Analysis of Failure of Unsaturated Soils

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Abstract: In this work, an analysis of failure condition in partially saturated soils using the extended Macari Runesson Sture Lade model is presented. The elastoplastic constitutive model is based on an extension of the well-know MRS Lade model by Sture et al., whereby the suction and effective stress tensor are introduced as an additional independent and dependent stress components, respectively. The condition for discontinuous bifurcation in partially saturated porous media and the localized failure predictions of the proposed material formulation for different suctions are also analyzed and discussed. The localization analysis with this model demonstrates that an increment of the suction is related to a destabilizing effect of the discontinuous bifurcation so the localized failure can be diffuse or continuous.

Key words: Unsaturated soil, elasto-plasticity, failure behavior.

1. Introduction

In the study of failure indicator, many authors such as Hill [1], Rudnicki and Rice [2], Nicot et al. [3] among others, have developed the localization condition related to a bifurcation condition and simultaneously balance of discontinuity or first order kinematic jump field in space and time (strain rate).

Localization occurs under different conditions stress states in the geo materials, from the simple case of uniaxial compression on samples soil or rock cylindrical, to more complex behaviors as the triaxial state with passive and active strain. Therefore, it is important to predict when and how these shear band and your orientation occur.

For frictional materials, is known that existing one step in which the resistance is lower than the peak. The response softening is usually accompanied by localized deformation [4].

Much effort has focused research on mathematical development and numerical models to capture the phenomena of experimentally observed bifurcation. In

many cases, the shear band can be an infinitesimal thickness with continuous jump of displacement called as strong discontinuity.

During the last few decades, several theoretical approaches have been proposed to deal with material instability at a constitutive level. Pore fluids (air and water) can influence bifurcation modes [5]. Most of these theories tried to generalize the concept of failure, giving it a much broader meaning [6]. Different strategies were used to identify unstable conditions: Nicot et al. [3] studied the sign of the second-order work, and Prunier et al. [7] analyzed the second-order work criteria at the material point level.

Actually, the intrinsic hydro-mechanical coupling of partially saturated porous media and the presence of the suction in the constitutive equations strongly affects the localized failure indicators. As a consequence, both the solutions for discontinuous bifurcation as well as the critical directions for localization depend not only on the mechanical non-linear properties of the material formulation (i.e., yield condition, non-associativity, hardening/softening evolution law), but also on the hydraulic features of the deformation history.

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It is the goal of the present contribution to investigate the deformation and the localization behavior of unsaturated soil and to exhibit the influence of the solid-fluid coupling on the localization analysis.

The hydro-mechanical coupling inherent in partially saturated porous media and the presence of the suction in the constitutive equations strongly affects the failure indicators. Consequently, the solution for the discontinuous bifurcation for the stress states and for critical suction leads to lack of uniqueness particularly in pre peak regime. These bifurcation conditions also depend on critical directions, the nonlinear mechanical properties such as yield function, no associatively and evolution law of hardening-softening variable, and of the hydraulic conditions during the deformation.

2. Constitutive Stress

The behavior of partially saturated porous media is described by a constitutive model with two independent elastoplastic stress variables. These are the matric suction $s = (p_a - p_w)$ and the net stress tensor $\sigma_n = \sigma - I p_a$, as implemented in the model of Sheng et al. [8], where σ is the total stress tensor, p_w and p_a are the water and the air pore pressure and I is the second-order identity tensor. The constitutive stress tensor takes the form

$$\sigma' = \sigma - I p_w = \sigma_n + I s \quad (1)$$

In many geotechnical applications, the air pressure remains constant and, as a consequence, water pore pressure instead of the suction can be treated as a variable in the model formulation. Nevertheless, the constitutive formulation in this work is based on the suction so it allows a most general model application.

3. Elastoplastic Constitutive Formulation

Based on the general constitutive relations of plasticity theory for infinitesimal strains, with no associated flow law and assuming additive decomposition of the total strain tensor in elastic ϵ_e and plastic part ϵ_p , according to the Prandtl theory

$$\dot{\epsilon} = \dot{\epsilon}_e + \dot{\epsilon}_p \quad (2)$$

$$\dot{\sigma}' = E : (\dot{\epsilon} - \dot{\epsilon}_p) \quad (3)$$

$$(\sigma' - \sigma'_o) : A : \epsilon_p \geq 0, \forall \sigma'_o \in B(\epsilon, s, \kappa) \quad (4)$$

$$\dot{\kappa} = h(\epsilon_p) \quad (5)$$

where, E is the four order constitutive tensor, $\dot{\epsilon}_p$ is the plastic portion of total strain rate tensor $\dot{\epsilon}$ and the function h is a first degree homogeneous vector function. Eq. (4) represents the second postulate of Drucker that defines local stability of the behaviour of a solid point subjected to a stress-strain state and that in the nonlinear problem is related to the maximum dissipation axiom of plastic. The variational form of the non associated flow rule can be reformulated using its rate form and the Kuhn-Tucker

$$\dot{\epsilon}_p = \sum_i^U \dot{\lambda}_i m_i^\sigma \quad \dot{\lambda}_i \geq 0 \quad F_i \dot{\lambda}_i = 0 \quad (6)$$

where, $m^\sigma = \frac{\partial Q}{\partial \sigma'}$ and $n^\sigma = \frac{\partial F}{\partial \sigma'}$ represent the gradients to the plastics potential Q_i and to the yield surface F_i , respectively, with respect to the constitutive stresses.

In elastoplastic constitutive formulation, the consistency condition during plastic loading in case of the partially saturated soils takes the form

$$\dot{F} = n_i^\sigma : \dot{\sigma}' + n_i^s \dot{s} + r_i \dot{\kappa} = 0, \quad i = 1, 2, \dots, U \quad (7)$$

where,

$$n_i^s = \frac{\partial F_i}{\partial s} ; \quad r_i = \frac{\partial F_i}{\partial \kappa_i}$$

Eq. (7) described the evolutions of the gradients of the yield function respect to the suction and the hardening/softening variables, respectively. After replacing Eq. (3) in Eq. (7), the explicit expression of the plastic multiplier can be obtained as

$$\dot{\lambda}_i = \frac{n_i^\sigma : E : \dot{\epsilon} + n_i^s \dot{s}}{n_i^\sigma : E : m_i^\sigma - r_i h_i} \quad (8)$$

Substituting Eq. (8) into the rate form of the flow rule leads to the compact form of the constitutive equation

$$\dot{\sigma}' = D_{ep} : \dot{\epsilon}' \quad (9)$$

Whereby, and as in Ref. [8], the extended strain rate

field is composed by the classical strain rate tensor and the suction rate which is treated as an extra strain rate field.

$$\dot{\epsilon}' = \begin{pmatrix} \dot{\epsilon} \\ \dot{s}I \end{pmatrix} \quad (10)$$

The material operator in Eq. (9) is defined as

$$D_{ep} = (E_{ep} \times E_s) \quad (11)$$

where,

$$E_{ep} = E - \sum_{i=1}^U \left[\frac{E : m^\sigma \times n^\sigma : E}{n^\sigma : E : m^\sigma + H} \right]_i \quad (12)$$

$$E_s = - \sum_{i=1}^U \left[\frac{E : m^\sigma \times n^s I}{n^\sigma : E : m^\sigma + H} \right]_i \quad (13)$$

Finally, we obtain the evolution of the total stress tensor

$$\dot{\sigma} = \dot{\sigma}' + I \dot{p}_w \begin{cases} \text{General case} \\ \dot{\sigma} = E_{ep} : \dot{\epsilon} + (-I_{sym} + E_s) : I \dot{s} \\ \text{If } \dot{p}_a = 0 \rightarrow \\ \dot{\sigma} = E_{ep} : \dot{\epsilon} - (-I_{sym} + E_s) : I \dot{p}_w \\ \text{If } \dot{p}_a = \dot{p}_w = 0 \rightarrow \dot{\sigma} = E_{ep} : \dot{\epsilon} \end{cases} \quad (14)$$

The equation shows the influence of the suction on the value of the total stress expressed by the suction tensor E_s in addition to the classical elastoplastic tensor E_{ep} .

4. Constitutive Model

The elastoplastic constitutive model for partially saturated soils developed is an extension of cone-cap model known as MRS Lade. The MRS Lade model is a model developed at the University of Colorado by Runesson and Sture [9] and Macari et al. [10], which is a further development of the three invariant model proposed by Lade [11] for non-cohesive soils. The extended model MRS Lade is formulated in the space of three invariant of stress and in which the suction is added as a variable additional stress. The formulation of the model is oriented to partially saturated soils, including moderate plasticity clays (CL), inorganic silt (ML), silty and clayey silt (CL-ML), and mixtures with variable contents of very fine sand. Within these types of soils are collapsible loess with features that

are causing serious damage to different types of road and hydraulic structures.

To simulate the complex behavior of partially saturated porous media model, the following assumptions are adopted.

- formulation of two surfaces of failure, a smooth curved surface for the cone that intersects with another curved surface of cap in the meridian plane;
- definition of the surface of failure lc (load-collapse), according to Gens [12] and Schrefler and Bolzon [13];
- the variable-hardening and softening of both surfaces are based on the work of plastic dissipation;
- rule of non-associatively non-association on the meridian plane and the associated association in the deviatoric plane in the region of the cone and associatively rule in the cap region;
- dependency of the yield surface in cone of the third stress invariant, assuming a non-circular shape in deviatoric plane;
- the yield condition is defined in terms of the first invariant of the effective stress tensor p , of the second and third invariant of the deviator stress tensor q and θ , respectively, and of the hardening/softening variables in the cone region k_{cono} . defining the effective pressure in terms of the net mean stress p_n and the suction s , the generic shape of the cone takes the form:

$$F_{cono}(p_n, q, \theta, s, \kappa_{cono}) = f(q, \theta) - \eta(\kappa_{cono})(p_n + s - p_c) = 0 \quad (15)$$

with

$$f(q, \theta) = q \left(1 + \frac{q}{q_a} \right)^m g(\theta) \quad (16)$$

$$p_c = r_{pc} s \quad (17)$$

$$g(\theta) = \frac{4(1-e^2)\cos^2(\theta) + (2e-1)^2}{2(1-e^2)\cos(\theta) + (2e-1)[4(1-e^2)\cos^2(\theta) + 5e^2 - 4e]^{1/2}} \quad (18)$$

where, m is a material constant that controls the curvature of the cone in the meridian plane (p_n, q), q_a is the deviatoric stress of reference, η is the friction internal angle defined as function of the hardening parameters k_{cono} , p_c is the cohesion of the material and r_{pc} is a parameter material. The invariants are

$$\begin{aligned}
 p_n &= -\frac{I_{1n}}{3} \\
 q &= \sqrt{3 J_{2D}} \\
 \cos 3\theta &= \frac{3\sqrt{3}}{2} \frac{J_{3D}}{\sqrt{(J_{2D})^3}}
 \end{aligned} \quad (19)$$

where, p_n is the net volumetric pressure, I_{1n} is the first invariant of the net stress tensor, q is the deviatoric invariant, J_{2D} and J_{3D} are the second and third invariant of the net deviatoric stress, respectively and θ is the Lode's angle. Finally, $g(\theta)$ is the Willam and Warnke function [14], which assures a continuous and smooth variation of the shear strength in the deviatoric plane according with eccentricity parameter e which fulfills the condition $1/2 \leq e \leq 1$. More details are shown in Refs. [15, 16]. Fig. 1 illustrates the failure surface in stress space.

5. Discontinuous Bifurcation Condition

This section will define the condition of localized failure modes in the form of discontinuous bifurcation. Localized failure modes are detected by the formation of spatial discontinuities or jumps in the kinematics field across singularity surface. The analysis of localization leads the same format and consequent relations as the propagation condition for plane acoustic waves in solid, as shown in Refs. [1, 17]. The formation of the weak discontinuity assumes that a second-order singularity appears in the strain field rate, while the displacement rates are still continuous

$$[[\dot{\mathbf{u}}]] = \dot{\mathbf{u}}^+ - \dot{\mathbf{u}}^- = 0 \quad (20)$$

$$[[\nabla_x \dot{\mathbf{u}}]] = \nabla_x \dot{\mathbf{u}}^+ - \nabla_x \dot{\mathbf{u}}^- \neq 0 \quad (21)$$

Here the double brackets indicate the jump. Applying the Maxwell's theorem (1873), the jump condition of the velocity gradient must be a rank one tensor

$$[[\nabla_x \dot{\mathbf{u}}]] = \dot{\gamma} \vec{\mathbf{M}} \times \vec{\mathbf{N}} \quad (22)$$

$\vec{\mathbf{M}}$ is the unit vector to determine the direction of the jump, $\vec{\mathbf{N}}$ is the normal to the discontinuity

surface and $\dot{\gamma}$ is the jump magnitude.

As defined by the classical continuum, the jump in the rate of strain tensor

$$[[\dot{\boldsymbol{\varepsilon}}]] = [[\nabla^{\text{sym}} \dot{\mathbf{u}}]] \quad (23)$$

According to Cauchy's law, the equilibrium condition in contour, the jump rate traction vector remains continuous across the surface singularity inside the solid. The equilibrium condition, considering the symmetry of the tensor $\mathbf{E}_{ijkl}^T = \mathbf{E}_{jilk}^T$, takes the form

$$[[\dot{\mathbf{t}}]] = \vec{\mathbf{N}} : [[\dot{\boldsymbol{\sigma}}]] = \dot{\gamma} \mathbf{Q}_{ep} \times \vec{\mathbf{M}} = \vec{\mathbf{0}} \quad (24)$$

with

$$\mathbf{Q}_{ep} = \vec{\mathbf{N}} \times \mathbf{E}_{ep} \times \vec{\mathbf{N}} \quad (25)$$

Thus, the discontinuity bifurcations begin when the localization tensor turns singular

$$\det(\mathbf{Q}_{ep}) = 0 \quad (26)$$

This last equation represents the condition of partially saturated media location field, admitting hydraulic continuity, agree with the continuous classical formulation. In the general case, the kinematic field and the material tensor are extended for effect hydraulic flow, which in the present formulation is taken into account by suction. Considering that both sides of the surface where the discontinuity occurs when the load are plastic, and admitting the existence of a jump in the field of the suction rates, the jump of the total stress is

Fig. 1 Extended MRS Lade model's failure envelope in stress space.

$$\begin{aligned} \llbracket \dot{\boldsymbol{\sigma}} \rrbracket = & \dot{\gamma}_u \mathbf{E}_{ep} : \left[(\tilde{\mathbf{N}} \times \tilde{\mathbf{M}}) + (\tilde{\mathbf{M}} \times \tilde{\mathbf{N}}) \right] \\ & + \dot{\gamma}_s [\mathbf{E}_s - \mathbf{I}_{sym}] : \left[(\tilde{\mathbf{N}} \times \tilde{\mathbf{M}}) + (\tilde{\mathbf{M}} \times \tilde{\mathbf{N}}) \right] \end{aligned} \quad (27)$$

where, $\dot{\gamma}_u$ and $\dot{\gamma}_s$ are jump magnitude of velocity gradient and of the suction gradient, respectively.

According to Cauchy's law, the equilibrium condition, take the form

$$\begin{aligned} \llbracket \dot{\mathbf{t}} \rrbracket = & \tilde{\mathbf{N}} : \llbracket \dot{\boldsymbol{\sigma}} \rrbracket = \dot{\gamma}_u \mathbf{Q}_{ep} \cdot \tilde{\mathbf{M}} + \dot{\gamma}_s \mathbf{Q}_s \cdot \tilde{\mathbf{M}} = \vec{0} \\ \mathbf{Q}_s = & \tilde{\mathbf{N}} \cdot (\mathbf{I}_{sym} - \mathbf{E}_s) \cdot \tilde{\mathbf{N}} \end{aligned} \quad (28)$$

Thus, the discontinuity bifurcations begin when the localization tensor turns into singular

$$\begin{aligned} \det(\mathbf{Q}_{ep}) &= 0 \\ \det(\mathbf{Q}_s) &= 0 \end{aligned} \quad (29)$$

These conditions, which may occur separately or simultaneously, represent the general location condition in the present formulation for partially saturated porous media.

6. Model Predictions

In this section, the solutions of the discontinuous bifurcation of the developed model analyzed for different paths of tensions and histories of deformations are presented. In particular, it discusses the influence of the suction on the behavior. The numerical analysis in this section focuses on the trials PSP (plane strain passive), PSA (plane strain active) and ASS (compressive axial symmetric).

We consider in elastic plastic continuous media a homogeneous strain and that the constitutive response is described by the function yield F and potential function Q . $\mathbf{E}_{ijkl}^e$ is the elastic tangential tensor component having the properties of mayor and minor symmetry and positive definite. We identify modes of instability bifurcation in the shear band, bifurcation continuous or strong and discontinuous bifurcation or weak.

In the continuous bifurcation, it is assumed that the plastic flow is continuous from both sides of the band at the onset of instability, and in the discontinuous

bifurcation is considered to exist within the band plastic flow in and out of elastic unloading.

These two modes have been identified by Rudnicki and Rice [2]. Under the concept of diffuse fissure (smeared crack), localized failure modes are associated with discontinuous bifurcations of equilibrium and lead to the loss of ellipticity of the equations governing the static equilibrium according to Willam and Etse [18].

6.1 PSP Test with Constant Stress

Fig. 2 shows the localization condition, for $\sigma = cte = -172$ KPa, where $\det \mathbf{Q}_{ep}$ as obtained numerically for the maximum load on passive plane strain test (PSP). It is noted that the localization tensor remains positive for the minimum suction condition of $s = 10$ kPa, while for suction of 100 kPa and 400 kPa verifies this condition with angles $\theta = 59^\circ$ and $\theta = 56^\circ$ respectively.

This means that the critical localization direction decreases with increasing of the suction, according to the experimental tests, where angle of friction increases for increase of the suction.

6.2 PSP Test with Stress Variable

Fig. 3 shows the localization condition for peak stress with variable confinement pressure for $\sigma_1 = -172$ kPa, $-1,892$ kPa, $-2,172$ kPa, $-3,172$ kPa and $-3,672$ kPa. The condition $\det(\mathbf{Q}_{ep}) = 0$ occurs to a confinement pressure $\sigma_1 = -3,172$ kPa with a angle $\theta =$

Fig. 2 Localization in PSP at peak.

58°, but not for $\sigma_1 = -3,672$ kPa. Therefore, increasing the confinement leads to suppression of the condition and location for a high confining pressure of the failure changes to diffuse. Numerical simulation shows that the region where $\det \mathbf{Q}_s \leq 0$ increases when the suction decreases. Now the condition analysis shows that $\det \mathbf{Q}_s \leq 0$ covers the whole regions $\det \mathbf{Q}_{elp}/\det \mathbf{Q}_{el} \rightarrow 0$, therefore, the condition of the location is controlled by classical singularity tensor.

Fig. 4 shows a sector on the yield surface from the theoretical formulation $q - p$ space where occurs location and angle corresponding to the PSP test and for suction of $s = 400$ kPa.

6.3 PSA Test with Suction Variable

Fig. 5 shows the localization condition for peak stress for PSA with $\sigma = cte = -172$ kPa.

It is noted that the fault condition or diffuse localized critical direction depend of the moisture content. In the case of plane strain active state is shown a decrease in the effect of suction and a diffuse failure for saturated soil condition.

6.4 Conventional Triaxial Compression

Fig. 6 shows the localization condition for conventional triaxial compression test for variable suction and constant confinement pressure of 172 kPa. Fig. 6 has also incorporated the result of the location condition to the MRS Lade model for triaxial compression test. The results obtained by Peric [4] show that the classical MRS Lade model no present the localized failure for this condition. In the case of extended MRS Lade model, we can observe/see the influence of the suction in the failure's behavior of the material and for/in the peak stress in the case of suction $s = 400$ kPa, with a critical angle $\theta = 61^\circ$.

In the case of extended MRS Lade model, we can see the influence of the suction in the failure's behaviour of the material and in the peak stress in the case of suction $s = 400$ kPa, with a critical angle $\theta = 61^\circ$.

6.5 Influence of the Third Invariant

We are going to evaluate the influence of the third invariant, depending on the value adopted by the eccentricity parameter e , for a given value of suction on

Fig. 3 Localization for PSP test with variable confinement.

Fig. 4 Localization angle on the yield surface.

Fig. 5 Localization for PSA.

Fig. 6 Localization for conventional triaxial compression.

Fig. 7 Model prediction, influence of the third invariant.

Fig. 8 Localization analysis, influence of third invariant.

the model prediction of both the stress-strain response as the volumetric behavior. Numerical simulations of the plain strain passive tests were made for values of eccentricity $e = 0.58$ and $e = 1.00$, with a confinement pressure and a suction of 100 kPa. Stress-strain response of the model was plotted in Fig. 7 and can see an increase in the peak stress and a decrease in

ductility when the eccentricity increases.

Fig. 8 shows the condition of localization at peak on the stress state in the case of plane strain passive, for values of the eccentricity $e = 0.58$ from $e = 1.00$. It can be seen that the angle of location undergoes a variation of 56° to 54° , i.e., the angle decreases with increasing of the eccentricity.

7. Conclusion

This analysis demonstrates that the extended MRS Lade model reproduces the most relevant forms of behavior of partially saturated soils.

Furthermore, the analysis of the localization condition also indicates that the level of suction produces a destabilization in this condition and changes the shape of diffuse localized failure.

The state of stress and load configuration play an important role in the localization condition for partially saturated porous media.

The results demonstrate the significant influence of the suction on the mode and the critical direction of failure. It also appears that the first discontinuous bifurcation destabilization occurs in pre-peak or hardening regime.

For conditions in plane strain, active or passive test demonstrates the influence of the suction on the condition of localization, critical direction and strain level.

It follows that the locating condition in partially saturated soils requires consideration of the tensor which takes into account the possibility of breaks in the pore pressure field.

It was shown that the eccentricity parameter value adopted in the numerical simulation produces the variation of the angle of localization.

Furthermore, the analysis involves having localized failure, with the formation of spatial discontinuities there a jump in the rate field deformations along a surface of discontinuity. This necessarily occurs in the soil mass a jump in the rate of void ratio in both directions, which inevitably induces a jump rate of the

relationship between the water pressure and the air that is a jump in the rate suction. It requires further study of the influence of intrinsic coupling in the condition of location.

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