and elastic extension-compression of the spring. With its use, some general regularities of the autovibrational process self-excitation and proceeding were established.

But the considered model undergoes qualitative alterations if the spring is long (Fig. 2). Then, its mass may be comparable or even larger the body mass, it ceases to be a simple elastic element, and becomes an elastic waveguide transmitting longitudinal extension-compression waves. Such device should be simulated by distributed systems with vibrations possessing modes arranged in an ordered (wave) fashion. In practice, such phenomena may appear, for example, in towing a transport facility on a water or solid surface (Fig. 3).

Similar processes also occur in the devices of deep drilling [2, 3]. At the drill string extraction from the bore-hole cavity, the drill bit grates with its surface and the string begins to play the role of a waveguide (Fig. 4a). However, apparently the most distinctive autovibrational wave processes are generated in drilling...
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Fig. 1 Model of the simplest autovibrational system.

Fig. 2 Model of an autovibrational system with a waveguiding elastic element.

Fig. 3 Schematic of dragging a body by a long elastic cable.

Fig. 4 Schematics of (a) longitudinal and (b) torsional wave autovibration excitation in long drill strings.

the deep vertical bore-holes [4-9]. When, as a result of non-linear frictional interaction between the rotating bit and the near bottom surface of the well, the bit begins to commit torsional vibrations and torsional waves begin to propagate along the drill string (Fig. 4b). Analysis of these vibrations was performed on the basis of the non-linear model of torsional wave pendulum in [10]. It was shown in this reference that the self-oscillations were realized inside some diapason of change of the system rotation velocity and transitions from stationary rotations to periodic rotational motions were accomplished in the forms of the Hopf bifurcations [1].

It can be concluded that analogous effects take place also in other self-vibrating systems. So, in this paper, the questions of analysis of stationary and transient self-oscillating processes in homogeneous and heterogeneous waveguides are considered. It is established that owing to the absence of the wave dispersion in the considered dynamic models, the transfer from study of a wave equation throughout the whole length of the waveguide to analysis of one non-linear differential equation with a delay argument can be fulfilled.

The basic attention is paid to the systems with small inertia where the vibrating body mass is much less the waveguide inertia. In this connection, the constructed differential equation has the small parameter before the senior (second) derivative. The equations of this type are called singularly perturbed [11, 12]. Their solutions have the shapes of saw-tooth functions and the vibrations described by them are called relaxational [13]. The second feature of the constructed solutions consists in the fact that the velocity function of the vibrating body has the quantized character in time with the quantum duration equaled the duration of the waveguide double length running by the wave [14].

A matter of definite interest is a heterogeneous waveguide consisting of segments with different acoustic stiffnesses. In this event, the wave signal propagates with different velocities inside every segment and at the interfaces between these segments, the wave experiences additional diffractions associated with smaller splitting the time quanta.

There are also the questions of vibration self-excitation in non-stationary waveguide systems which are not practically studied. These states correspond to the regimes of speeding and braking and should be specially analyzed.
The paper suggested is dedicated to the analysis of the foregoing effects. It is organized as follows: Section 2 is dedicated to the statement of the problem of auto-oscillations of a homogeneous waveguide; Section 3 describes models of energy exchange in autovibrational systems; Section 4 introduces the multipoint boundary value problem for multilink waveguides; Section 5 presents results of numerical analysis; Section 6 gives conclusions.

2. Constitutive Equation of Auto-oscillations of a Homogeneous Waveguide

For the purpose of theoretical simulating the phenomenon of self-excitation of a waveguide vibration, the wave model of a dragging device with elastic cable of length \( L \) (Fig. 3) is used. It should be remarked that this model can be easily extended to other mechanic or electronic waveguide systems. In the considered case the right-hand end of the cable is considered to move with constant velocity \( v \) along the immovable \( OX \) axis. The longitudinal vibrations of the dragged body are excited through its frictional interaction with the horizontal surface.

To describe the body motion, introduce also the \( O_1x \) axis moving with speed \( v \). Then, the distance travelled by the body along the \( Ox \) axis is \( vt + u(0, t) \), where \( vt \) is the distance covered by the \( O \) point; \( t \) is the time; \( u = u(x, t) \) is the elastic displacement of the cable element along the \( Ox \) axis.

By treating the elastic cable as an elastic waveguide, its axial vibrations can be described by the wave equation

\[
\rho A \frac{\partial^2 u}{\partial t^2} - E A \frac{\partial^2 u}{\partial x^2} = 0
\]  

where, \( A \) is the cross-section area of the cable, \( \rho \) is its material density, and \( E \) is its elasticity modulus.

Eq. (1) can be brought to the standard form:

\[
\frac{\partial^2 u}{\partial t^2} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0
\]  

Here, \( \alpha = \sqrt{E / \rho} \) is the velocity of the longitudinal wave.

The solution to Eqs. (1) and (2) is

\[
u(x, t) = f(x - \alpha t) + g(x + \alpha t)\]  

which expressed through the phase variables \( x - \alpha t \) and \( x + \alpha t \). In Eq. (3), \( f(x - \alpha t) \) is the longitudinal elastic wave emanated towards right end \( x = L \) from the body, \( g(x + \alpha t) \) is the wave propagating to the body from the right end \( x = L \) of the cable.

Since the end \( x = L \) is moving with constant velocity \( v \), it can be considered as clamped one for the elastic displacement. Then,

\[
u(L, t) = 0 \text{ or } f(L - \alpha t) + g(L + \alpha t) = 0 \]  

To deduce the boundary condition at the left end \( x = 0 \), consider the dynamic equilibrium of the forces applied to it. So, one has

\[
F^u + F^v + F^\alpha = 0
\]  

where, \( F^u = -m \ddot{u} \) is the inertia force acting on the body, \( F^v = F^v(v + \dot{u}) \) is the friction force formed between the body and horizontal surface, it will be defined later. Here, the dot over a symbol denotes derivative with respect to time.

The elastic force \( F^\alpha \) is calculated with the help of equality

\[
F^\alpha = EA \frac{\partial u}{\partial x}|_{x=0}
\]  

where, the strain \( \frac{\partial u}{\partial x} \) value is calculated as follows:

\[
\frac{\partial u}{\partial x}|_{x=0} = \frac{\partial}{\partial x} [f(x - \alpha t) + g(x + \alpha t)]_{x=0}
\]  

Since the independent variables \( u \) and \( t \) are connected by the phase variables \( x - \alpha t \) and \( x + \alpha t \), the partial derivative \( \frac{\partial u}{\partial x} \) can be expressed via the \( \frac{\partial u}{\partial t} \) derivative. Really, it issues from Eqs. (3) and (4):

\[
g(L + \alpha t) = -f(L - \alpha t)
\]  

However,

\[
g(x + \alpha t) = g\left[L + \alpha \left(t - \frac{L - x}{\alpha}\right)\right]
\]  

From Eqs. (8) and (9), it follows

\[
g\left[L + \alpha \left(t - \frac{L - x}{\alpha}\right)\right] = -f\left[L - \alpha \left(t - \frac{L - x}{\alpha}\right)\right]
\]  

So,

\[
g(x + \alpha t) = -f(2L - x - \alpha t)
\]
Introduce the notations \( x - \alpha t = p \), \( 2L - x - \alpha t = q \).

Then, instead of Eq. (3), the presentation is gained:

\[
\begin{align*}
 u(x) &= f(x - \alpha t) - f(2L - x - \alpha t) = f(p) - f(q) \quad (11)
\end{align*}
\]

With its use, the derivatives can be calculated:

\[
\begin{align*}
 \frac{\partial u}{\partial x} &= \frac{\partial f(p(x,t))}{\partial x} - \frac{\partial f(q(x,t))}{\partial x} = \\
 &= \frac{\partial f(p)}{\partial p} \frac{\partial p}{\partial x} - \frac{\partial f(q)}{\partial q} \frac{\partial q}{\partial x} = \frac{\partial f(p)}{\partial p} + \frac{\partial f(q)}{\partial q}
\end{align*}
\]

\[
\begin{align*}
 \frac{\partial u}{\partial t} &= \frac{\partial f(p(x,t))}{\partial t} - \frac{\partial f(q(x,t))}{\partial t} = \\
 &= \frac{\partial f(p)}{\partial p} \frac{\partial p}{\partial t} - \frac{\partial f(q)}{\partial q} \frac{\partial q}{\partial t} = -\alpha \frac{\partial f(p)}{\partial p} + \alpha \frac{\partial f(q)}{\partial q}
\end{align*}
\]

(12)

Through correlation these equalities, one can represent

\[
\begin{align*}
 \frac{\partial u}{\partial x} &= \frac{1}{a} \frac{\partial f(p)}{\partial p} + \frac{1}{a} \frac{\partial f(q)}{\partial q}
\end{align*}
\]

(13)

and express the \( F^\alpha \) force in terms of the derivatives with respect to \( t \):

\[
\begin{align*}
 F^\alpha &= \frac{EA}{a} \left[ -\frac{\partial}{\partial t} f(-\alpha t) + \frac{\partial}{\partial t} f(-\alpha t + 2L) \right]
\end{align*}
\]

After performing these substitutions and transformations, Eq. (5) will look like the following non-linear relationship:

\[
\begin{align*}
 m \ddot{v}(-\alpha t) - \ddot{f}(-\alpha t + 2L) + \\
 + \frac{EA}{a} \left[ f(-\alpha t) - \dot{f}(-\alpha t + 2L) - F^\alpha (v + \dot{u}) \right] = 0
\end{align*}
\]

(14)

Here, \( f = f(0, t) \), \( u = u(0, t) \). Types of the \( F^\alpha \) function will be discussed later.

Solutions of the constructed equation display a series of features stemming from the type of the \( F^\alpha(v + \dot{u}) \) non-linear function. In the first place, it has a stationary solution for every value of \( v \). Indeed, if \( \dot{u} = 0 \) and \( \ddot{u} = 0 \), then Eq. (2) is reduced to the form

\[
\begin{align*}
 \ddot{u}/dx^2 &= 0
\end{align*}
\]

with stationary solution \( du/dx = c_1 \), \( u = c_2 + c_3 x \).

The constants \( c_1 \) and \( c_2 \) are found from boundary conditions (4) and (5)

\[
\begin{align*}
 u(L) &= 0, \quad EA \frac{du}{dx} = -F^\alpha (v)
\end{align*}
\]

or

\[
\begin{align*}
 c_2 + c_3 L &= 0, \quad c_1 = -F^\alpha (v)/EA
\end{align*}
\]

Thus,

\[
\begin{align*}
 u(x) &= (L - x) \cdot F^\alpha (v)/EA
\end{align*}
\]

(15)

Secondly, a diapason \( v_s \leq v \leq v_i \) exists (Fig. 5), where stable periodic solutions occur in addition to stationary ones (15), which become unstable. Outside this diapason, stationary solutions (15) in the form of balanced motion \( v = const \), \( u(0) = L \cdot F^\alpha (v)/EA \) are stable. The states \( v = v_s \), \( v = v_i \), where the stationary motion is changed by autovibration and vice versa, are called the bifurcations of limit cycle birth and limit cycle loss or the Hopf (Poincare-Andronov-Hopf) bifurcations [1].

In parallel with these two traits, the third one exists for small ineritance (mass \( m \)) of the body in comparison with large ineritance (or acoustic stiffness \( \rho c^2 \alpha E \)) of the waveguide and values of \( F^\alpha (v) \). So, as indicated in Ref. [13], the problem of integration the equation with small coefficient before the senior derivative is singularly perturbed, the autovibrations are of relaxation type, and have nearly discontinuous velocities.

Beginning from the classic works by Poincare and Liapunov, the so called regular type of equations

\[
\begin{align*}
 \ddot{x} = F(t, x, \dot{x}, \epsilon) \quad (0 \leq t \leq 1)
\end{align*}
\]

(16)

was analyzed in details. Here, it is assumed that right-hand term regularly depends on the parameter \( \epsilon \) in the vicinity of \( \epsilon = 0 \) and the solutions are studied inside the segment \( 0 \leq t \leq 1 \). However the equation solutions become less regular and more diversified when the small parameter \( 0 < \epsilon << 1 \) occurs before the second derivative

\[
\begin{align*}
 \epsilon \ddot{x} = F(t, x, \dot{x}) \quad (0 \leq t \leq 1)
\end{align*}
\]

(17)

In this case, the influence of the left-hand member on the solution becomes significant only for large values of \( \dot{x} \), related to the states of fast change in the system motion. So, the distinguishing property of these type
equations is that they have periodic solutions in the shape of nearly broken straight lines or saw tooth curves.

Ultimately, one more feature of Eq. (14) is that it includes the delay argument $-\alpha(t - 2L/\alpha)$. By virtue of this, the system remembers the perturbations imposed previously on it with the $2L/\alpha$ delay and is self-adjusted to quantized vibrations with time quantum $\Delta \tau = 2L/\alpha$ \cite{14}. So, it is of interest to follow the evolution of the autovibration modes with the change in parameter $\nu$.

Yet, the principal cause of the irregular dynamics agitation is non-linear frictional interaction between the movable body and foundation surface which is responsible for the antidumping and excitation of relaxation oscillations. Thus, mechanism of this kind interaction should be considered especially.


To see the effects of the body (or autovibrational system) oscillation generation, one must examine the complete dynamical system and simulate a model describing inflow of energy into the system and its outflow. In mechanical systems, these exchanges are connected with action of the forces of internal and external friction. In electric systems, where this role is played by electric resistance, relaxation vibrations are widely met and employed in measuring devices, telecontrol, automatics and other divisions of electronics. Different generators, such as the blocking generators, multivibrators, RC-generators and others, are used for their excitations.

In Fig. 6, some relay element characteristics are shown which have widespread application in practice. Horizontal segments of these correlations point to existence of insensitivity segments. The functions in Figs. 6c and 6d are not single-valued, that is, have hysteresis character.

Dependence of the electron tube anode current $i_x$ on the grid voltage $u_g$ is represented in Fig. 7a. Usually, this curve is approximated by two-link (Fig. 7b) or three-link (Fig. 7c) functions, depending on the regime of the electron tube use.

The tunnel diode and dynistor used in contemporary radio systems have the characteristics presented in Figs. 8a and 8b, correspondingly. Here $I$ is the electric current strength, $u$ is the voltage. These and other analogical nonlinearities, possessing the sections with negative inclinations of derivatives, require external sources of energy for their realization. The similar outline has the diagram of the viscous friction force $F_{\nu}$ dependence on the velocity gradient $\nu$ in a non-Newtonian liquid (Fig. 8c).

Fig. 6 Non-linear characteristics of relay elements.

Fig. 7 Dependence of current $i_x$ on voltage $u_g$ in electron tubes.
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In the case of discrete systems, the separation of motion regimes into quasi-static and periodic ones is determined by their temporal behavior only. However, when the case in point is associated with waveguiding systems, the models of their state change are characterized not only by their temporal state transition but also by substitutions of their spatial dynamic modes for quasi-static ones. Because of this, the temporal behavior of discrete auto-vibratory systems was at the focus of scientific attention for a long time and assumed well-defined mathematic formalization, while the auto-vibratory states of waveguiding systems remain practically unexplored.

In Fig. 9, the diagrams of functions, representing typical non-linearities, meeting in many mechanical auto-vibrational systems, are represented. The characteristic of the dry (Coulombic) friction has the outline, shown in Fig. 9a, where $v(\omega)$ are the relative linear (rotary) velocities of rubbing surfaces. In many cases this correlation can be approximated by the so-called $z$-characteristic (Fig. 9b). Yet, these frictional curves are used in simplified statements of the problems. After prolonged scientific discussions, the friction model shown in Fig. 9c was assumed to be the most trustworthy. Here, the friction force $F^z$ (moment $M^z$) can vary in wide diapasons between the static ($F^{\text{stat}}$, $M^{\text{stat}}$) and extremal ($F^{\text{max}}$, $M^{\text{max}}$) values. The similar characteristics are used for the friction (cutting) simulation in lathes. They are usually used for description of the cutting moment $M^z$ applied to the drill string bit [4-7].

Investigations of self-induced vibrations were begun in the 20th years of the last century in the works by Van-Der-Pole. Relaxation vibrations of this type are comprehensively studied thanks to the possibility of application of the analogue simulation method to their analysis.

4. Multipoint Boundary Value Problem for Multilink Waveguides

The considered problem acquires additional peculiarities if the waveguide is not homogeneous and is combined from segments with dissimilar mechanical properties. Then, on the one hand, in different segments, the waves propagate with different velocities; for another, the waves endure additional fragmentations after reflection-refraction phenomenon at the two-segment link-up point. In this instance, the waves $f(x - at)$ and $g(x + at)$ transform in their propagation and Eq. (14) loses its meaning. If, for example, the waveguide consists of two segments (Fig. 10), then, to solve the problem, it is necessary to build the function $g(0 + at)$ at the left end $x = 0$ for every specific case and to use the appropriate functions $g(at)$, $\dot{g}(at)$ in Eq. (14) instead of the variables $f(-at + 2L)$, $\dot{f}(-at + 2L)$.

Let us assume, for example, that the waveguide consists of two sections, as shown in Fig. 11 with lengths $l_1$, $l_2$ and mechanical characteristics $\alpha_1, \rho_1, A_1$ and $\alpha_2, \rho_2, A_2$, accordingly. Then, the components of the $f(x - at)$ wave, propagating from the point $x = 0$ and reaching the point $x = l_1$, will endure the action of impact reflection-refraction (transmission). To calculate the intensities of the reflected and transmitted waves, consider the process of diffraction of the wave component of length $\alpha_1\Delta t$ during the time interval $\Delta t$. Separate the waveguide elements in the incident, reflected and transmitted waves,
Fig. 9 Models of non-linear friction used in simulation of mechanical systems.

Fig. 10 Schematic of a two-section waveguide.

Fig. 11 Wave diffraction at the interface point of the waveguide system.

which take part in this interaction and have velocities \( \dot{u}'_i, \dot{u}'_r, \dot{u}'_t \) and length \( \alpha_i \Delta t, \alpha_r \Delta t, \alpha_t \Delta t \), respectively (Fig. 11). Here, the top indices \( i, r, t \) denote the incident, reflected and transmitted waves, and the lower ciphers 1, 2 indicate the waveguide section number. Considering the \( \dot{u}'_i \) velocity to be known, one can calculate the \( \dot{u}'_r, \dot{u}'_t \) velocities. For this purpose, use the condition of conservation of total momentum of all the elements before and after impact.

It can be represented in the form

\[
\Delta Q'_i = \Delta Q'_r + \Delta Q'_t = \dot{u}'_i \cdot \rho, A_i \alpha_i \Delta t
\]

\[
\Delta Q'_r = \dot{u}'_r \cdot \rho, A_r \alpha_r \Delta t
\]

\[
\Delta Q'_t = \dot{u}'_t \cdot \rho, A_t \alpha_t \Delta t
\]

Supplementing Eq. (19) by condition of the \( \dot{u} \) velocity continuity

\[
\dot{u}'_i + \dot{u}'_r = \dot{u}'_t
\]

one gains the system of two algebraic equations for calculation of \( \dot{u}'_r \) and \( \dot{u}'_t \). Its solution is

\[
\dot{u}'_r = \frac{\alpha_t \rho, A_t - \alpha_i \rho, A_i \dot{u}'_i}{\alpha_i \rho, A_i + \alpha_t \rho, A_t} \dot{u}'_i
\]

\[
\dot{u}'_t = \frac{2 \alpha_t \rho, A_t}{\alpha_i \rho, A_i + \alpha_t \rho, A_t} \dot{u}'_i
\]
The elastic displacements \( w' \), \( w'' \) in the reflected and transmitted waves can be found from the conditions of continuity for elastic stresses and displacements at \( x = l_i \):

\[
\begin{align*}
  w' &= \frac{\alpha_i \rho_i A_i - \alpha_j \rho_j A_j}{\alpha_i \rho_i A_i + \alpha_j \rho_j A_j} w''_i \\
  w'' &= \frac{2 \alpha_i \rho_i A_i}{\alpha_i \rho_i A_i + \alpha_j \rho_j A_j} w''_i
\end{align*}
\]

On examination of diffraction of the \( g(x+\alpha t) \) wave at the point \( x = l_i \), the \( g'_i(x+\alpha t) \) wave in the second section, arriving to this point, is regarded to be incident and known, but the reflected \( g''_i(x-\alpha t) \) and transmitted \( g'_i(x+\alpha t) \) ( \( x \leq l_i \) ) waves should be determined. Their kinematical characteristics are calculated by the foregoing techniques through the use of the formulae below:

\[
\begin{align*}
  w''_i &= \frac{\alpha_i \rho_i A_i - \alpha_j \rho_j A_j}{\alpha_i \rho_i A_i + \alpha_j \rho_j A_j} w''_i \\
  w'_i &= \frac{2 \alpha_i \rho_i A_i}{\alpha_i \rho_i A_i + \alpha_j \rho_j A_j} w''_i
\end{align*}
\]

Eqs. (21)-(23) are derived for general cases of joining the waveguide sections with different parameters \( \alpha, \rho, A \). To confirm their plausibility, one can verify them in limiting cases with obvious boundary conditions. For example, Eqs. (21)-(23) attest that when two sections have identical geometrical and mechanical properties, then \( w'_i = 0 \), \( w''_i = w''_i', \) \( u'_i = 0 \), \( u''_i = u''_i' \) and the wave passes by the interface point without any transformation. If the second section is absolutely rigid, its acoustic stiffness \( \alpha_i \rho_i A_i \rightarrow \infty \) and the equalities \( u'_i = 0 \), \( u''_i = u''_i' \), \( u'_i = 0 \) become valid. In this event the incident wave is completely reflected. When the acoustic stiffness of the second section vanishes, then the conditions of the free end \( u'_i = -u''_i \), \( u''_i = 0 \) are realized. In this event, conditions \( 2u'_i = 2u''_i \), \( u''_i = 2u''_i \) make no sense, since the second section becomes immaterial.

In consequence of diffraction of the waves \( f'_i(x-\alpha t) \) and \( g'_i(x+\alpha t) \) (Fig. 10), the superposition of the transmitted \( f'_i(x-\alpha t) \) and reflected \( f''_i(x-\alpha t) \) waves will produce the \( f_i(x-\alpha t) \) function in the second section and the sum \( g'_i(x+\alpha t) + g'_i(x+\alpha t) \) will generate the \( g_i(x+\alpha t) \) function in the first one. Carrying out these operations, one can obtain the initial conditions for the \( f_i(x-\alpha t) \) wave at the boundary \( x = l_i \) of the segment \( l_i \leq x \leq L \) in the form:

\[
\begin{align*}
  f_i(l_i - \alpha t) &= \frac{2 \alpha_i \rho_i A_i}{\alpha_i \rho_i A_i + \alpha_j \rho_j A_j} f'_i(l_i - \alpha t) + \\
  &+ \frac{\alpha_j \rho_j A_j - \alpha_i \rho_i A_i}{\alpha_i \rho_i A_i + \alpha_j \rho_j A_j} g'_i(l_i + \alpha t) \\
  f_i(l_i - \alpha t) &= \frac{2 \alpha_i \rho_i A_i}{\alpha_i \rho_i A_i + \alpha_j \rho_j A_j} f'_i(l_i - \alpha t) + \\
  &+ \frac{\alpha_j \rho_j A_j - \alpha_i \rho_i A_i}{\alpha_i \rho_i A_i + \alpha_j \rho_j A_j} g'_i(l_i + \alpha t)
\end{align*}
\]

The initial conditions for the \( g_i(x+\alpha t) \) wave propagating in the segment \( 0 \leq x \leq l \), are constructed similarly:

\[
\begin{align*}
  g_i(l_i + \alpha t) &= \frac{2 \alpha_i \rho_i A_i}{\alpha_i \rho_i A_i + \alpha_j \rho_j A_j} f'_i(l_i - \alpha t) + \\
  &+ \frac{\alpha_j \rho_j A_j - \alpha_i \rho_i A_i}{\alpha_i \rho_i A_i + \alpha_j \rho_j A_j} g'_i(l_i + \alpha t) \\
  g_i(l_i + \alpha t) &= \frac{2 \alpha_i \rho_i A_i}{\alpha_i \rho_i A_i + \alpha_j \rho_j A_j} f'_i(l_i - \alpha t) + \\
  &+ \frac{\alpha_j \rho_j A_j - \alpha_i \rho_i A_i}{\alpha_i \rho_i A_i + \alpha_j \rho_j A_j} g'_i(l_i + \alpha t)
\end{align*}
\]

The system of Eqs. (2) and (3), boundary conditions (4), (5) and continuity conditions (24), (25) governs the three-point boundary problem relative to the independent variable \( x \) with the boundary conditions at the points \( x = 0 \), \( x = l_i \) and \( x = L \). These equations, coupled with appropriate initial conditions, make up the Cauchy problem. It can be solved by the Runge-Kutta method.

The foregoing example of vibration self-excitation in the elongated system, permitting the passage of elastic longitudinal waves, is rather a multi-purpose formal problem which can be easily rearranged for analysis of similar processes in mechanical or electric (electronic) waveguiding systems. Indeed, inasmuch as only several parameters characterize the treated waveguide system, it does not present a real challenge.
to perform corresponding replacements of their values for other mechanical or electric systems. Among them are the mass $m$, wave velocity $\alpha$, elasticity modulus $E$, acoustic stiffness $n = \alpha \rho A$ and friction function $F^x(\dot{v} + \dot{u})$. So, one might expect that autovibrational processes in different mechanical and physical waveguide systems exhibit the properties of similarity and the regularities of their manifestation, established for one of them, are typical for others.

5. Relaxation Autovibration in a Mechanical Waveguiding System

In mechanics, the waveguiding autovibrational systems are rarely met. The sole example of the singularly perturbed problem in this field, which plays a large role in practical applications, is the problem about the torsional auto-oscillations of long drill strings at their rotation with angular velocity $\omega$ (Fig. 4b). Such vibrations are generated as a consequence of nonlinear frictional interaction of their bits with the bore-hole bottom surfaces at the rock cutting. Because of the fact that the bit (vibrating body) mass is much less than the drill string (waveguide) mass, the system has small inertance, the coefficient before the inertia member (the second derivative) of the corresponding vibration equation is very small and its solution has the shape of a broken line. In connection with the discontinuous character of the relaxation vibrations, they are dangerous for the strength of the bit and drill string. Yet, the structure of the equation, describing these phenomena, is complicated, there are no universal methods for their investigation, and because of this, they are poorly understood.

$$M^x = M_{in} = \left( m \frac{a \dot{k}(\omega + \dot{\phi}) + a \dot{k}'(\omega + \dot{\phi}) + a \dot{k}''(\omega + \dot{\phi}) + a \dot{k}'''(\omega + \dot{\phi})}{1 + a \dot{k}''(\omega + \dot{\phi})} \right)^{1/2}$$

Table 1  Correspondence of mechanical parameters in longitudinal and torsional waveguides.

<table>
<thead>
<tr>
<th>Parameter of the problem</th>
<th>Wave velocity</th>
<th>Elasticity parameter</th>
<th>Cross-section parameter</th>
<th>Acoustic stiffness</th>
<th>Measure of the body inertance</th>
<th>Function of energy dissipation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Longitudinal waveguide</td>
<td>$\alpha$ (m·s$^{-1}$)</td>
<td>Elasticity module in tension, $E$ (Pa)</td>
<td>Cross-section area, $A$ (m$^2$)</td>
<td>$n = \alpha \rho A$ (kg·s$^{-1}$)</td>
<td>Mass of the body, $m$ (kg)</td>
<td>Friction force $F^x(\dot{v} + \dot{u})$ (N)</td>
</tr>
<tr>
<td>Torsional waveguide</td>
<td>$\beta$ (m·s$^{-1}$)</td>
<td>Elasticity module in shear, $G$ (Pa)</td>
<td>Moment of inertia of the cross-section area, $I$ (m$^4$)</td>
<td>$n = \beta \rho I$ (kg·m$^2$·s$^{-1}$)</td>
<td>Moment of inertia of the body, $J$ (kg·m$^2$)</td>
<td>Moment of friction force $M^x(\omega + \dot{\phi})$ (N·m)</td>
</tr>
</tbody>
</table>

To state the problem on the drill string torsion autovibration, it is necessary to perform substitutions of the parameters cited in Table 1. With their applications to correlation (4), the appropriate equation of the drill bit rotation dynamics is constructed.

Firstly, consider the homogeneous drill string 8,000 m in length. The characteristic parameters used for its analysis are selected as follows: $G = 8.077 \times 10^{10}$ Pa, $\rho = 7.8 \times 10^3$ kg/m$^3$. External and internal radii of the tube cross-section are $r_1 = 0.0841$ m and $r_2 = 0.0741$ m, then $I_z = 3.12 \times 10^{-5}$ m$^4$.

One of the main features, influencing on the process of the bit torsion vibration, is the law of the friction moment $M^x$ dependence on the total velocity $\omega + \dot{\phi}$ of its rotation. The shape of function $M^x(\omega + \dot{\phi})$ is determined by many factors, what is more, the values of their parameters vary during the drilling process.

In this reason, it is conceivable that no universal functions of this kind can be chosen for analysis of the system dynamics. The most commonly encountered relationships between $M^x$ and $\omega + \dot{\phi}$ are represented by the Coulomb friction law shown in Fig. 9c. It is used in our investigation for analysis of general regularities of autovibration proceeding. In its diagram, the vertical segment determines the static friction moment $M_{in}$, it is realized in the absence of sliding between bodies. After achieving some limit value $M_{in}$, the static friction moment $M_{\infty}$ is replaced by the dynamic friction moment $M_{ds}$, which is accompanied by sliding between rubbing surfaces. Then, the friction moment $M^x$ can be represented with the aid of the following approximate function [10]:

$$M^x = M_{in} - \left( \frac{a \dot{k}(\omega + \dot{\phi}) + a \dot{k}'(\omega + \dot{\phi}) + a \dot{k}''(\omega + \dot{\phi}) + a \dot{k}'''(\omega + \dot{\phi})}{1 + a \dot{k}''(\omega + \dot{\phi})} \right)^{1/2}$$
where the coefficients \( a_i (i=1,2,...,9) \) are found by the trial-and-error method. For the considered cases they have the following values: \( a_1 = 2,400 \text{ N·m·s}, \ a_2 = 225 \text{ s}^2, \ a_3 = 15,000 \text{ N·m·s}^3, \ a_4 = 1 \text{ N·m·s}^5, \ a_5 = 4 \text{ N·m·s}^7, \ a_6 = -130 \text{ N·m·s}^8, \ k = 0.025, \ m = 1,000, \ M_{lim} = -41,250 \text{ N·m}, \ M_{min} = -8.25 \times 10^4 \text{ N·m·s} \) (Fig. 9c).

The analysis of the bit dynamics was performed by integrating the appropriate equation by the Runge-Kutta method with the initial conditions \( \phi(0) = 0, \ \dot{\phi}(0) = 0 \) for different values of \( \omega \). The integration step was selected to be \( \Delta t = 7.769 \times 10^{-6} \text{ s} \).

The calculation results permit us to formulate some regularities. On the one hand, in the process of functioning, the drill string can be either in the states of stationary rotation or of torsional self-induced elastic oscillation, depending on the chosen regime of drilling (Fig. 5). As this takes place, the value \( \omega_b \) of the angular velocity \( \omega \) corresponding to the bifurcation state of the limit cycle birth equals the value \( \omega = 0.71 \text{ rad/s} \), which conforms to the minimum point of the \( M^b(\omega+\dot{\phi}) \) diagram (Fig. 9c). The regimes of motion with \( \omega < \omega_b \) are characterized by the stationary rotation without any oscillation when the system changes from its initial state \( \phi(0) = 0, \ \dot{\phi}(0) = 0 \) to some quasi-static equilibriums state \( \phi(t) = \phi_c, \ \dot{\phi}(t) = 0 \) and self-induced vibrations do not take place. But during the system transition from outside to inside this diapason through the value \( \omega = \omega_b \), the Hopf bifurcation occurs and limit cycles appear together with the unstable stationary solutions \( \phi(t) = \text{const}, \ \dot{\phi}(t) = 0 \).

The mode of the bit angular vibration in the result of bifurcation of the limit cycle birth (\( \omega_b = 0.71 \text{ rad/s} \)) is shown in Fig. 12. It is realized with comparatively large period \( T \approx 115 \text{ s} \) and swing \( D \approx 35 \text{ rad} \). But the more interesting feature of this process is that the auto-oscillations are of the relaxational (nearly discontinuous) type and include time segments of fast and slow motions inside every period.

The diagram of angular velocity \( \dot{\phi}(t) \) in the time diapason \( 380 \leq t \leq 520 \text{ s} \) is presented in Fig. 13. It illustrates a principally new, subtler peculiarity, which is unique only to waveguiding systems [14]. This feature consists in the fact that the self-excited oscillations proceed in the manner of quantized time and the time quantum duration \( \Delta \tau \) is equal to the time segment of the wave passing the path from the bit to the top end of the DS and backward, i.e., \( \Delta \tau = 2L/\beta \).

Similar properties can be traced in the sectional DS dynamics. In Figs. 14-16, the results for the DS of \( L = 1,000 \text{ m} \) with two sections of the lengths \( l_i = L/3, \ l_i = 2L/3 \) are presented. The tube of the first section has the geometrical and mechanical properties assumed in above. The second section tube has diameters \( d_1 = 0.089 \text{ m}, \ d_2 = 0.101 \text{ m} \).

Firstly, note that in all cases, the bifurcation values \( \omega_b \) are determined by the position of the minimum point in the diagram of the \( M^b(\omega) \) function, for this reason the \( \omega_b \) value did not change and constitutes \( \omega_b = 0.71 \text{ rad/s} \). Secondly, the auto-oscillations again have relaxation mode (Fig. 14) with fast and slow motions (Fig. 15). Thirdly, the \( \dot{\phi}(t) \) function change also proceeds in small-scale quantized manner with time quantum \( \Delta \tau = 2L/\beta \) (Fig. 16) but this time it experiences additional fractioning \( \Delta \tau_i = \Delta \tau/3 \) provoked by existence of interface between the first and second links of the DS.

![Fig. 12](image-url) The mode of the bit auto-oscillation (\( L = 8,000 \text{ m}, \ \omega_b = 0.71 \text{ rad/s} \)).
Together with the phenomena of self-excitation of stationary auto-oscillations, the problem about transient processes under conditions of the moving body speeding up or braking represents certain interest. It can be imagined that if a DS begins to rotate with small constant angular acceleration $\varepsilon$, then it will gradually pass through the bifurcation velocity $\omega_b$, enter into the diapason of auto-oscillations and afterwards again go out through the point $\omega_l$ of limit cycle loss to the domain of pure rotation without oscillations (Fig. 5). But the situation changes when acceleration $\varepsilon$ is not small. Then, owing to existence of the system inertance, all the observed effects can take place, though with some delay (Fig. 17 for
homogeneous DS 1.000 m in length and angular acceleration $\varepsilon = 0.05 \text{ rad/s}^2$). The found effect becomes more visible with further $\varepsilon$ enlargement and for the values $\varepsilon \geq 0.05 \text{ rad/s}^2$ the auto-oscillation phenomena do not occur at all.

6. Conclusions

The analysis of the limit cycle birth bifurcations in the models of homogeneous and sectional waveguiding systems is presented in this paper. The constitutive differential equations with delay argument are constructed, which are shown to be singularly perturbed. Based on analysis of an applied example associated with self-excitation of deep drill string torsion oscillation, one can draw the following conclusions:

1. The auto-oscillations of homogeneous and sectional DSs prevail at low values of the their angular velocity $\omega$, the boundaries of the $\omega$ segments of their self-excitation do not depend on the number of the DS sections and are determined by the outline of the friction moment function;
2. The autovibrations are of the relaxation type and contain fast and slow motions;
3. The self-excited oscillations proceed in the manner of quantized time. The time quantum durations equal the time of the torsional wave propagating through the doubled length of the DS;
4. The velocity-time quanta in sectional drill strings have additional fragmentations caused by the multiple diffractions of the torsional waves at the points of the section joints.

References