Surface Elastic Waves of Semi-infinite Superlattice: On the Acoustic-Electromagnetic-Quantum Analogies

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Abstract: In this work, the propagation of transverse elastic waves and the existence and behavior of surface acoustic modes associated with the interface between an isotropic medium and a semi-infinite of periodically layered media, consisting of alternating isotropic layers composed by the elements, W/Al are investigated. The dispersion relations for such waves are obtained analytically and solved numerically. It has shown that the existence of surface modes is dependent upon the relative parameters of the isotropic medium and the semi-infinite superlattice. Furthermore, we study the mathematical analogy between acoustic, electromagnetic waves and quantum confined states. A general mathematical expression that described physical phenomena of different nature is obtained.

Key words: Surface states, acoustical properties, low-dimensional structures (superlattices).

1. Introduction

An important tool for the development of physics is the use of analogies between distinct fields. These analogies allow us to describe, with the same mathematical theory, physical phenomena of different nature. Analogies have been used to obtain a wide variety of significant results in the area of the solid state [1-5]. One of these prominent analogies is the one that settles down between the confined quantum states and electromagnetic and acoustic propagation waves. We can obtain a complete parallelism between their wave equation, and a related mathematical development can be used to describe them.

The superlattice (SL) is a structure in the form of alternating plane-parallel layers of two materials differing in characteristics (Fig. 1), depending on the physical property of interest; dielectric, elastic impedance or quantum confinement. The frequency spectrum of SLs contains a system of both a great number of frequency bands and gaps corresponding to forbidden frequencies of modes or the dispersion curves $\omega(k)$. In order to characterize such spectra qualitatively and to illustrate their main quantitative features, we can use a simple model that permits to explain the existence of the allowed and forbidden energy bands; it is Kronig-Penney model [6]. For the quantum case, this model considers a crystalline solid where the ions are in the points of the net, and the electrons can move through the whole solid. Then, the electrons are in an approximately periodic potential. By means of wells and barriers, this periodic potential can be modeled (the superlattice). The interaction between electrons and the potential brings the appearance of well-known bands, allowed and forbidden. Those allowed single-particle states, each with real values of the wave number $k$, are Bloch waves, and are free or delocalized in the sense that the wave functions extend throughout the sample. Surface states exist when symmetry of infinite periodic superlattice is broken. In this case, when we try to match the boundary conditions, the wave number $k$ has to become complex, allowing wave function exponentially decays away from the...
surface, then a new state appears, one for each band. Since the superficial state has a frequency that corresponds to states prohibited in the volume subsequently the mode is spatially localized at the sample edge. The superficial states have been observed experimentally in finite superlattices by Ohno [7]. Their existence was predicted in the quantum mechanics for Tamm [8]. Bloss obtained an analytic equation for these states in a semi-infinite system of wells and barriers [9]. Surface waves have been first considered in the context of solid state and condensed matter physics, but they are presented in several areas of science, physics, chemistry, biology, and show properties that have no counterpart in the bulk [10].

The Kronig-Penney model that helps us to understand the physical quantum properties of the crystalline solids has been used by Ortega-Montiel and Vázquez et al [5], to study the propagation of longitudinal electromagnetic waves in a system of alternating conducting layers arranged periodically as a system of wells and barriers. This study is analogous with the quantum [9] and electromagnetic cases [5]. In this work, we present a study of the acoustic longitudinal wave propagation, along the perpendicular direction to the multilayer surface, in a periodic structure composed of two alternating elastic materials differing in elastic module and sound velocities. It shows a great similarity in the mathematical treatment with the quantum and electromagnetic Kronig-Penney models.

2. Semi-infinite Superlattice and the Motion Equation

We consider a system formed by three materials, stacked along the x direction; it consists of a semi-infinite slab, c, and semi-infinite SL of alternating plane-parallel layers of thickness a and b, the semi-infinite SL period is \( L = a + b \), as shown in Fig. 1. The material parameters that characterize each medium are: mass densities, \( \rho_i \), sound velocities, \( c_i \), and acoustic impedance \( Z_i = \rho_i c_i \) with \( i = a, b, c \).

The analogies between the differential equations, shown in Table 1, for quantum, electromagnetic and acoustic cases, and applying similar boundary conditions lead to an analogous solution of the propagation problem. These physically different cases are isomorphic one to each other from a mathematical point of view [11]. This enables to write the dispersion relations for all these collective excitations in one common mathematical expression in which two parameters take different values for every kind of physical system.

The results presented here were calculated within the continuum theory of elasticity [12]. First, we consider transverse wave propagation in an infinitely extended layered structure. The elastic equation of motion [13], for waves propagating in the x-z plane, in the first medium a, for the displacement \( u_y \), may be written as:

\[
\frac{\partial^2}{\partial x^2} u_y(x, z, t) = c_{ta}^2 \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u_y(x, z, t)
\]  

(1)

Where, \( c_{ta} \) is the sound velocity in the medium a. The solution in form of a plane wave propagating along x direction in the medium a is

\[
u_y(x, z, t) = \exp[i(k_z - \omega t)] u_y(x)
\]  

(2)

Here, \( k_z \) is the component of the wave vector, which is parallel to the surface. When we substitute Eq. (2) in Eq. (1) taking \( k_a = \sqrt{k_z^2 - \omega^2/c_{ta}^2} \), we then obtain:

\[
k_a^2 u_y(x) = c_{ta}^2 \frac{\partial^2 u_y(x)}{\partial x^2}
\]  

(3)

We can also take as a general solution of Eq. (3):

\[
u_y(x) = A_+ e^{k_a x} + A_- e^{-k_a x}
\]  

(4)

Since we have periodicity in layered structure, we
Table 1  Analogies between quantum, electromagnetic and acoustic cases.

<table>
<thead>
<tr>
<th>Acoustic</th>
<th>Quantum</th>
<th>Electromagnetic</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \frac{d^2}{dx^2} + \frac{1}{c^2} [e^{c^2k_z^2} - \omega^2] ] ( u(x) = 0 )</td>
<td>[ \frac{d^2}{dx^2} + \frac{2m}{\hbar^2} [E - V(x)] ] ( \psi(x) = 0 )</td>
<td>[ \frac{d^2}{dx^2} - q^2 ] ( \phi(x) = 0 )</td>
</tr>
<tr>
<td>( u(x) = A_ke^{ikx} + A_{-}e^{-kx} )</td>
<td>( \psi(x) = A_ke^{ikx} + A_{-}e^{-kx} )</td>
<td>( \phi(x) = A_ke^{iqx} + A_{-}e^{-iqx} )</td>
</tr>
<tr>
<td>( k = \frac{1}{\sqrt{c^2} [\omega^2 - c^2k_z^2]} )</td>
<td>( K = \frac{2m}{\hbar^2} [E - V(x)] )</td>
<td>( k = \frac{1}{\sqrt{c^2} [\omega^2 - \omega_f^2]} )</td>
</tr>
</tbody>
</table>

apply the Bloch condition:
\[ u_y(x) = e^{iqx}u_y(q, x) \] (5)
where, \( q \) is the Bloch wave vector. \( u_y(q, x) \) satisfies:
\[ u_y(q, x) = u_y(q, x + L) \] (6)

We can get \( u_y(q, x) \) from Eqs. (4) and (5):
\[ u_y(q, x) = e^{-iqx}[A_ke^{ikx} + A_{-}e^{-kx}] \] (7)

The function \( u_y(q, x) \) is periodic with period \( L \), if we replace \( x \) by \( xL \), we can write:
\[ u_y(q, x) = e^{-iqx}[A_ke^{ikx} + A_{-}e^{-kx}] \] (8)

Substituting Eqs. (5) and (8) in Eq. (6), we consequently have:
\[ u_{yn}(x) = e^{iqx}[A_ke^{ikx} + A_{-}e^{-kx}] \] (9)

Considering:
\[ nL < x < nL + a \]

In a similar way, we can get an expression for the displacement in the b regions, using:
\[ k_b = \sqrt{\frac{\omega^2}{c^2} - k_z^2} \]

Therefore we write:
\[ u_{yn}(x) = e^{iqx}[B_{+}e^{k_b(x-nL-a)} + B_{-}e^{-k_bx-nL-a}] \] (10)

Considering:
\[ nL + a < x < (n + 1)L \]

Here we want to outline that in the last equations \( n \) labels the layer unit cell starting from cell 0 at the interface. Notice that for this infinite superlattice, \( x \) is only defined for each cell, and bounded as \( 0 < x < L = a + b \).

The elastic wave, that is transmitted through the infinite superlattice, should fulfill the boundary conditions, this results in the continuity of the displacements, \( u_{yn}(x) \), of the acoustic waves and the continuity of the normal components of the stress, \( \sigma_n(x) = \rho c^2 [\partial u_{yn}(x)/\partial x] \), across different interfaces [14].

Applying the boundary conditions: continuity of \( u_{yn}(x) \) and \( n(x) \) at \( x = nL + a \) and \( x = nL \), derives in the next set of equations:
\[ A_{+}e^{k_{a}x} + A_{-}e^{-k_{a}x} = B_{+} + B_{-} \] (11)
\[ k_{a}k_{a}c_{a}^2A_{+}e^{k_{a}x} + A_{-}e^{-k_{a}x} = k_{b}k_{b}c_{b}^2B_{+} + B_{-} \] (12)
\[ A_{+} + A_{-} = e^{-iqx}[B_{+}e^{k_{b}x} + B_{-}e^{-k_{b}x}] \] (13)
\[ k_{a}k_{a}c_{a}^2A_{+} - A_{-} = e^{-iqx}[B_{+}e^{k_{b}x} + B_{-}e^{-k_{b}x}] \] (14)

From this set of equations, Eqs. (11)-(14), and the solvability conditions, we obtain the following dispersion relation for propagating modes:
\[ \cos(qL) = \cos(k_{a}a)\cos(k_{b}b) + \frac{1}{2}F + \frac{1}{2}F \sinh(k_{a}a)\sinh(k_{b}b) \] (15)
where,
\[ F = \frac{k_{a}k_{b}c_{a}c_{b}}{k_{a}k_{b}c_{a}c_{b}} \] (16)

We now look for surface wave solutions. The surface modes arise when the periodicity of the semi-infinite SL is interrupted and additional modes appear. The surface modes have frequencies that correspond to bands prohibited in the volume. These modes are confined to the crystalline border. They spread for the frontier diminishing exponentially toward the interior of the structure. The solution in form of a surface mode along direction \( x \) is a decaying wave function away for the surface at \( x = 0 \) into the superlattice. In order to find these states, we considered a semi-infinite superlattice with a and b layers in contact with a semi-infinite slab. For the homogeneous slab of material c with
for $x < 0$ the wave function decays exponentially as 

$$u_{yn}(x) = Ce^{k_c x}$$  \hspace{1cm} (17)$$

Applying again the boundary conditions: continuity of $u_{yn}(x)$ and $u_n(x)$ but now at $x = 0$ and $x = L$, we can obtain the surface states equations [5],

$$k_a \rho_a c_a^2 \left[ k_b^2 \rho_b c_b^2 + k_c^2 \rho_c c_c^2 \right] - k_c \rho_c c_c^2 \left[ k_b^2 \rho_b c_b^2 + k_c^2 \rho_c c_c^2 \right] \tanh(k_c b)$$

$$= k_a \rho_a c_a^2 \cot(k_a a)$$  \hspace{1cm} (18)$$

The Eq. (18) must be solved to obtain the dispersion curves. There are several different types of solutions, in Eq. (18) we take that $k_a$ will be pure imaginary and $k_b$ will be real. Solving this equation, we find the frequencies of surface mode states for the acoustic system studied.

3. Results

Comparing the Eqs. (15) and (18) for acoustic waves, with the Eqs. (18) and (31) of Ref. [5] for electromagnetic waves and with the Eqs. (12) and (20) of Ref. [9] for confined quantum states, we can write the following general equations:

$$\cos(qL) = \cosh(ka)\cosh(Kb) + \frac{1}{2} \frac{k + K}{k + K} \sinh(ka)\sinh(Kb)$$  \hspace{1cm} (19)$$

$$\frac{k^2 - 1}{k^2 + 1} \left[ k^2 + 1 \right] \sinh(k_b \gamma) + \frac{k^2 - 1}{k^2 + 1} \tanh(k_c \gamma) = k_c \cot(k_a a)$$  \hspace{1cm} (20)$$

The corresponding expressions for $k$ and $K$ for acoustic, electromagnetic and quantum systems are provided in Table 2.

Now, for acoustic waves, we apply the above theoretical results for the surface states of a semi-infinite SL and semi-infinite homogeneous elastic medium. For all cases, we have taken $k_a a = 1$. The appropriate choice of parameters for media a, b and c, enables a surface state with desired energy position. We consider for the calculations a semi-infinite SL of W/Al alternating layers, W as medium a and Al as medium b. In order to achieve a complete study, we try three different cases for the medium c: First one, the material of medium c is different to media a and b; second one, the material of medium c is equal to medium b; and the last one, for $x = 0$, we have a free surface. We also explore the effect of the stiffness $\gamma$ of the substrate c.

3.1 Semi-infinite Layered Structure, a-b, in Contact with a Semi-infinite Clad c

Here we consider for the medium c, in the structure illustrated in Fig. 1, six different materials, Pt, Au, Ge, Si, Epoxi and Polyethylene. For each of these systems, we show only the first band and the corresponding surface state. In Fig. 2, we show, for W/Al infinite SL, the exact solution to Eq. (15) (solid lines) and the solution to Eq. (18) for the surface state (dotted lines) for each of the different materials for medium c. Additionally, we used the reduce frequency $\omega a/c_a$ as a function of the width of the medium b normalized with the width of the medium a. The corresponding parameters for the different media are shown in Table 3. In Fig. 2 we observe that the bulk band width is diminished as $b/a$ is increased at fixed $a$. The position of the center of the permitted band remains unchanged. For the case of surface modes, we can see that for a

| Table 2 | Parameters for acoustic, electromagnetic and quantum systems. |
|---|---|---|
| **Acoustic** | **Quantum** | **Electromagnetic** |
| $k$ | $\rho_a c_a^2 \left( \frac{\omega^2}{c_a^2} - k_z^2 \right)$ | $\left( \frac{2m}{\hbar^2} \right)^{1/2} \sqrt{\varepsilon}$ | $1 \frac{v_a}{\beta_a \mu_a \hbar} \sqrt{\omega^2 - \omega_p^2}$ |
| $K$ | $\rho_b c_b^2 \left( \frac{\omega^2}{c_b^2} - k_z^2 \right)$ | $\left( \frac{2m}{\hbar^2} \right)^{1/2} \sqrt{V_0 - \varepsilon}$ | $1 \frac{v_b}{\beta_b \mu_b \hbar} \sqrt{\omega_p^2 - \omega^2}$ |
| $K'$ | $\rho_c c_c^2 \left( \frac{\omega^2}{c_c^2} - k_z^2 \right)$ | $\left( \frac{2m}{\hbar^2} \right)^{1/2} \sqrt{V_0 - \varepsilon}$ | $1 \frac{v_c}{\beta_c \mu_c \hbar} \sqrt{\omega_p^2 - \omega^2}$ |
high (Pt, Au, Ge, Si) or low (Epoxi and polyethylene) acoustic impedance $Z$ causes that the corresponding surface states to appear above or below the infinite SL bulk band. On the other hand, the values for the $\omega$ frequencies for the surface modes keep almost constant for enough large values of $b/a$, and grow or diminished when $b/a$ is reduced until they cease to exist for small values of $b/a$.

When medium c is the same material of the medium b, we have a simple solution for the dispersion relation; it is given by [13]:

$$\omega^2 = \left[ k_x^2 + \left( \frac{mn}{a} \right)^2 \right] c_{ta}^2 \cdot n = 0, 1, 2, 3, \ldots$$ (21)

We can see that the values for the $\omega$ frequencies, for the surface modes, do not depend at all of width changes of material b.

3.2 Semi-Infinite Layered Structure, a-b, in Contact with a Free Surface

In this case, the Eq. (18) is not longer valid, here; the boundary condition is that the surface $x = 0$ is stress free, and the corresponding dispersion relation is [13]

$$\frac{k_a \rho_a c_t^2}{k_b \rho_b c_b^2} \tanh (k_b b) + \tanh (k_a a) = 0$$ (22)

The Fig. (3) shows the surface mode values that we obtain from the solution of Eq. (22). We can see similar dependence with the width of the medium b for the values for the $\omega$ frequencies for the surface modes shown in the case of the medium c with small acoustic impedance $Z$ (epoxi and polyethylene).

3.3 Substrate Stiffness

We shall discuss the effect of the stiffness related with the medium c. For this purpose, the media forming the layers of the semi-infinite SL and the medium c are characterized by their mass densities $\rho_i$ and their elastic constants $C_{44i}$, here, $c_{ti}^2 = C_{44i}/\rho_i$. The sound velocity in the medium c is taken to be equal to $\beta$ times to those of the medium b (Al), $(c_{tc} = \beta c_{tb})$. We define the parameter $\gamma$ as:

$$\omega^2 = k^2 + \left( \frac{mn}{a} \right)^2 \frac{c_{ta}^2}{c_{tb}^2} \cdot n = 0, 1, 2, 3, \ldots$$
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Fig. 4 Normalized frequencies $\omega a/c_a$ of bulk band for infinite periodic superlattice (solid line) and for the surface state of a semi-infinite periodic layer structure in contact with an elastic half space with several stiffness of the substrate (dashed lines) as a function of normalized medium width $b/a$.

The parameter $\gamma$ represents the stiffness of the substrate, which enables us to change smoothly from the case of a free semi-infinite SL surface ($\gamma = 0$) to the case of a rigidly bound semi-infinite SL surface ($\gamma \rightarrow \infty$). Fig. 4 shows the cases of a soft medium $c$, for the weak $\gamma$ values ($\gamma = 0.1$ and 1), and a stiff medium $c$, for the strong $\gamma$ values ($\gamma = 4$ and 6). We can notice that the dependence of the frequencies of surface modes, as a function of acoustic impedance of medium $c$ ($Z_c$), gives similar behavior as a function of stiffness parameter $\gamma$, as seen in Figs. 3 and 4. Behavior of high (Pt, Au, Ge, Si) acoustic impedance $Z$ media are similar to strong $\gamma$ values and low (Epoxi and Polyethylene) acoustic impedance $Z$ media are similar to weak $\gamma$ values.

4. Conclusions

We modeled a system of conducting alternating a and b layers of elastically isotropic materials as a system of materials with different acoustic impedance, particularly we studied here a W/Al semi-infinite SL, in contact with a semi-infinite homogeneous elastic medium $c$. We used the continuum theory of elasticity; we found a dispersion relation considering various types of material boundaries in the structure. With the results presented in this work for transverse acoustic waves, we obtained a complete parallelism with electromagnetic and quantum cases. All these physically different phenomena are isomorphic one to each other from the mathematical point of view. This enables to write an analytical general expression for dispersion relations, in which a few parameters take different values for every kind of physical system. The analogy constitutes a mathematical equivalence that allows the acoustic, electromagnetic and quantum systems to be solved with the same analytical methodology. The most interesting application of the analogy is the use of the same computer code to solve acoustic, electromagnetic and quantum propagation problems, but, of course, with different dispersion relations. We obtained the behavior of the system when the termination of the semi-infinite SL is varied with different semi-infinite homogeneous elastic media. The frequency values of surface modes are strongly dependent on the semi-infinite SL termination; we used six different materials (Pt, Au, Ge, Si, epoxi and polyethylene) and simulate different stiffness for the medium $c$ by means of a couple of parameters, $\beta$ and $\gamma$ that simulates from stress-free to rigid boundary conditions. In addition, the acoustic impedance $Z$ or material stiffness $\gamma$ of media behave similarly to the barrier height in the quantum system and to the electronic densities in the electromagnetic system. We used in all the calculations $k, a = 1$, but the surface modes can be obtained even with $k, a = 0$. In this limit, the displacement profile remains as a surface wave in the perpendicular direction inside the layered medium but does not oscillate along the surface direction. The calculations presented here are limited to transverse waves, but it is easy to extend these ones to longitudinal acoustic waves.

References

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