On Concomitants of Record Values from Generalized Farlie-Gumbel-Morgenstern Distribution

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In this paper, we derive the distributions of concomitants of record values from generalized Farlie-Gumbel-Morgenstern family of bivariate distributions. We derive the single and the product moments of the concomitants for the general case. The results are then applied to the case of the two-parameter exponential marginal distributions. Using concomitants of record values we derive the best linear unbiased estimators of parameters of the marginal distributions. Moreover, two methods for obtaining predictors of concomitants of record values are presented. Finally, a numerical illustration is performed to highlight the theoretical results obtained.

Keywords: Generalized Farlie-Gumbel-Morgenstern family, Concomitants, Record values, Best linear unbiased estimator, Best linear unbiased predictor.

Introduction

The Farlie-Gumbel-Morgenstern (FGM) family of bivariate distributions has found extensive use in practice especially in the context of reliability and lifetime tests. Balakrishnan and Lai (2009) showed several applications for the FGM in the literature. The FGM family of distributions, was primarily introduced by Morgenstern (1956) for Cauchy marginals, and the same structure was investigated by Gumbel (1960) for exponential marginals and then generalized by Farlie (1960). The FGM distribution is characterized by the marginal distribution functions $F_X(x)$ and $F_Y(y)$ of random variables X and Y , respectively, and the association parameter α . The cumulative distribution function (cdf) $F_{X,Y}(x,y)$ of the FGM distribution is given by

$$F_{X,Y}(x,y) = F_X(x)F_Y(y) \{1 + \alpha(1 - F_X(x))(1 - F_Y(y))\}. \quad (1)$$

The generalizations of FGM family of bivariate distributions received a great deal of attention of many researchers. For example, Huang and Kotz (1999) presented a polynomial-type one parameter extensions of the FGM bivariate distribution with uniform margins. Bairamov and Kotz (2002) propose an extension of FGM bivariate distributions to increase the correlation between the variables and they provide various theorems characterizing symmetry of the FGM distributions. Also Bairamov et al (2001) considered a generalization of FGM family by introducing additional parameters which allow to increase the dependence between the

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variables. Bairamov et al (2001) considered the following form, (with uniform marginals)

$$H_{\alpha}(x, y) = xy\{1 + \alpha(1 - x^{p_1})^{q_1}(1 - y^{p_2})^{q_2}\}, \quad p_1, p_2 \geq 1, q_1, q_2 > 1, 0 \leq x, y \leq 1,$$

the admissible range of α is

$$-\min\left\{1, \frac{1}{p_1 p_2} \left(\frac{1 + p_1 q_1}{p_1(q_1 - 1)}\right)^{q_1 - 1} \left(\frac{1 + p_2 q_2}{p_2(q_2 - 1)}\right)^{q_2 - 1}\right\} \leq \alpha \leq \min\left\{\frac{1}{p_1} \left(\frac{1 + p_1 q_1}{p_1(q_1 - 1)}\right)^{q_1 - 1}, \frac{1}{p_2} \left(\frac{1 + p_2 q_2}{p_2(q_2 - 1)}\right)^{q_2 - 1}\right\},$$

and the range for the correlation coefficient ρ between the variables is given by

$$\begin{aligned} -12\tau(q_1, p_1)\tau(q_2, p_2) \min\left\{1, \frac{1}{p_1 p_2} \left(\frac{1 + p_1 q_1}{p_1(q_1 - 1)}\right)^{q_1 - 1} \left(\frac{1 + p_2 q_2}{p_2(q_2 - 1)}\right)^{q_2 - 1}\right\} &\leq \rho \leq \\ 12\tau(q_1, p_1)\tau(q_2, p_2) \min\left\{\frac{1}{p_1} \left(\frac{1 + p_1 q_1}{p_1(q_1 - 1)}\right)^{q_1 - 1}, \frac{1}{p_2} \left(\frac{1 + p_2 q_2}{p_2(q_2 - 1)}\right)^{q_2 - 1}\right\}, \end{aligned}$$

where $\tau(a, b) = \frac{\Gamma(a+1)\Gamma(2/b)}{b\Gamma(a+1+2/b)}.$

In this case, the correlation coefficient between the variables can exceed 1/2 compared with the original FGM where the correlation coefficient cannot be greater than 1/3. Extension of this range of correlation coefficient is clearly purposive in practical applications. If a system of two components have correlation larger than 0.5, this distribution is purposive to model systems data.

In the present article, we study the concomitants of record values arising from a generalized form of Bairamov et al (2001), where the joint cdf and the joint probability density function (pdf) of X and Y are given respectively by

$$F(x, y) = F_X(x)F_Y(y)\{1 + \alpha(1 - F_X^{p_1}(x))^{q_1}(1 - F_Y^{p_2}(y))^{q_2}\}, \quad p_1, p_2 \geq 1, q_1, q_2 > 1, \quad (2)$$

$$f(x, y) = f_X(x)f_Y(y)\{1 + \alpha(1 - F_X^{p_1}(x))^{q_1-1}[1 - (1 + p_1 q_1)F_X^{p_1}(x)](1 - F_Y^{p_2}(y))^{q_2-1}[1 - (1 + p_2 q_2)F_Y^{p_2}(y)]\}, \quad (3)$$

To our knowledge there are no studies concerning the records and their concomitants of the general form in (2) and (3).

Let $\{(X_{(i)}, Y_{[i]}), i \geq 1\}$ be independent and identically distributed random variables from some continuous bivariate distribution. Let $\{X_{(n)}, n \geq 1\}$ be the sequence of upper record values arising from the sequence of X 's. Then the Y -variable paired with the X -value which is qualified as the n th record is called the concomitant of the n th record value and is denoted by $Y_{[n]}$. In many situations the only available observations are bivariate record values, i.e., records and their concomitants, and hence we must make inferences based on records and their concomitants. Such situations often occur in industrial stress, life time experiments, sporting matches, weather data recording and some other experimental fields. Some properties of concomitants of record values were discussed in Houchens (1984), Arnold et al. (1998) and Ahsanullah and Nevzorov (2000). For a general review of concomitants of ordered random variables see Raqab et al. (2002). However, the concomitants in case of record values have not been extensively studied as compared with the concomitants of order statistics. This branch is

relatively new in the field of ordered random variables.

If $\{X_{(n)}, n \geq 1\}$ is the sequence of upper record values then the pdf, $g_n(\cdot)$, of the n th upper record value can be obtained by using the following expression given by Ahsanullah (1995)

$$g_n(x) = \frac{1}{\Gamma n} [-\log(1 - F_X(x))]^{n-1} f_X(x), \quad (4)$$

where $f_X(x)$ and $F_X(x)$ are the pdf and the cdf of X respectively. Then the pdf of the concomitant $Y_{[n]}$ of the n th upper record value, for $n \geq 1$ is given by

$$f_{[n]}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) g_n(x) dx, \quad (5)$$

where $f_{Y|X}(y|x)$ is the conditional pdf of Y given $X=x$ of the parent bivariate distribution. Ahsanullah (1995) has given the joint distribution of m th and n th upper record values for $m < n$ as

$$g_{m,n}(x_1, x_2) = \frac{[-\log(1 - F_X(x_1))]^{m-1}}{\Gamma m} \frac{[-\log\left(\frac{1 - F_X(x_2)}{1 - F_X(x_1)}\right)]^{n-m-1}}{\Gamma(n-m)} \frac{f_X(x_1)f_X(x_2)}{1 - F_X(x_1)}, \quad x_1 < x_2. \quad (6)$$

The joint pdf of concomitants of m th and n th upper record values for $m < n$ is given by

$$f_{[m,n]}(y_1, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} f_{Y|X}(y_1|x_1) f_{Y|X}(y_2|x_2) g_{m,n}(x_1, x_2) dx_1 dx_2, \quad (7)$$

The rest of this paper is organized as follows: In Sections 2, we derive the distribution of concomitants of record values arising from the generalized FGM in (2). The single and product moments are present in section 3. In Section 4, we investigate the results obtained in Sections 2 and 3 for the two-parameter exponential marginal distributions. Best linear unbiased estimators (BLUEs) based on concomitants of record values of some parameters involved in the distribution are derived in Section 5. In Section 6, we present two different methods for obtaining predictors of future concomitants of record values. Finally, in Section 7, numerical illustrations are presented to highlight the theoretical results obtained.

Concomitants of Record Values

In this section, we derive the distribution of concomitants of record values arising from the generalized FGM given in (2).

Theorem 1 Let $(X_{(i)}, Y_{[i]}), i=1,2,\dots$ be a sequence of independent observations from (2). If $Y_{[n]}$ is the concomitant of the n th record value on the X sequence of observations, then the pdf of $Y_{[n]}$ is given by

$$f_{[n]}(y) = f_Y(y) \left\{ 1 + \alpha \sum_{i=0}^{q_1-1} \binom{q_1-1}{i} (-1)^i \left[\sum_{j=0}^{ip_1} \frac{\binom{ip_1}{j} (-1)^j}{(j+1)^n} - (1+p_1q_1) \sum_{k=0}^{ip_1+p_1} \frac{\binom{ip_1+p_1}{k} (-1)^k}{(k+1)^n} \right] (1 - F_Y^{p_2}(y))^{q_2-1} [1 - (1+p_2q_2)F_Y^{p_2}(y)] \right\}. \quad (8)$$

where p_1, p_2, q_1 and q_2 are integer numbers.

proof The conditional pdf of Y given $X=x$ is given by

$$f(y|x) = f_Y(y) \{1 + \alpha(1 - F_X^{p_1}(x))^{q_1-1} (1 - F_Y^{p_2}(y))^{q_2-1} [1 - (1+p_1q_1)F_X^{p_1}(x)][1 - (1+p_2q_2)F_Y^{p_2}(y)]\}. \quad (9)$$

Using (9) and (4) in (5) and noticing that $\int_{-\infty}^{\infty} g_n(x)dx = 1$, we get

$$f_{[n]}(y) = f_Y(y) \left\{ 1 + \alpha B \int_{-\infty}^{\infty} (1 - F_X^{p_1}(x))^{q_1-1} [1 - (1 + p_1 q_1) F_X^{p_1}(x)] \frac{f_X(x) [-\log(1 - F_X(x))]^{n-1}}{\Gamma n} dx \right\}.$$

where $B = (1 - F_Y^{p_2}(y))^{q_2-1} [1 - (1 + p_2 q_2) F_Y^{p_2}(y)]$.

Using the transformation $-\log(1 - F_X(x)) = t$, we have

$$f_{[n]}(y) = f_Y(y) \left\{ 1 + \alpha B \int_0^{\infty} (1 - (1 - e^{-t})^{p_1})^{q_1-1} [1 - (1 + p_1 q_1)(1 - e^{-t})^{p_1}] \frac{t^{n-1} e^{-t}}{\Gamma n} dt \right\}.$$

Applying the binomial theorem we get

$$f_{[n]}(y) = f_Y(y) \left\{ 1 + \alpha B \sum_{i=0}^{q_1-1} \binom{q_1-1}{i} (-1)^i \int_0^{\infty} [(1 - e^{-t})^{ip_1} - (1 + p_1 q_1)(1 - e^{-t})^{ip_1+p_1}] \frac{t^{n-1} e^{-t}}{\Gamma n} dt \right\}.$$

again applying the binomial theorem and integrating with respect to t , after simplifications the proof is complete.

Theorem 2 Let $(X_{(i)}, Y_{[i]}), i=1,2,\dots$ be a sequence of independent observations from (2), then the joint pdf of $Y_{[m]}$ and $Y_{[n]}$ for $m < n$ is given by

$$f_{[m,n]}(y_1, y_2) = f_Y(y_1) f_Y(y_2) [1 + \alpha B_1 I_1 + \alpha B_2 I_2 + \alpha^2 B_1 B_2 I_3] \quad (10)$$

where

$$B_i = (1 - F_Y^{p_2}(y_i))^{q_2-1} [1 - (1 + p_2 q_2) F_Y^{p_2}(y_i)], \quad i=1,2.$$

$$I_1 = \sum_{i=0}^{q_1-1} \binom{q_1-1}{i} (-1)^i \left[\sum_{j=0}^{ip_1} \frac{\binom{ip_1}{j} (-1)^j}{(j+1)^m} - (1 + p_1 q_1) \sum_{k=0}^{ip_1+p_1} \frac{\binom{ip_1+p_1}{k} (-1)^k}{(k+1)^m} \right], \quad (11)$$

$$I_2 = \sum_{i=0}^{q_1-1} \binom{q_1-1}{i} (-1)^i \left[\sum_{j=0}^{ip_1} \frac{\binom{ip_1}{j} (-1)^j}{(j+1)^n} - (1 + p_1 q_1) \sum_{k=0}^{ip_1+p_1} \frac{\binom{ip_1+p_1}{k} (-1)^k}{(k+1)^n} \right], \quad (12)$$

and

$$I_3 = \sum_{i=0}^{q_1-1} \binom{q_1-1}{i} (-1)^i \left\{ \sum_{j=0}^{ip_1} \frac{\binom{ip_1}{j} (-1)^j}{(j+1)^{n-m}} \left[\sum_{t=0}^{q_1-1} \binom{q_1-1}{t} (-1)^t \left[\sum_{r=0}^{tp_1} \frac{\binom{tp_1}{r} (-1)^r}{(r+j+1)^m} - (1 + p_1 q_1) \sum_{s=0}^{tp_1+p_1} \frac{\binom{tp_1+p_1}{s} (-1)^s}{(s+j+1)^m} \right] \right] \right. \\ \left. - (1 + p_1 q_1) \sum_{k=0}^{ip_1+p_1} \frac{\binom{ip_1+p_1}{k} (-1)^k}{(k+1)^{n-m}} \left[\sum_{t=0}^{q_1-1} \binom{q_1-1}{t} (-1)^t \left[\sum_{r=0}^{tp_1} \frac{\binom{tp_1}{r} (-1)^r}{(r+j+1)^m} - (1 + p_1 q_1) \sum_{s=0}^{tp_1+p_1} \frac{\binom{tp_1+p_1}{s} (-1)^s}{(s+j+1)^m} \right] \right] \right\} \quad (13)$$

proof By using (6) and (9) in (7), and noticing that $\int_{-\infty}^{\infty} \int_{-\infty}^{x_2} g_{m,n}(x_1, x_2) dx_1 dx_2 = 1$, we get

$$f_{[m,n]}(y_1, y_2) = f_Y(y_1)f_Y(y_2)[1 + \alpha B_1 J_1 + \alpha B_2 J_2 + \alpha^2 B_1 B_2 J_3], \quad (14)$$

where

$$J_i = \int_{-\infty}^{\infty} \int_{x_1}^{\infty} (1 - F_X^{p_1}(x_i))^{q_1-1} [1 - (1 + p_1 q_1) F_X^{p_1}(x_i)] g_{m,n}(x_1, x_2) dx_2 dx_1, \quad i = 1, 2,$$

$$J_3 = \int_{-\infty}^{\infty} \int_{x_1}^{\infty} (1 - F_X^{p_1}(x_1))^{q_1-1} [1 - (1 + p_1 q_1) F_X^{p_1}(x_1)] (1 - F_X^{p_1}(x_2))^{q_1-1} [1 - (1 + p_1 q_1) F_X^{p_1}(x_2)] g_{m,n}(x_1, x_2) dx_2 dx_1.$$

Using (6) and applying the transformations $-\log(1 - F_X(x_1)) = u$, $-\log(1 - F_X(x_2)) = v$, we get

$$J_1 = \frac{1}{\Gamma m \Gamma(n-m)} \int_0^{\infty} \int_u^{\infty} (1 - (1 - e^{-u})^{p_1})^{q_1-1} [1 - (1 + p_1 q_1)(1 - e^{-u})^{p_1}] u^{m-1} (v-u)^{n-m-1} e^{-v} dv du.$$

Using the transformation $v-u=s$, we get

$$J_1 = \frac{1}{\Gamma m \Gamma(n-m)} \int_0^{\infty} \int_0^{\infty} (1 - (1 - e^{-u})^{p_1})^{q_1-1} [1 - (1 + p_1 q_1)(1 - e^{-u})^{p_1}] u^{m-1} s^{n-m-1} e^{-s} e^{-u} ds du.$$

Integrating with respect to s

$$J_1 = \frac{1}{\Gamma m} \int_0^{\infty} (1 - (1 - e^{-u})^{p_1})^{q_1-1} [1 - (1 + p_1 q_1)(1 - e^{-u})^{p_1}] u^{m-1} e^{-u} du,$$

applying the binomial theorem

$$J_1 = \sum_{i=0}^{q_1-1} \binom{q_1-1}{i} (-1)^i \int_0^{\infty} \left[\sum_{j=0}^{ip_1} \frac{\binom{ip_1}{j} (-1)^j e^{-ju}}{\Gamma m} - (1 + p_1 q_1) \sum_{k=0}^{ip_1+p_1} \frac{\binom{ip_1+p_1}{k} (-1)^k e^{-ku}}{\Gamma m} \right] u^{m-1} e^{-u} du.$$

Integrating with respect to u , and after simplifications we obtain $J_1 = I_1$. Proceeding in a similar manner we get $J_2 = I_2$ and $J_3 = I_3$. Upon substituting the values of J_1, J_2 and J_3 the proof is complete.

The Moments of Concomitants of Record Values

From (8) the k th moment of the concomitant of the n th upper record value is given by

$$E[(Y_{[n]})^k] = \mu^{(k)} + \alpha A_n \int_{-\infty}^{\infty} y^k f_Y(y) (1 - F_Y^{p_2}(y))^{q_2-1} [1 - (1 + p_2 q_2) F_Y^{p_2}(y)] dy, \quad (15)$$

where $\mu^{(k)} = E[Y^k]$, and

$$A_b = \sum_{i=0}^{q_1-1} \binom{q_1-1}{i} (-1)^i \left[\sum_{j=0}^{ip_1} \frac{\binom{ip_1}{j} (-1)^j}{(j+1)^b} - (1 + p_1 q_1) \sum_{k=0}^{ip_1+p_1} \frac{\binom{ip_1+p_1}{k} (-1)^k}{(k+1)^b} \right]. \quad (16)$$

From (10), the product moment of concomitants of the m th and n th upper record values for $m < n$ is given by

$$\begin{aligned} E[Y_{[m]} Y_{[n]}] &= \mu^2 + \alpha \mu \{I_1 + I_2\} \int_{-\infty}^{\infty} y f_Y(y) (1 - F_Y^{p_2}(y))^{q_2-1} [1 - (1 + p_2 q_2) F_Y^{p_2}(y)] dy \\ &+ \alpha^2 I_3 \left\{ \int_{-\infty}^{\infty} y f_Y(y) (1 - F_Y^{p_2}(y))^{q_2-1} [1 - (1 + p_2 q_2) F_Y^{p_2}(y)] dy \right\}^2, \end{aligned} \quad (17)$$

where $\mu = E[Y]$.

Exponential Marginals

In the present and the subsequent sections, we shall investigate concomitants of record values for the bivariate random variable (X, Y) , having generalized FGM distribution given by (2) with two-parameter exponential marginals, as follows,

$$f_X(x) = \frac{1}{\lambda_1} \exp\left(\frac{-(x - \mu_1)}{\lambda_1}\right), \quad x \geq \mu_1 \quad \text{and} \quad F_X(x) = 1 - \exp\left(\frac{-(x - \mu_1)}{\lambda_1}\right), \quad (18)$$

$$f_Y(y) = \frac{1}{\lambda_2} \exp\left(\frac{-(y - \mu_2)}{\lambda_2}\right), \quad y \geq \mu_2 \quad \text{and} \quad F_Y(y) = 1 - \exp\left(\frac{-(y - \mu_2)}{\lambda_2}\right). \quad (19)$$

The correlation coefficient between the two variables X, Y is given by

$$\begin{aligned} \rho &= \alpha \sum_{i=0}^{q_1-1} \binom{q_1-1}{i} (-1)^i \left[\sum_{j=0}^{ip_1} \frac{\binom{ip_1}{j} (-1)^j}{(j+1)^2} - (1 + p_1 q_1) \sum_{k=0}^{ip_1+p_1} \frac{\binom{ip_1+p_1}{k} (-1)^k}{(k+1)^2} \right] \\ &\times \sum_{r=0}^{q_2-1} \binom{q_2-1}{r} (-1)^r \left[\sum_{s=0}^{rp_2} \frac{\binom{rp_2}{s} (-1)^s}{(s+1)^2} - (1 + p_2 q_2) \sum_{t=0}^{rp_2+p_2} \frac{\binom{rp_2+p_2}{t} (-1)^t}{(t+1)^2} \right] \\ &= \alpha \gamma \quad (\text{say}), \end{aligned} \quad (20)$$

where

$$\begin{aligned} \gamma &= \sum_{i=0}^{q_1-1} \binom{q_1-1}{i} (-1)^i \left[\sum_{j=0}^{ip_1} \frac{\binom{ip_1}{j} (-1)^j}{(j+1)^2} - (1 + p_1 q_1) \sum_{k=0}^{ip_1+p_1} \frac{\binom{ip_1+p_1}{k} (-1)^k}{(k+1)^2} \right] \\ &\times \sum_{r=0}^{q_2-1} \binom{q_2-1}{r} (-1)^r \left[\sum_{s=0}^{rp_2} \frac{\binom{rp_2}{s} (-1)^s}{(s+1)^2} - (1 + p_2 q_2) \sum_{t=0}^{rp_2+p_2} \frac{\binom{rp_2+p_2}{t} (-1)^t}{(t+1)^2} \right]. \end{aligned}$$

Table 1 shows the admissible values of the dependence parameter α and the correlation coefficient ρ for the exponential marginal distributions with respect to different values of $p_1, p_2 \geq 1, q_1, q_2 > 1$. We tabled a list of the cases of strongest positive correlation coefficients for $q_1, q_2 > 1$.

Table 1

The admissible values of α and ρ

q_1	q_2	p_1	p_2	α		ρ	
				Lower bound	Upper bound	Lower bound	Upper bound
2	2	10	10	-0.04410	0.21000	-0.08575	0.40831
2	2	13	13	-0.0255	0.15976	-0.08575	0.41498
2	2	14	14	-0.02189	0.14796	-0.06141	0.41502
2	2	15	15	-0.01898	0.13778	-0.05709	0.41437
2	3	10	11	-0.04560	0.21000	-0.07788	0.35870
2	3	11	13	-0.04127	0.19008	-0.07442	0.34278
2	3	16	18	-0.01671	0.12891	-0.04884	0.37665

Table 1 continued

q_1	q_2	p_1	p_2	α		ρ	
				Lower bound	Upper bound	Lower bound	Upper bound
3	2	16	18	-0.01673	0.11420	-0.04854	0.33122
3	2	16	14	-0.02168	0.14655	-0.05561	0.37588
3	2	10	9	-0.02168	0.23457	-0.08497	0.35368
3	3	19	19	-0.01503	0.12261	-0.08497	0.34559
4	4	23	23	-0.01133	0.10646	-0.03245	0.30485
5	5	26	26	-0.00937	0.09682	-0.02691	0.27802
6	6	29	29	-0.00780	0.08830	-0.02285	0.25882
7	7	32	32	-0.00655	0.08093	-0.01977	0.24423
10	10	37	37	-0.00511	0.07148	-0.01540	0.21539

Figures 1-6 show the admissible values of the correlation coefficient ρ for the exponential marginal distributions with respect to different values of $p_1, p_2 \geq 1$, for some different values of $q_1, q_2 > 1$.

From Figures 1-6, we see that the largest upper bound of the positive correlation coefficient is greater than $\rho=0.25$ (classic FGM) for $2 \leq q_1, q_2 \leq 6$. For the cases of $q_1 = q_2$ the upper bound of the positive correlation coefficient attains its maximum when $p_1 = p_2$, also for the cases of $q_1 < (>) q_2$ the upper bound of the positive correlation coefficient attains its maximum when $p_1 < (>) p_2$. However, the upper bound of the positive correlation coefficient decreases as q_1, q_2, p_1, p_2 increase while the admissible ranges of ρ shrinks as q_1, q_2, p_1, p_2 tends to infinity. Also, the strongest positive correlation coefficient $\rho=0.41503$ is attained for the case of $q_1 = q_2 = 2$ when $p_1 = p_2 = 14$, while the negative lower bound of correlation coefficient for these values is -0.06141 .

Figure 1. Bounds of correlations coefficient ρ for $q_1=q_2=2$, as a function of p_2 for given p_1 .

Figure 2. Bounds of correlations coefficient ρ for $q_1=2, q_2=3$, as a function of p_2 for given p_1 .

Figure 3. Bounds of correlations coefficient ρ for $q_1=q_2=3$, as a function of p_2 for given p_1 .

Figure 4. Bounds of correlations coefficient ρ for $q_1=q_2=4$, as a function of p_2 for given p_1 .

Figure 5. Bounds of correlations coefficient ρ for $q_1=q_2=6$, as a function of p_2 for given p_1 .

Figure 6. Bounds of correlations coefficient ρ for $q_1=q_2=10$, as a function of p_2 for given p_1 .

Estimation of the Location and scale Parameters of the Exponential Margins

In this section we discuss the estimation of the location and the scale parameters μ_1, λ_1, μ_2 and λ_2 when the association parameter α is either known or unknown.

Ahsanullah (1980) derived the BLUEs of μ_1 and λ_1 based on the first n record values drawn from the marginal distribution of X as

$$\hat{\mu}_1 = \frac{1}{n-1}(nX_{(1)} - X_{(n)}), \quad (21)$$

$$\hat{\lambda}_1 = \frac{1}{n-1}(X_{(n)} - X_{(1)}). \quad (22)$$

Now we want to estimate μ_2 and λ_2 using the concomitants of record values.

Estimation of μ_2 and λ_2 When α is Known

Let $U = (X - \mu_1)/\lambda_1$ and $V = (Y - \mu_2)/\lambda_2$ be the standard exponential random variables. Clearly upon substitution with $F_U(u) = (1 - e^{-u})$ and $F_V(v) = (1 - e^{-v})$ into (8) and (10), we obtain the pdf of the concomitant of the n th upper record value and the joint pdf of the concomitants of the m th and n th upper record values with standard exponential margins, respectively.

We have,

$$\begin{aligned}
& \int_{-\infty}^{\infty} v^k f_V(v) (1 - F_V^{p_2}(v))^{q_2-1} [1 - (1 + p_2 q_2) F_V^{p_2}(v)] dv \\
&= k! \sum_{i=0}^{q_2-1} \binom{q_2-1}{i} (-1)^i \left[\sum_{j=0}^{ip_2} \frac{\binom{ip_2}{j} (-1)^j}{(j+1)^{k+1}} - (1 + p_2 q_2) \sum_{r=0}^{ip_2+p_2} \frac{\binom{ip_2+p_2}{r} (-1)^r}{(r+1)^{k+1}} \right], \quad k \geq 1,
\end{aligned} \quad (23)$$

and

$$\mu^{(k)} = E[V^k] = k!. \quad (24)$$

Substituting (23) and (24) into (15), we obtain the k th moment of the concomitant of the n th upper record value

$$E[(V_{[n]})^k] = k! + \alpha A_n k! \sum_{i=0}^{q_2-1} \binom{q_2-1}{i} (-1)^i \left[\sum_{j=0}^{ip_2} \frac{\binom{ip_2}{j} (-1)^j}{(j+1)^{k+1}} - (1 + p_2 q_2) \sum_{r=0}^{ip_2+p_2} \frac{\binom{ip_2+p_2}{r} (-1)^r}{(r+1)^{k+1}} \right].$$

Hence, for $k=1$, we obtain

$$E[V_{[n]}] = 1 + \alpha A_n \sum_{i=0}^{q_2-1} \binom{q_2-1}{i} (-1)^i \left[\sum_{j=0}^{ip_2} \frac{\binom{ip_2}{j} (-1)^j}{(j+1)^2} - (1 + p_2 q_2) \sum_{r=0}^{ip_2+p_2} \frac{\binom{ip_2+p_2}{r} (-1)^r}{(r+1)^2} \right] = \varepsilon_n \text{ (say)} \quad (25)$$

Thus the variance of $V_{[n]}$ is given by

$$\begin{aligned}
\text{var}(V_{[n]}) &= \left\{ 1 + 2\alpha A_n \sum_{i=0}^{q_2-1} \binom{q_2-1}{i} (-1)^i \left[\sum_{j=0}^{ip_2} \frac{\binom{ip_2}{j} (-1)^j}{(j+1)^3} - (1 + p_2 q_2) \sum_{r=0}^{ip_2+p_2} \frac{\binom{ip_2+p_2}{r} (-1)^r}{(r+1)^3} \right] \right. \\
&\quad \left. - 2\alpha A_n \sum_{i=0}^{q_2-1} \binom{q_2-1}{i} (-1)^i \left[\sum_{j=0}^{ip_2} \frac{\binom{ip_2}{j} (-1)^j}{(j+1)^2} - (1 + p_2 q_2) \sum_{r=0}^{ip_2+p_2} \frac{\binom{ip_2+p_2}{r} (-1)^r}{(r+1)^2} \right] \right. \\
&\quad \left. - \left(\alpha A_n \sum_{i=0}^{q_2-1} \binom{q_2-1}{i} (-1)^i \left[\sum_{j=0}^{ip_2} \frac{\binom{ip_2}{j} (-1)^j}{(j+1)^2} - (1 + p_2 q_2) \sum_{r=0}^{ip_2+p_2} \frac{\binom{ip_2+p_2}{r} (-1)^r}{(r+1)^2} \right] \right)^2 \right\} = \rho_{n,n} \text{ (say)}.
\end{aligned} \quad (26)$$

From (17), the product moment of concomitants of the m th and n th upper record values for $m < n$ is given by

$$\begin{aligned}
E[V_{[m]} V_{[n]}] &= 1 + \alpha \{I_1 + I_2\} \sum_{i=0}^{q_2-1} \binom{q_2-1}{i} (-1)^i \left[\sum_{j=0}^{ip_2} \frac{\binom{ip_2}{j} (-1)^j}{(j+1)^2} - (1 + p_2 q_2) \sum_{r=0}^{ip_2+p_2} \frac{\binom{ip_2+p_2}{r} (-1)^r}{(r+1)^2} \right] \\
&\quad + \alpha^2 I_3 \left(\sum_{i=0}^{q_2-1} \binom{q_2-1}{i} (-1)^i \left[\sum_{j=0}^{ip_2} \frac{\binom{ip_2}{j} (-1)^j}{(j+1)^2} - (1 + p_2 q_2) \sum_{r=0}^{ip_2+p_2} \frac{\binom{ip_2+p_2}{r} (-1)^r}{(r+1)^2} \right] \right)^2
\end{aligned} \quad (27)$$

Hence, the covariance of $V_{[m]}$ and $V_{[n]}$, $m < n$ is given by

$$\begin{aligned}
\text{cov}(V_{[m]}V_{[n]}) &= 1 + \alpha \{I_1 + I_2\} \sum_{i=0}^{q_2-1} \binom{q_2-1}{i} (-1)^i \left[\sum_{j=0}^{ip_2} \frac{\binom{ip_2}{j} (-1)^j}{(j+1)^2} - (1 + p_2 q_2) \sum_{r=0}^{ip_2+p_2} \frac{\binom{ip_2+p_2}{r} (-1)^r}{(r+1)^2} \right] \\
&\quad + \alpha^2 I_3 \left\{ \sum_{i=0}^{q_2-1} \binom{q_2-1}{i} (-1)^i \left[\sum_{j=0}^{ip_2} \frac{\binom{ip_2}{j} (-1)^j}{(j+1)^2} - (1 + p_2 q_2) \sum_{r=0}^{ip_2+p_2} \frac{\binom{ip_2+p_2}{r} (-1)^r}{(r+1)^2} \right] \right\}^2 \\
&\quad - \left\{ 1 + A_m \sum_{i=0}^{q_2-1} \binom{q_2-1}{i} (-1)^i \left[\sum_{j=0}^{ip_2} \frac{\binom{ip_2}{j} (-1)^j}{(j+1)^2} - (1 + p_2 q_2) \sum_{r=0}^{ip_2+p_2} \frac{\binom{ip_2+p_2}{r} (-1)^r}{(r+1)^2} \right] \right\} \\
&\quad \times \left\{ 1 + A_n \sum_{i=0}^{q_2-1} \binom{q_2-1}{i} (-1)^i \left[\sum_{j=0}^{ip_2} \frac{\binom{ip_2}{j} (-1)^j}{(j+1)^2} - (1 + p_2 q_2) \sum_{r=0}^{ip_2+p_2} \frac{\binom{ip_2+p_2}{r} (-1)^r}{(r+1)^2} \right] \right\} \\
&= \rho_{m,n} \text{ (say),}
\end{aligned} \tag{28}$$

where A_m and A_n are defined by (16), and I_1, I_2 and I_3 are defined in Equations (11)- (13) respectively.

Let $\mathbf{Y}_{[n]}$ denote the vector of concomitants of the first n record values, that is $\mathbf{Y}_{[n]} = (Y_{[1]}, Y_{[2]}, \dots, Y_{[n]})'$, where $Y_{[i]} = \lambda_2 V_{[i]} + \mu_2$, $i = 1 \dots n$. From (25), we can write

$$E[\mathbf{Y}_{[n]}] = \lambda_2 \mathbf{\epsilon}_n + \mu_2 \mathbf{1}, \tag{29}$$

where $\mathbf{\epsilon}_n = (\epsilon_1, \dots, \epsilon_n)'$ denotes the column vector of expected values of the concomitant of upper record values from the standard exponential distribution and $\mathbf{1}$ is $n \times 1$ vector whose components are all 1's.

The variance covariance matrix of $\mathbf{Y}_{[n]}$ is given by

$$D[\mathbf{Y}_{[n]}] = \lambda_2^2 \mathbf{\Sigma} \tag{30}$$

where $\mathbf{\Sigma} = (\rho_{ij})$, and ρ_{ij} are determined by (26) and (28), $i, j = 1, \dots, n$.

Clearly $\mathbf{\epsilon}_n, \rho_{n,n}$, and $\rho_{m,n}$ are known constants provided that α, m and n are known.

Proceeding as in David and Nagaraja (2003), the BLUEs $\hat{\mu}_2$ of μ_2 and $\hat{\lambda}_2$ of λ_2 are given by

$$\hat{\mu}_2 = -\mathbf{\epsilon}_n' \Gamma \mathbf{Y}_{[n]} = \sum_{i=1}^n a_i Y_{[i]}, \tag{31}$$

$$\hat{\lambda}_2 = \mathbf{1}' \Gamma \mathbf{Y}_{[n]} = \sum_{i=1}^n b_i Y_{[i]}, \tag{32}$$

where $\Gamma = \mathbf{\Sigma}^{-1}(\mathbf{1}\mathbf{\epsilon}_n' - \mathbf{\epsilon}_n\mathbf{1}')/\Delta$, $\Delta = (\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1})(\mathbf{\epsilon}_n'\mathbf{\Sigma}^{-1}\mathbf{\epsilon}_n) - (\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{\epsilon}_n)^2$, and $a_i, b_i, i = 1, 2, \dots, n$ are constants.

The variances and covariance of $\hat{\mu}_2$ and $\hat{\lambda}_2$ are given respectively by

$$\text{var}(\hat{\mu}_2) = \lambda_2^2 \mathbf{\epsilon}_n' \mathbf{\Sigma}^{-1} \mathbf{\epsilon}_n / \Delta, \text{var}(\hat{\lambda}_2) = \lambda_2^2 \mathbf{1}' \mathbf{\Sigma}^{-1} \mathbf{1} / \Delta, \tag{33}$$

and

$$\text{cov}(\hat{\mu}_2, \hat{\lambda}_2) = -\lambda_2^2 \mathbf{e}_n' \Sigma^{-1} \mathbf{1} / \Delta.$$

Estimation of μ_2 and λ_2 When α is Unknown

Following Chacko and Thomas (2006), if α is unknown, we may replace α in (31) and (32) by a rough moment type estimator. If r is the sample correlation coefficient between $X_{(i)}$ and $Y_{[i]}$, $i = 1, 2, \dots$, then the rough moment type estimator $\tilde{\alpha}$ for α is obtained by equating r with the correlation coefficient given by (20). Thus

$$\tilde{\alpha} = \begin{cases} -\min\left\{1, \frac{1}{p_1 p_2} \left(\frac{1+p_1 q_1}{p_1(q_1-1)}\right)^{q_1-1} \left(\frac{1+p_2 q_2}{p_2(q_2-1)}\right)^{q_2-1}\right\}, & \text{if } r \leq \gamma \min\left\{1, \frac{1}{p_1 p_2} \left(\frac{1+p_1 q_1}{p_1(q_1-1)}\right)^{q_1-1} \left(\frac{1+p_2 q_2}{p_2(q_2-1)}\right)^{q_2-1}\right\} \\ \min\left\{\frac{1}{p_1} \left(\frac{1+p_1 q_1}{p_1(q_1-1)}\right)^{q_1-1}, \frac{1}{p_2} \left(\frac{1+p_2 q_2}{p_2(q_2-1)}\right)^{q_2-1}\right\}, & \text{if } r \geq \gamma \min\left\{\frac{1}{p_1} \left(\frac{1+p_1 q_1}{p_1(q_1-1)}\right)^{q_1-1}, \frac{1}{p_2} \left(\frac{1+p_2 q_2}{p_2(q_2-1)}\right)^{q_2-1}\right\} \\ r\gamma^{-1}, & \text{otherwise.} \end{cases} \quad (34)$$

Predictors of Concomitants of Record Values

One would wish to use past data for predicting a future observation. In this section we discuss the prediction of future concomitants of record values. Let $(X_{(i)}, Y_{[i]}), i = 1, 2, \dots, n$ represent the first observed n upper record values and their concomitants. We present two different methods for obtaining the m th predicted concomitant, $m > n$. For the first method, we obtain the best linear unbiased predictor (BLUP) $Y_{[m]}^*$ of $Y_{[m]}, m > n$, while the second method we use the conditional distribution of $Y_{[m]}$ given $X_{[m]}$ for obtaining the predictor which we call the conditional predictor $Y_{[m]}^{\tilde{*}}$.

The BLUP of $Y_{[m]}$

Using the generalized linear regression model, see Goldberger (1962), the BLUP $Y_{[m]}^*$ of $Y_{[m]}, m > n$ is

$$Y_{[m]}^* = \hat{\mu}_2 + \hat{\lambda}_2 \varepsilon_m + w' \Sigma^{-1} (\mathbf{Y}_{[n]} - \hat{\mu}_2 \mathbf{1} - \hat{\lambda}_2 \mathbf{e}_n), \quad (35)$$

where ε_m is the expected value of $V_{[m]}$, $\hat{\mu}_2$ and $\hat{\lambda}_2$ are the BLUE of μ_2 and λ_2 , respectively, w' is the vector of covariances of the prediction observation with the vector of observed concomitants of record values from the standard exponential distribution. i.e. $(\rho_{1,m}, \dots, \rho_{n,m})$, Σ is the standard variance-covariance matrix, $\mathbf{Y}_{[n]}$ is the vector of observed concomitants of record values, and $\mathbf{e}_n$ is the vector of expected values of the concomitant of record values from the standard exponential distribution.

The Conditional Predictor of $Y_{[m]}$

Another method for obtaining a predicted value $Y_{[m]}^{\tilde{*}}$ of the m th concomitant $Y_{[m]}, m > n$, can be applied by using the predicted m th record value and the conditional cdf of Y given $X=x$. Ahsanullah (1980) derived the BLUP of m th record value, $X_{(m)}^*, m > n$, based on the first n record values drawn from the marginal distribution of X as

$$X_{(m)}^* = \frac{1}{n-1} \{(m-1)X_{(n)} - (m-n)X_{(1)}\}. \quad (36)$$

The conditional cdf of $Y_{[i]}$ given $X_{(i)} = x, i = 1, 2, \dots$, is given by

$$F_{Y_{[i]}|X_{(i)}}(y|x) = F(y|x) = F_Y(y)\{1 + \alpha(1 - F_Y^{p_2}(y))^{q_2}(1 - F_X^{p_1}(x))^{q_1-1}[1 - (1 + p_1q_1)F_X^{p_1}(x)]\}. \quad (37)$$

Let $(U_{(i)}, V_{[i]}), i=1,2,\dots,n$ represent the first observed n upper record values from the standard exponential distribution and their concomitants. The cdf of $V_{[i]}$ given $U_{(i)} = u$ is given by

$$F_{V_{[i]}|U_{(i)}}(v|u) = (1 - e^{-v})\{1 + \alpha(1 - (1 - e^{-v})^{p_2})^{q_2}(1 - (1 - e^{-u})^{p_1})^{q_1-1}[1 - (1 + p_1q_1)(1 - e^{-u})^{p_1}]\}. \quad (38)$$

Suppose that the m th predicted record value and its concomitant is $(X_{(m)}^*, Y_{[m]}^*)$ where $m > n$. Setting $F(v_{[m]}^* | u_{(m)}^*) = R$, where R is a random number, we can solve (38) in $v_{[m]}^*$ given $u_{(m)}^*$, where

$$u_{(m)}^* = (x_{(m)}^* - \hat{\mu}_1) / \hat{\lambda}_1, \quad (39)$$

and

$$v_{[m]}^* = (y_{[m]}^* - \hat{\mu}_2) / \hat{\lambda}_2, \quad (40)$$

Substituting with (21), (22) and (36) in (39) we find that

$$u_{(m)}^* = m. \quad (41)$$

Notice that the value of $v_{[m]}^*$ depends on the value of the random number R , and since $0 < R < 1$, so we can replace R by its mean (0.5). Thus substituting with $u_{(m)}^* = m$, and $R = 0.5$ in (38), we get

$$F(v_{[m]}^* | m) = (1 - e^{-v_{[m]}^*})\{1 + \alpha(1 - (1 - e^{-v_{[m]}^*})^{p_2})^{q_2}(1 - (1 - e^{-m})^{p_1})^{q_1-1}[1 - (1 + p_1q_1)(1 - e^{-m})^{p_1}]\} = 0.5, \quad (42)$$

thus solving (42) in $v_{[m]}^*$ and substituting with its value in (40), we obtain the predicted value

$$\tilde{y}_{[m]}^* = \hat{\lambda}_2 v_{[m]}^* + \hat{\mu}_2. \quad (43)$$

Remarks:

For improving $\tilde{y}_{[m]}^*$ and reducing the sensitivity of $v_{[m]}^*$ to R , we apply the following algorithm, using a variance reduction technique (see, Wilson (1984)),

Algorithm 1:

(1) Generate a sequence of s paired random numbers $(R_1, 1 - R_1) \dots (R_s, 1 - R_s)$.

(2) Solve (42) for $R = R_i$ to obtain $v_{[m]}^*$ and for $R = 1 - R_i$ to obtain $v_{[m]}^{'}$.

(3) Compute $v_{[m]} = \frac{v_{[m]}^* + v_{[m]}^{'}}{2}$.

(4) $v_{[m]}^* = \frac{1}{s} \sum_{i=1}^s v_{[m]}.$

Numerical illustration

We calculated the coefficients a_i and b_i in the BLUEs $\hat{\mu}_2$ and $\hat{\lambda}_2$ of μ_2 and λ_2 , respectively, given by (31) and (32) for $i=1(1)10$ and taking arbitrary values for α in the admissible range $(-0.02, 0.1, 0.12)$, for $p_1=p_2=14$ and $q_1=q_2=2$ (the strongest positive correlation coefficient case). The results are presented in Tables 2-4.

From Tables 2&3, we can see that $\text{var}(\hat{\mu}_2)$ and $\text{var}(\hat{\lambda}_2)$ decreases as the value of the association parameter α and the number of concomitants increase.

Table 2

The coefficients $a_i, i = 1, \dots, n$ in the BLUE $\hat{\mu}_2 = \sum_{i=1}^n a_i y_{[i]}$.

n	α	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	$\text{var}(\hat{\mu}_2) / \lambda_2^2$
2	-0.02	1.6748	1.5808									571.0
	0.1	1.6748	2.1446									35.36
	0.12	1.6748	2.2385									25.39
3	-0.02	-9.4516	.6765	9.7750								170.0
	0.1	3.1368	-.2434	-1.8930								15.08
	0.12	2.7791	-.1993	-1.5798								11.39
4	-0.02	-7.8751	-1.4271	4.3223	5.9799							109.2
	0.1	2.8114	.2198	-.9076	-1.1236							11.44
	0.12	2.5186	.1655	-.7611	-.9230							8.924
5	-0.02	-7.856	-1.9782	3.2481	4.7375	2.8485						99.90
	0.1	2.8097	.3193	-.7290	-.8839	-.5161						10.90
	0.12	2.5214	.2347	-.6295	-.7298	-.3967						8.597
6	-0.02	-7.8210	-1.9442	3.2817	4.7717	2.8837	-.1718					99.86
	0.1	2.7876	.3046	-.7439	-.9040	-.5432	.0989					10.89
	0.12	2.4928	.2187	-.6461	-.7550	-.4341	.1236					8.570
7	-0.02	-6.9578	-1.4374	3.4756	4.8820	3.1139	.2458	-2.3220				93.65
	0.1	2.5391	.2163	-.7746	-.9424	-.6263	-.0431	.6311				10.26
	0.12	2.2699	.1505	-.6658	-.7878	-.5144	-.0171	.5646				8.054
8	-0.02	-5.5813	-.7864	3.4838	4.7098	3.1780	.6887	-1.5416	-3.1510			81.14
	0.1	2.0993	.0971	-.7639	-.9209	-.6642	-.1768	.3947	.9347			8.925
	0.12	1.8859	.0598	-.6504	-.7687	-.5515	-.1416	.3468	.8196			7.007
9	-0.02	-4.3111	-.2574	3.3537	4.3920	3.0985	.9950	-.8902	-2.2510	-3.1295		68.40
	0.1	1.6411	-.0097	-.7230	-.8578	-.6533	-.2594	.2052	.6475	1.0093		7.418
	0.12	1.4818	-.0235	-.6121	-.7152	-.5444	-.2163	.1779	.5633	.8884		5.826
10	-0.02	-3.3596	.1064	3.1944	4.0824	2.9765	1.1782	-.4334	-1.5967	-2.3478	-2.800	58.31
	0.1	1.2692	-.0887	-.6766	-.7886	-.6221	-.3007	.0784	.4398	.7366	.9527	6.142
	0.12	1.1483	-.0867	-.5710	-.6567	-.5185	-.2525	.0668	.3799	.6455	.8449	4.814

Table 3

The coefficients $b_i, i = 1, \dots, n$ in the BLUE $\hat{\lambda}_2 = \sum_{i=1}^n b_i y_{[i]}$.

n	α	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	$\text{var}(\hat{\lambda}_2) / \lambda_2^2$
2	-0.02	-1.6748	-1.6748									605.6
	0.1	-1.6748	-1.6748									27.79
	0.12	-1.6748	-1.6748									19.13
3	-0.02	10.317	-.3632	-9.9536								189.8
	0.1	-2.2211	.4394	1.7817								9.841
	0.12	-1.8568	.3800	1.4768								6.898
4	-0.02	8.7000	1.7944	-4.3610	-6.1334							125.9
	0.1	-1.9175	.0072	.8621	1.0481							6.676
	0.12	-1.6138	.0397	.7131	.8610							4.744
5	-0.02	8.6808	2.3463	-3.2853	-4.8892	-2.8526						116.5
	0.1	-1.9158	-.0965	.6758	.7980	.5384						6.087
	0.12	-1.6168	-.0357	.5698	.6505	.4322						4.356
6	-0.02	8.6063	2.2741	-3.3565	-4.9617	-2.9272	.3651					116.3
	0.1	-1.9278	-.1045	.6677	.7871	.5237	.0539					6.082
	0.12	-1.6250	-.0403	.5651	.6433	.4216	.0353					4.354
7	-0.02	7.6228	1.6966	-3.5775	-5.0874	-3.1896	-.1107	2.6458				108.3
	0.1	-1.7838	-.0534	.6855	.8094	.5719	.1362	-.3659				5.871
	0.12	-1.5060	-.0039	.5757	.6608	.4645	.1104	-.3016				4.207
8	-0.02	6.0948	.9738	-3.5866	-4.8963	-3.2607	-.6025	1.7795	3.4984			92.84
	0.1	-1.4923	.0254	.6784	.7952	.5970	.2246	-.2095	-.6181			5.285
	0.12	-1.2681	.0523	.5661	.6490	.4875	.1875	-.1667	-.5075			3.805
9	-0.02	4.6975	.3920	-3.4436	-4.5467	-3.1733	-.9393	1.0630	2.5086	3.4418		77.42
	0.1	-1.1763	.0993	.6501	.7515	.5894	.2817	-.0786	-.4196	-.6974		4.566
	0.12	-1.0050	.1065	.5412	.6142	.4828	.2361	-.0567	-.3406	-.5785		3.304
10	-0.02	3.6562	-.0061	-3.2692	-4.2079	-3.0398	-1.1398	.5631	1.7925	2.5863	3.0646	65.34
	0.1	-.9136	.1550	.6173	.7027	.5674	.3108	.0110	-.2729	-.5048	-.6730	3.929
	0.12	-.7823	.1487	.5137	.5751	.4655	.2604	.0175	-.2181	-.4163	-.5643	2.853

Table 4

$\text{cov}(\hat{\mu}_2, \hat{\lambda}_2) / \lambda_2^2$.

n	2	3	4	5	6	7	8	9	10
$\alpha = -0.02$	-587.83	-179.53	-117.17	-107.77	-107.70	-100.63	-100.63	-72.718	-61.68
$\alpha = 0.1$	-31.0789	-12.000	-8.6061	-8.0415	-8.0505	-7.6868	-6.8014	-5.7599	-4.8586
$\alpha = 0.12$	-21.7648	-8.6789	-6.3694	-6.0130	-6.0206	-5.7453	-5.0968	-4.3279	-3.6519

In order to illustrate numerically the estimators obtained and the predicted concomitants, we generated 8 observations of record values and their concomitants from (2) with exponential margins in (18) with $p_1=p_2=14$, $q_1=q_2=2$, $\alpha=0.1$, $\mu_1=4$, $\lambda_1=2$, $\mu_2=3$, $\lambda_2=1$, as follows: (4.4987,4.3577), (5.1835,4.8029), (7.4557,3.1489), (7.9276,6.1529), (8.9763,5.3594), (9.9653,3.3535), (12.8987,3.5689), (13.1135,4.5541).

We assume that we have only 8 or 6 observations and we require to predict the 9th concomitant value ($m=n+1$ or $m=n+3$). For calculating the BLUP in (35) or the conditional predictor in (43) we must first calculate μ_2 and λ_2 . Assuming α is known, using (31), (32) and the coefficients a_i and $b_i, i=1, \dots, n$ given in Tables 2&3 for $\alpha=0.1$, we get

$$\begin{aligned} \text{for } n=8, \hat{\mu}_2 &= 3.0557 \text{ and } \hat{\lambda}_2 = 1.0351, \\ \text{for } n=6, \hat{\mu}_2 &= 3.1262 \text{ and } \hat{\lambda}_2 = 1.0299. \end{aligned}$$

First method (BLUP):

Calculating $\varepsilon_{9,w}, \Sigma^{-1}$ and ε_n and substituting with the corresponding $\hat{\mu}_2$ and $\hat{\lambda}_2$ in (35) we get

$$\begin{aligned} \text{for } n=8, Y_{[9]}^* &= 4.1566, \\ \text{for } n=6, Y_{[9]}^* &= 4.2187. \end{aligned}$$

Second method (The conditional predictor):

Using (42) and (43), we get

$$\begin{aligned} \text{for } n=8, Y_{[9]}^{\tilde{*}} &= 3.7783, \\ \text{for } n=6, Y_{[9]}^{\tilde{*}} &= 3.8451. \end{aligned}$$

while using Algorithm 1, with $s=50$, we have

$$\begin{aligned} \text{for } n=8, Y_{[9]}^{\tilde{\tilde{*}}} &= 4.0631, \\ \text{for } n=6, Y_{[9]}^{\tilde{\tilde{*}}} &= 4.1355. \end{aligned}$$

We see from the above results that the predicted values of the 9th concomitant using both methods are almost the same. We may conclude that the conditional predictor is simple to apply and requires less calculations than the BLUP and gives satisfactory results. Moreover it can be improved by applying Algorithm 1.

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