The Scope of the Structural Completeness in the Class of all Over-Systems of the Classical Functional Calculus with Identity

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Abstract: This paper is the continuation of the paper [13]. Namely, in [13], the scope of the structural completeness in the class of all over-systems of the classical predicate calculus, has been established. In this paper we establish the scope of the structural completeness in the class of all over-systems of the classical functional calculus with identity.

Key words: Structural completeness

1 Preliminaries

Let $\to, \sim, \lor, \land, \equiv$ denote the connectives of implication, negation, disjunction, conjunction and equivalence, respectively. Let $i, j, k, n \in \mathcal{N}$, where $\mathcal{N} = \{1, 2, \ldots\}$. $X \subseteq Y$ denotes that $X$ is a subset of the set $Y$. $X \subset Y$ denotes that $X \subseteq Y$ and $Y \neq X$. $\emptyset$ denotes the empty set.

By $A_0 = \{p_1^1, p_1^2, \ldots, p_1^k, p_2^k, \ldots\}$ we denote the set of all propositional variables. Hence, $S_0$ is the set of all well-formed formulas, which are built in the usual manner by propositional variables and by means of logical connectives. $\mathcal{M}_2$ denotes the well-known classical two-valued matrix. $Z_2$ denotes the set of all formulas valid in the classical two-valued matrix $\mathcal{M}_2$ (see [6], [9], [15]).

Symbols $x_1, x_2, \ldots$ are individual variables. $V$ denotes the set of all individual variables. Symbols $P^n_k$ are $n$-ary predicate letters. The set of all atomic formulas of the form $P^n_k(x_1, \ldots, x_n)$, is denoted by $At_1$. The symbols $\forall x_i, \exists x_i$ are quantifiers. The set $S_1$ of all well-formed formulas, is constructed in the usual way, by the symbols listed above.

Now we describe the set $S_I$ of all well-formed formulas for a functional calculus with identity. Symbols $x_1, x_2, \ldots$ are individual variables. $V_I$ denotes the set of all individual variables. Symbols $a_1, a_2, \ldots$ are individual constants. $U_I$ denotes the set of all individual constants. Symbols $f^n_k$ are $n$-ary functional letters. Now we define the set $T_I$ of all terms. Namely, $U_I \cup V_I \subseteq T_I$ and if $t_1, \ldots, t_n \in T_I$, then $f^n_k(t_1, \ldots, t_n) \in T_I$. We assume here that $P^2_1$ denotes the predicate letter of identity. We use also the symbol $I$, as the predicate letter of identity. Namely, we write sometimes $I(t_k, t_n)$, instead of $P^2_1(t_k, t_n)$.
and we write sometimes \(I(t_k, t_n)\), instead of \(t_k = t_n\).

The set of all atomic formulas of the form \(P^I_k(t_1, ..., t_n)\), is denoted by \(At_i\). \(\land, \lor, \forall, \exists\) are quantifiers. The set \(Si\) of all well-formed formulas, is constructed in the usual manner, by the symbols listed above. \(P^I_i(\phi)\) denotes the set of all predicate letters occurring in \(\phi\), where \(\phi \in Si\), \(P^I_i(X)\) denotes the set of all predicate letters occurring in formulas, which belong to the set \(X\), where \(X \subseteq Si\). \(U_i(\phi)\) denotes the set of all individual constants occurring in \(\phi\), where \(\phi \in Si\). \(\bar{S}_i = \{\phi \in Si: v_f_i(\phi) = \emptyset\}\). 

Next \(\land\alpha = \land_{x_i}, ..., \land_{x_k}\alpha\), if \(v_f_i(\alpha) = \{x_1, ..., x_k\}\) and \(\lor\alpha = \lor_{x_i}, ..., \lor_{x_k}\alpha\), if \(v_f_i(\alpha) = \{x_1, ..., x_k\}\). Hence, \(\land\alpha = \alpha\), if \(v_f_i(\alpha) = \emptyset\) and \(\lor\alpha = \alpha\), if \(v_f_i(\alpha) = \emptyset\). \(G_i(\alpha)\) denotes the set of all connectives occurring in \(\alpha\), where \(\alpha \in Si\). \(P^I_i(S_i)\) denotes the set of all predicate letters, occurring in \(S_i\).

\(S^f_i\) is the set of all well-formed formulas, which are in prenex-conjunctive normal form, where \(S_i^f \subseteq Si\) (see [6] pp. 85 - 97, [2] pp. 186 - 194 and [1] pp. 35 - 42 and pp. 130 - 132). We use \(\Rightarrow, \neg, \Leftrightarrow, \land, \lor, \forall, \exists\) as metalogical symbols. \(S^f_i = \{\phi \in S_i: P^I_i(\phi) = \{I\}\}\), \(\bar{S}^I_i = \{\phi \in S^f_i: v_f_i(\phi) = \emptyset\}\), \(S^c = \{\phi \in S^f_i: \sim \notin G_i(\phi)\}\), \(S^c_{\bar{I}} = \{\phi \in S^c: v_f_i(\phi) = \emptyset\}\). \(R_i\), denotes the set of all rules over \(S_i(i \in \{0, 1, I\})\) (see [8] p.37). For any \(X \subseteq S_i\) and \(R \subseteq R_i\), \(Cn(R, X)\) is the smallest subset of \(S_i\), containing \(X\) and closed under the rules of \(R \subseteq R_i\). Next, the couple \(\langle R, X \rangle\) is called a system, whenever \(X \subseteq S_i\) and \(R \subseteq R_i\). By \(r_i^I(i \in \{0, 1\})\) we denote the rule of substitution for propositional and predicate letters, respectively. Namely, \(\langle (\alpha), \beta \rangle \in r^I_0 \iff h^e(\alpha) = \beta\), for some endomorphism \(h^e: S_i \rightarrow S_i\), which is an extension of the function \(e: At_i \rightarrow S_i\), \(e \in \epsilon_i(i \in \{0, 1\})\). Next, \(r^I_i\) denotes the rule of substitution for predicate letters in functional calculus with identity. Namely, \(\langle (\alpha), \beta \rangle \in r^I_i \iff h^e(\alpha) = \beta\), for some endomorphism \(h^e: S_i \rightarrow S_i\), which is an extension of the function \(e: At_i \rightarrow S_i\), \(e \in \epsilon_i\). Of course \(e(I(t_1, t_2)) = I(t_1, t_2)\) (for details see [5], [7], [8], see also [10], [12]).

Next, here and later, \(r_0\) denotes Modus Ponens for propositional calculus, \(r_0^I\) denotes Modus Ponens for predicate calculus and \(r_0^I\) denotes Modus Ponens for functional calculus with identity, respectively.

Next, \(r_+^I\) denotes the generalization rule for predicate calculus and \(r_0^I\) denotes the generalization rule for functional calculus with identity. Thus, 

\(R_0 = \{r_0\}, R_0^I = \{r_0^I, r_+^I\}\) and \(R_+^I = \{r_0^I, r_+^I\}\).

\(L_2\) denotes the set of all formulas valid in the classical predicate calculus and \(L_2^I\) denotes the set of all formulas valid in the classical functional calculus with identity.

Now we define the function \(i: S_i \rightarrow S_0\) as follows: 

\((a_1)i(p^c_k(t_1, ..., t_n)) = p^c_k(p^c_k \in At_0)\),

\((a_2)i(I(t_k, t_n)) = p_1(p_1 \in At_0)\),

\((a_3)i(\phi \rightarrow \psi) = i(\phi) \rightarrow i(\psi)\),

\((a_4)i(\phi \lor \psi) = i(\phi) \lor i(\psi)\),

\((a_5)i(\phi \land \psi) = i(\phi) \land i(\psi)\),

\((a_6)i(\phi \equiv \psi) = i(\phi) \equiv i(\psi)\),

\((a_7)i(\neg \phi) = \neg i(\phi)\),

\((a_8)i(\land_{x_k} \phi) = i(\phi)\),
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\[(a_{i})k(V_{a_{i}} \phi) = i(\phi).\]

Let \(\phi \in S_{1}\) and \(\alpha \in At_{1}\) and let \(v: At_{0} \rightarrow |\mathfrak{M}_2|\) be arbitrary, but fixed valuation in the matrix \(\mathfrak{M}_2\) such that \(h^v(\text{id}(\phi)) = 1\).

Then:

\[(\text{I}) \quad e_{\phi}(\alpha) = \begin{cases} \Lambda \phi \land \alpha, & \text{if } v(\alpha) = 0 \\ \Lambda \phi \rightarrow \alpha, & \text{if } v(\alpha) = 1 \\ \alpha, & \text{if } \alpha \in S' \end{cases}\]

Now we repeat the well-known properties of the operation of consequence. Let \(X \subseteq S_{1}\) and \(R \subseteq R_{S_{1}}\). Thus,

\[(c_{1})X \subseteq Cn(R,X),\]
\[(c_{2})X \subseteq Y \Rightarrow Cn(R,X) \subseteq Cn(R,Y),\]
\[(c_{3}) R \subseteq R' \Rightarrow Cn(R,X) \subseteq Cn(R',X),\]
\[(c_{4})Cn\{R,Cn(R,X)\} \subseteq Cn(R,X),\]
\[(c_{5})Cn(R,X) = \bigcup\{Cn(R,Y); Y \in Fin(X)\},\]

Where \(Y \in Fin(X)\) denotes that \(Y\) is the finite subset of \(X\).

Now we repeat the well-known definitions of the permissible rule, the derivable rule and the structural rule (see [5], [8]). Let \(X \subseteq S_{1}\) and \(R \subseteq R_{S_{1}}\). Thus,

\[r \in \text{Perm}(R,X) \text{ iff } (\forall \Pi \subseteq S_{1})(\forall \phi \in S_{1})[(\Pi, \phi) \in r \& \Pi \subseteq Cn(R,X) \Rightarrow \phi \in Cn(R,X)]\]
\[r \in \text{Der}(R,X) \text{ iff } (\forall \Pi \subseteq S_{1})(\forall \phi \in S_{1})[(\Pi, \phi) \in r \Rightarrow \phi \in Cn(R,X \cup \Pi)]\]
\[r \in \text{Struct}\{S_{1}\} \text{ iff } (\forall \Pi \subseteq S_{1})(\forall \phi \in S_{1})(\forall e \in e_{\Pi})[(\Pi, e) \in r \Rightarrow (h^e(\Pi), h^e(\phi)) \in r]\]

Next,

\[(R,X) \in SCpl_{S_{1}} \text{ iff } \]
\[\text{Struct}_{S_{1}} \cap \text{Perm}(R,X) \subseteq \text{Der}(R,X), \]
\[Z_{1} = \{\phi \in S_{1}; i(\phi) \in Z_{2}\}, \]
\[Z_{2} = \{\phi \in Z_{2}; vf_{1}(\phi) = 0\}\]

2 The Well-Known Theorems

It is well-known fact that on the ground of the classical functional calculus with identity, the following theorems are valid (see [2] and [8]):

**THEOREM 1.** Let \(\alpha \in S_{1}\) and \(X \subseteq S_{1}\). Then, \(\beta \in Cn(R_{0+}, L_{2}^{1} \cup X \cup \{\alpha\}) \Rightarrow \alpha \rightarrow \beta \in Cn(R_{0+}, L_{2}^{1} \cup X)\).

**THEOREM 2.** Let \(\alpha, \beta, \delta, \phi, \psi \in S_{1}\) and \(Q_{1}, \ldots, Q_{n} \in \{A_{x_{1}}, \ldots, A_{x_{i}}, V_{x_{i+1}}, \ldots, V_{x_{n}}\}\). Then the following formulas are valid on the ground of the classical functional calculus with identity:

\[(\text{II}) \Lambda \phi \rightarrow \phi\]
\[(\text{III}) \sim \phi \equiv \phi\]
\[(\text{IV}) (\phi \rightarrow \delta) \rightarrow [(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow (\delta \land \psi))]\]
\[(\text{V}) \sim (\phi \land \psi) \equiv (\phi \rightarrow \sim \psi)\]
\[(\text{VI}) Q_{1} \ldots Q_{n} (\phi \rightarrow \psi) \equiv (\phi \rightarrow Q_{1} \ldots Q_{n} \psi), \text{ if } x_{1}, \ldots, x_{n} \notin vf_{1}(\phi)\]
\[(\text{VII}) A_{x_{k}} (\phi \rightarrow \psi) \equiv (V_{x_{k}} \phi \rightarrow \psi), \text{ if } x_{k} \notin vf_{1}(\psi)\]
\[(\text{VIII}) \sim V_{x_{k}} \sim \phi \equiv \sim \sim A_{x_{k}} (\phi)\]
\[(\text{IX}) \phi \rightarrow V_{x_{k}} \phi\]
\[(\text{X}) \alpha \rightarrow (\beta \rightarrow (\alpha \land \beta))\]
\[(\text{XI}) \alpha \rightarrow (\alpha \lor \beta)\]
\[(\text{XII}) \alpha \rightarrow (\beta \lor \alpha)\]
\[(\text{XIII}) (\alpha \rightarrow \beta) \rightarrow [\alpha \rightarrow (\beta \lor \delta)]\]
\[(\text{XIV}) (\alpha \rightarrow \delta) \rightarrow [\alpha \rightarrow (\beta \lor \delta)]\]

**THEOREM 3.** \(Cn(R_{0+}, \{r_{i}'\}, L_{2}^{1}) = L_{2}^{1}\).

3 The Scope of the Structural Completeness in the Class of all Over-Systems of the Classical Functional Calculus with Identity
Lemma 1. Let $X \subseteq S_I$, $\text{Cn}(R_{0+}^l, L_2^l \cup X) = Z_3$ and $(\forall \alpha \in Z_3^l) \lnot \alpha \in Z_3$. Then, $(\forall \beta_0 \in S^l) \lnot \beta_0 \in Z_3$.

Proof. At first, we assume that $X \subseteq S_I$, (1)

$\text{Cn}(R_{0+}^l, L_2^l \cup X) = Z_3$, (2)

$(\forall \alpha \in Z_3^l) \lnot \alpha \in Z_3$, (3)

and $(\forall \beta_0 \in S^l) \lnot \beta_0 \in Z_3$. (4)

From (2), by the definition of the set $L_2^l$, it follows that $(\forall t_1 \in T_I) [\bigwedge I (t_1, t_1) \in Z_3]$. (5)

At first, we consider the following cases:

(a) $(\forall \nu: A_0 \rightarrow |\mathcal{M}_2|)[h^v(i(\beta_0))] = 1$.

Hence, from (1) - (4), it follows that

$\beta_0 \in Z_3 \lor \lnot \beta_0 \in Z_3$. (6)

Next, we must consider the following cases:

(b) $(\exists \nu_1: A_0 \rightarrow |\mathcal{M}_2|)[h^{v_1}(i(\beta_0))] = 1$.

and

(c) $(\exists \nu_2: A_0 \rightarrow |\mathcal{M}_2|)[h^{v_2}(i(\beta_0))] = 0$.

In the cases (b) and (c), from (4) and (5), it follows that

$(c_1)(\forall \nu: A_0 \rightarrow |\mathcal{M}_2|)[h^v(i(\alpha_0))] = 1 \Rightarrow h^v(i(\beta_0)) = 1$.

or

$(c_2)(\exists \nu: A_0 \rightarrow |\mathcal{M}_2|)[h^v(i(\alpha_0))] = 1 \land h^v(i(\beta_0)) = 0$.

where

$\alpha_0 = \bigwedge I (t_1, t_1)$. (7)

Of course, in (c), from (1) - (4), one can obtain the following cases:

$I_4 \sim (\alpha_0 \rightarrow \beta_0) \in Z_3$.

Lemma 2. Let $\alpha_1 \in S_I^*, X \subseteq S_I$, $\text{Cn}(R_{0+}^l, L_2^l \cup X) = Z_3$, $S^l \subseteq Z_3$, $t_1, t_2 \in T_I$.

Then, $\sim \alpha_1 \in Z_2^l \land \alpha_1 \neq \sim I (t_1, t_2) \Rightarrow \forall \alpha_1 (\alpha_1) \in Z_3$.

Proof. Let

$\alpha_1 \in S_I^*, (1)$

$X \subseteq S_I$, (2)

$\text{Cn}(R_{0+}^l, L_2^l \cup X) = Z_3$, (3)

$S^l \subseteq Z_3, t_1, t_2 \in T_I$. (4)

Suppose that

$\sim \alpha_1 \in Z_2^l \land \alpha_1 \neq \sim I (t_1, t_2)$.
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\[ h^{e_{a_1}}(\alpha_1) \in Z_3 \]. (5)

Hence,

\[ \sim \alpha_1 \notin \neg Z_2^{I}. \] (6)
\[ \alpha_1 \neq \sim I(t_1, t_2). \] (7)
\[ h^{e_{a_1}}(\alpha_1) \notin Z_3. \] (8)

From (6), it follows that

\[ (\exists v : At_0 \rightarrow [\neg Z_2] [h^v(i(\alpha_1)) = 1]. \] (9)

From (1), (7), one can obtain the following cases:

(1.1) \[ \alpha_1 = I(t_1, t_2) \]
or
(1.2) \[ \alpha_1 = P_k^n(t_1, \ldots, t_n) \]
or
(1.3) \[ \alpha_1 = \neg P_k^n(t_1, \ldots, t_n) \]
or
(1.4) \[ \alpha_1 = \phi_1 \lor \phi_2 \]
or
(1.5) \[ \alpha_1 = \phi_1 \land \phi_2 \]
or
(1.6) \[ \alpha_1 = Q_1 \ldots Q_k \phi_1, \]
where

\[ n, k \in N \] (10)

and

\[ Q_1, \ldots, Q_k \in \{\land X_1, \ldots, \land X_i, V_{X_{i+1}}, \ldots, V_{X_k}\} \]. (11)

In (1.1), from (3), (4), (7), (9) and by (1), one can obtain that

\[ h^{e_{a_1}}(\alpha_1) \in Z_3. \] (12)

In (1.2) and in (1.3), from (3) and (9), by (1), it follows that

\[ h^{e_{a_1}}(\alpha_1) \in Z_3. \] (13)

In (1.4) one can assume inductively that

\[ (a_i) h^{e_{a_1}}(\phi_1) \in Z_3 \]
or
\[ (a_2) h^{e_{a_1}}(\phi_2) \in Z_3. \]

From the properties of (I) and from THEOREM 2, it follows that

\[ h^{e_{a_1}}(\phi_1) = h^{e_{a_1}}(\phi_1) \lor h^{e_{a_1}}(\phi_2). \] (14)

Hence, from (3) and from (1.4), using THEOREM 2 (XI) and THEOREM 2 (XII), in (a_1) and in (a_2), we obtain that

\[ h^{e_{a_1}}(\alpha_1) \in Z_3. \] (15)

In (1.5) one can assume inductively that

\[ h^{e_{a_1}}(\phi_1) \in Z_3. \] (16)

and

\[ h^{e_{a_1}}(\phi_2) \in Z_3. \] (17)

From the properties of (I) and from THEOREM 2, it follows that

\[ h^{e_{a_1}}(\phi_1 \land \phi_2) = h^{e_{a_1}}(\phi_1) \land h^{e_{a_1}}(\phi_2). \] (18)

From (1), (3) and from (1.5), using THEOREM 2 (X), (16) - (18), we get that

\[ h^{e_{a_1}}(\alpha_1) \in Z_3. \] (19)

In (1.6) one can assume inductively that

\[ h^{e_{a_1}}(\phi_1) \in Z_3. \] (20)

Hence, from (1), (3), (1.6) and using THEOREM 2 (IX), one can obtain that

\[ h^{e_{a_1}}(\alpha_1) \in Z_3. \] (21)

Thus, in the cases (1.1) - (1.6), it follows that

\[ h^{e_{a_1}}(\alpha_1) \in Z_3. \] (22)

what contradicts (8).

\[ \square \]

Lemma 3. Let \[ \alpha_1, \beta_1 \in S_I, X \subseteq S_I, Cn(R_{0+}, L_{1}^{I} \cup X) = Z_3, Z_3 \subset S_I, (\forall \alpha \in Z_{3}^{I})[\alpha \in Z_3 \lor \sim \alpha \in Z_3] \] and

\[ (\forall e \in e_{1}^{I})[h^{e}(\alpha_1) \in Z_3 \Rightarrow h^{e}(\beta_1) \in Z_3]. \]

Then,

\[ \land \alpha_1 \rightarrow \beta_1 \in Z_3. \]

Proof. At first we assume that

\[ \alpha_1, \beta_1 \in S_I, (1) \]
\[ X \subseteq S_I, (2) \]
\[ Cn(R_{0+}, L_{1}^{I} \cup X) = Z_3, (3) \]
\[ Z_3 \subset S_I, (4) \]
\[ (\forall \alpha \in Z_{3}^{I})[\alpha \in Z_3 \lor \sim \alpha \in Z_3], (5) \]
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\[(\forall e \in \mathcal{E})[h^e(\alpha_1) \in Z_3 \Rightarrow h^e(\beta_1) \in Z_3].(6)\]

Suppose that

\[\Lambda \alpha_1 \rightarrow \beta_1 \notin Z_3.(7)\]

From (1) - (7), by the well-known **THEOREM of Replacement** (see [2], pp. 192-194 and [1], pp.128-132), one can obtain

\[\alpha_1^*, \beta_1^* \in S^*.(8)\]

\[(\forall e \in \mathcal{E})[h^e(\alpha_1^*) \in Z_3 \Rightarrow h^e(\beta_1^*) \in Z_3].(9)\]

\[\Lambda \alpha_1^* \rightarrow \beta_1^* \notin Z_3.(10)\]

From (3), (9) and (10), it follows that

\[\Lambda \alpha_1^*, \alpha_1^* \notin Z_3.(11)\]

From (3), by the definition of the set \(L_2^1\), one can obtain that

\[(\forall k \in \mathcal{N})(\forall t_k \in T_I)[\Lambda I(t_k, t_k) \in Z_3] .(12)\]

From (2) - (5), by **Lemma 1**, it follows that

\[(\forall \beta_0 \in \bar{S}^i)[\beta_0 \in Z_3 \forall \beta_0 \in Z_3].(13)\]

Next, we consider the following cases:

(a) \( (\exists t_1, t_2 \in T_I)[\Lambda I(t_1, t_2) \in Z_3]\)

or

(b) \( (\forall t_1, t_2 \in T_I)[\Lambda I(t_1, t_2) \in Z_3]\).

In (a), from (13), it follows that

\[(\exists t_1, t_2 \in T_I)[\sim \Lambda I(t_1, t_2) \in Z_3].(14)\]

By the definition of the set \(\bar{Z}_2^1\), it follows that

\[\forall t_1, t_2 \in T_I)(\forall t_k \in T_I)(\forall \delta \in \bar{S}_i)\]

\[[\Lambda I(t_k, t_k) \rightarrow (\sim \Lambda I(t_1, t_2) \rightarrow \delta) \in \bar{Z}_2^1].(15)\]

From (3), (5), (12), (14) and (15), it follows that

\[(\forall \delta \in \bar{S}_i)[\delta \in Z_3 \forall \sim \delta \in Z_3].(16)\]

Hence, from (3) and (8), it follows that

\[(b_1) \Lambda \alpha_1^* \rightarrow \beta_1^* \in Z_3\]

or

\[(b_2) \alpha_1^* \in Z_3.\]

Of course, \(b_1\) contradicts (10) and \(b_2\) contradicts (11).

Thus, the case \(\alpha_1\) is excluded. In the case \(\alpha_2\), it follows that

\[(\forall t_1, t_2 \in T_I)\Lambda I(t_1, t_2) \in Z_3].(17)\]

Hence,

\[\bar{S}_i^I \subseteq Z_3.(18)\]

From (3) and (10), it follows that

\[\sim \Lambda \alpha_1^* \notin Z_3.(19)\]

Hence, from (3), (5), (9), (10) and (11), it follows that

\[\Lambda \alpha_1^* \notin \bar{Z}_2^1.(20)\]

and

\[\sim \Lambda \alpha_1^* \notin \bar{Z}_2^1.(21)\]

Hence, by the definition of the set \(\bar{Z}_2^1\), one can obtain that

\[\sim \alpha_1^* \notin \bar{Z}_2^1.(22)\]

From (22), by the definition of the set \(\bar{Z}_2^1\), one can obtain that

\[(\exists v: At_0 \rightarrow [\mathcal{W}_2])(h^v(\text{if}(\alpha_1^*)) = 1].(23)\]

From (1), (2), (3), (8), (18), (22) and (23), by **Lemma 2**, one can obtain that

\[\forall t_1, t_2 \in T_I) [\alpha_1^* \neq \sim I(t_1, t_2) \Rightarrow \]

\[h^e(\alpha_1^*) \in Z_3].(24)\]

From (3) and (17), it follows that

\[\forall \beta_1^* \in \bar{S}_i)[\forall t_1, t_2 \in T_I)[\alpha_1^* = \sim I(t_1, t_2) \Rightarrow \]

\[\Lambda \alpha_1^* \rightarrow \beta_1^* \in Z_3].(25)\]

Hence, from (3), (8), (10), (17) and (24), it follows that

\[h^e(\alpha_1^*) \in Z_3.(26)\]

Hence, from (3) and (9), it follows that

\[h^e(\beta_1^*) \in Z_3.(27)\]

From (3) and (8), it follows that

\[(1.7) \beta_1^* = I(t_1, t_2)\]

or

\[(1.8) \beta_1^* = \sim I(t_1, t_2)\]

or

\[(1.9) \beta_1^* = P_k^n(t_1, ..., t_n)\]

or
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(1.10) \( \beta_1^\ast = \sim P_k^n(t_1, \ldots, t_n) \)

or

(1.11) \( \beta_1^\ast = \phi_1 \lor \phi_2 \)

or

(1.12) \( \beta_1^\ast = \phi_1 \land \phi_2 \)

or

(1.14) \( \beta_1^\ast = Q_1 \ldots Q_k \phi \),

where \( n, k \in \mathbb{N} \) and \( Q_1, \ldots, Q_k \in \mathbb{V} \).

(28)

In (1.7) and (1.8), from (27), by (I), one can obtain that

\( \beta_1^\ast \in Z_3. \)

(29)

Hence, from (3), it follows that

\( \land \alpha_1^\ast \rightarrow \beta_1^\ast \in Z_3. \)

(30)

In (1.9) and (1.10), from (3) and (27), by (I), one can obtain that

\( \land \alpha_2^\ast \rightarrow \beta_1^\ast \in Z_3. \)

(31)

In (1.11), from (I), (3) and (27), by (I), one can assume inductively that

\( (I_1) \land \alpha_1^\ast \rightarrow \phi_1 \in Z_3 \)

or

\( (I_2) \land \alpha_1^\ast \rightarrow \phi_2 \in Z_3 \).

In (1.11) and (I), from (3) and (27), by (I), one can obtain that

\( \land \alpha_1^\ast \rightarrow \beta_1^\ast \in Z_3. \)

(32)

In (1.11) and (I), from (3) and (27), by (I), one can assume inductively that

\( \land \alpha_1^\ast \rightarrow \phi_1 \in Z_3 \)

and

\( \land \alpha_1^\ast \rightarrow \phi_2 \in Z_3. \)

Thus, in (1.12), from (3), (34), (35) and by THEOREM 2 (IV), it follows that

\( \land \alpha_1^\ast \rightarrow \phi_1 \in Z_3. \)

(36)

In (1.14), from (3), (27), by (I), one can assume inductively that

\( \land \alpha_1^\ast \rightarrow \beta_1^\ast \in Z_3. \)

(37)

Hence, in (1.14), from (3), (28), THEOREM 2 (VI) and THEOREM 2 (IX), it follows that

\( \land \alpha_1^\ast \rightarrow Q_1 \ldots Q_k \phi \in Z_3. \)

(38)

In (1.11) and (I), from (3), by THEOREM 2 (XIII), it follows that

\( \land \alpha_1^\ast \rightarrow \beta_1^\ast \in Z_3. \)

(39)

In consequence, in (1.7)-(1.14), one can obtain the contradiction with (10). This completes the proof.

Lemma 4. Let \( Cn(R_0^+, L_2^I \cup X) = Z_3, Z_3 \subset S_I. \)

Then, \( \langle R_0^+, L_2^I \cup X \rangle \in SCpSIS \Rightarrow (\forall \alpha \in Z_3^I)[\alpha \in Z_3 \lor \sim \alpha \in Z_3]. \)

Proof. Let

\( Cn(R_0^+, L_2^I \cup X) = Z_3. \)

(1)

\( Z_3 \subset S_I. \)

(2)

\( \langle R_0^+, L_2^I \cup X \rangle \in SCpSIS. \)

(3)

Suppose that

\( (\exists \alpha_1 \in \bar{Z}_2^I)[\alpha_1 \notin Z_3 \land \sim \alpha_1 \notin Z_3]. \)

(4)

Hence, let

\( A_1 = \{\alpha_1, \sim \alpha_1\}. \)

(5)

where

\( \alpha_1 \in \bar{Z}_2. \)

(6)

At last, suppose that

\( - (\forall e \in e') (\forall \alpha) \in A_1 (\exists \alpha \in A_1)[[h^e(\sim \alpha_1 \equiv \sim \alpha_j \rightarrow \sim \alpha_i) \in Z_3 \Rightarrow h^e(\sim \alpha_j) \in Z_3]. \)

(7)

From (7) it follows that

\( (\exists e_1 \in e') (\forall \alpha) \in A_1 (\exists \alpha \in A_1)[[h^{e_1}(\sim \alpha_1 \equiv \sim \alpha_j \rightarrow \sim \alpha_i) \in Z_3 \Rightarrow h^{e_1}(\sim \alpha_i) \notin Z_3 \land h^{e_1}(\sim \alpha_j) \notin Z_3]. \)

(8)

Hence, from (5), it follows that
In the case \((a_i)\), from (5) and (8), it follows that there exists \(e_i \in e_i^i\) such that
\[
\{h^{e_i}(\sim a_i_1 \equiv (\sim a_1 \rightarrow \sim a_1)) \in Z_3 \Rightarrow h^{e_i}(\sim a_1_1) \\
\epsilon Z_3\} \& h^{e_i}(\sim a_1) \notin Z_3
\]
and
\[
\{h^{e_i}(a_1_1 \equiv (a_1 \rightarrow \sim a_1)) \in Z_3 \Rightarrow h^{e_i}(a_1_1) \\
\epsilon Z_3\} \& h^{e_i}(a_1) \notin Z_3.
\]
(9)

From (1) and (9), it follows that
\[
h^{e_i}(\sim a_1) \notin Z_3.
\]
(11)

From (1) and (10), it follows that
\[
h^{e_i}(\sim a_1) \in Z_3.
\]
(12)
what contradicts (11).

In the case \((a_2)\), from (5) and (8), it follows that there exists \(e_2 \in e_i^i\) such that
\[
\{h^{e_2}(\sim a_2_1 \equiv (\sim a_2 \rightarrow \sim a_2)) \in Z_3 \Rightarrow h^{e_2}(\sim a_2_1) \\
\epsilon Z_3\} \& h^{e_2}(\sim a_2) \notin Z_3
\]
and
\[
\{h^{e_2}(a_2_1 \equiv (a_2 \rightarrow \sim a_2)) \in Z_3 \Rightarrow h^{e_2}(a_2_1) \\
\epsilon Z_3\} \& h^{e_2}(a_2) \notin Z_3.
\]
(13)

From (1), (13), it follows that
\[
h^{e_2}(a_2) \in Z_3.
\]
(15)

From (1), (14), it follows that
\[
h^{e_2}(a_2) \notin Z_3.
\]
(16)
what contradicts (15).

Thus,
\[
(\forall e \in e_i^i)(\forall a_j \in A_i)(\exists a_i \in A_i)
\]

Hence, from (5), we obtain the following cases:

I) \(a_i = a_j = a_1\)

II) \(a_i = a_j = \sim a_1\)

III) \(a_i \neq a_j \& a_i = a_1\)

IV) \(a_i \neq a_j \& a_i = \sim a_1\).

In the case II), from (1), (5) and (17), it follows that

\[
(\forall e \in e_i^i)[h^e(\sim a_1) \equiv (\sim a_1 \rightarrow \sim a_1) \in Z_3 \Rightarrow h^e(\sim a_1) \in Z_3 \Rightarrow h^e(\sim a_1) \\
\epsilon Z_3]
\]
(18)

Hence,
\[
(\forall e \in e_i^i)[h^e(\sim a_1) \in Z_3].
\]
(19)

From (1) and (19), it follows that
\[
\sim a_1 \in Z_3.
\]
(20)
what, together with (6) and (17), contradicts (4).

In the case III), from (1), (5) and (17), it follows that

\[
(\forall e \in e_i^i)[h^e(\sim a_1) \equiv (a_1 \rightarrow a_1) \in Z_3 \Rightarrow h^e(a_1) \\
\epsilon Z_3 \Rightarrow h^e(a_1) \in Z_3]
\]
(21)

From (21) one can obtain that
\[
(\forall e \in e_i^i)[h^e(a_1) \in Z_3].
\]
(22)
Hence,
\[
a_1 \in Z_3.
\]
(23)

Thus, (23) together with (6) and (17), contradicts (4).

In the case III), from (1), (5) and (17), it follows that

\[
(\forall e \in e_i^i)[h^e(\sim a_1) \equiv (a_1 \rightarrow \sim a_1) \in Z_3 \Rightarrow h^e(a_1) \\
\epsilon Z_3 \Rightarrow h^e(\sim a_1) \in Z_3]
\]
(24)

Hence, from (1), it follows that
\[
(\forall e \in e_i^i)[h^e(a_1) \in Z_3 \Rightarrow h^e(\sim a_1) \in Z_3].
\]
(25)
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Let

\[ r = \{ (h^e(\alpha_1), h^e(\neg\alpha_1)) : e \in \varepsilon_l^1 \} \] (26)

Hence, from (1) and (25), it follows that

\[ r \in \text{Struct}_{S_1} \cap \text{Perm}(R_{0,1}^L, L_2^L \cup X) \text{.} \] (27)

Hence, from (1) and (3), it follows that

\[ r \in \text{Der}(R_{0,1}^L, L_2^L \cup X) \text{.} \] (28)

From (1), (26) and (28), by **THEOREM 1**, it follows that

\[ (\forall e \in \varepsilon_l^1) [ h^e(\neg\alpha_1) \in Z_3 ] \] (29)

From (29) it follows that

\[ \neg\alpha_1 \in Z_3 \text{,} \] (30)

what, together with (6) and (17), contradicts (4).

In the case IV), from (1), (5) and (17), it follows that

\[ (\forall e \in \varepsilon_l^1) [ h^e(\alpha_1 \equiv (\neg\alpha_1 \rightarrow \alpha_1)) \in Z_3 \]
\[ \Rightarrow h^e(\neg\alpha_1) \in Z_3 \Rightarrow h^e(\alpha_1) \in Z_3 \] \] (31)

Hence, from (1), it follows that

\[ (\forall e \in \varepsilon_l^1) [ h^e(\neg\alpha_1) \in Z_3 \Rightarrow h^e(\alpha_1) \in Z_3 ] \] (32)

Let

\[ r = \{ (h^e(\neg\alpha_1), h^e(\alpha_1)) : e \in \varepsilon_l^1 \} \] (33)

From (1), (32) and (33), it follows that

\[ r \in \text{Struct}_{S_1} \cap \text{Perm}(R_{0,1}^L, L_2^L \cup X) \text{.} \] (34)

From (1), (3) and (34), it follows that

\[ r \in \text{Der}(R_{0,1}^L, L_2^L \cup X) \text{.} \] (35)

Hence, from (1) and (33), by **THEOREM 1**, one can obtain that

\[ (\forall e \in \varepsilon_l^1) [ h^e(\alpha_1) \in Z_3 ] \] (36)

From (36), we obtain that

\[ \alpha_1 \in Z_3 \text{,} \] (37)

what, together with (6) and (17), contradicts (4). This completes the proof.

\[ \square \]

Finally (see also [11] and [14]):

**Theorem.** Let \( X \subseteq S_1 \) and \( \text{Cnt}(R_{0,1}^L, L_2^L \cup X) = Z_3 \). Then,

\[ \langle R_{0,1}^L, L_2^L \cup X \rangle \in \text{SCpl}_{S_1} \Leftrightarrow \]

\[ (\forall \alpha \in Z_3 \forall \neg\alpha \in Z_3 \text{.} \] (38)

**Proof.** By **Lemma 3** and **Lemma 4**.

\[ \square \]

**Remark.** The notion of the structural rule in propositional calculus was defined in [3] by J. Los and R. Suszko.

In [4] W. A. Pogorzelski introduced the notion of the structural completeness of the propositional calculus.

In [5] W. A. Pogorzelski and T. Prucnal introduced the notion of the structural completeness of the predicate calculus (see also [7] and [8], p. 103).

**References**


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