Free Vibration of Geometrically Nonlinear Functionally Graded Micro-switches under Electrostatic Force and Temperature Change

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Abstract: The free vibration characteristics of functionally graded micro-switches under combined electrostatic, axial residual stress and temperature change is investigated, with an emphasis on the effect of geometric nonlinear deformation due to mid-plane stretching, the influence of volume fraction profile parameter and temperature change. The micro-switch considered in this study is made of either homogeneous material or non-homogeneous functionally graded material with two material phases. Taking the temperature-dependency of the effective material properties into consideration, the Voigt model is used to simulate the material properties of the FGMs (functionally graded materials). The principle of virtual work is used to derive the nonlinear governing differential equation. The eigenvalue problem which describes free vibration of the micro-beam at its statically deflected state is then solved using DQM (differential quadrature method). The natural frequencies of clamped-clamped micro-switches are obtained. The solutions are validated through direct comparisons with experimental results reported in previous studies. A parametric study is conducted to show the effects of geometric nonlinearity, material composition, temperature change and geometrical parameters for the natural frequencies.

Key words: Micro-switch, free vibration, functionally graded materials, temperature change.

1. Introduction

With MEMS (micro-electro-mechanical systems) are used more and more widely due to their unique advantages, such as small size, lower power consumption, lower operation cost, increased reliability and higher precision, numerous analytical, numerical and experimental studies have been conducted on the pull-in instability of the MEMS devices [1]. As an important building block in MEMS, micro-switches overcome the limitations of conventional electromechanical and solid state switches. They have gained tremendous attentions in both research and industry communities.

A typical electrostatic micro-switch comprises of an electrostatically driven micro-beam actuator and a

fixed electrode. When an electrical potential difference is created between the two electrodes, the induced electrostatic charge gives rise to electrostatic force. Counteracting the electrostatic force is the elastic force, which tries to restore the movable electrode to its original position. Meanwhile, an instantaneous excitation will induce free vibration of the micro-beam at the deflected state and the natural frequencies of the system are influenced by the static deflection [2-4].

To overcome the difficulty for a single layer to meet all material and economical requirements for a MEMS structural layer, Witvrouw and Mehta [5] proposed a novel non-homogenous functionally graded layer polycrystalline-SiGe (poly-SiGe) layer in MEMS in order to achieve the desired electrical and mechanical properties. With their associates, they also conducted an experimental study for the characterization and strain gradient optimization of PECVD (plasma

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enhanced chemical vapor deposition) poly-SiGe layers for MEMS applications [6]. Recently, Jia et al. [7, 8] studied the pull-in instability and free vibration of electrostatically actuated FGM (functionally graded material) micro-beams allowing for the geometric nonlinearity and intermolecular Casimir force, but without considering the temperature change. Mohammadi-Alasti et al. [9] investigated the mechanical behavior of a cantilever FGM micro-beam subjected to a nonlinear electrostatic pressure and a temperature change, giving no consideration of geometric nonlinearity and Casimir force.

The objective of this paper is to present a free vibration analysis for both the homogeneous and non-homogeneous micro-switches under a combined electrostatic and temperature change within the framework of von Karman-type geometric nonlinearity and Euler-Bernoulli beam theory. The DQM (differential quadrature method) is used to numerically determine the natural frequencies for clamped-clamped micro-switches. The present analysis focuses on the effects of geometric nonlinearity, material composition, and temperature change on the natural frequencies. To the authors' best knowledge, no previous work has been conducted in open literature. The paper is organized as follows: Section 2 gives the theoretical formulation; Section 3 introduces the solution procedures; Section 4 presents the results and discussions; and Section 5 gives conclusions.

2. Theoretical Formulation

2.1 FGM Beam Model

Fig. 1 shows the structure of a typical micro-switch where the key components include a fixed electrode modeled as a ground plane and a movable electrode modeled as a micro-beam of length L , width b , and thickness h , separated by a dielectric spacer with an initial gap g_0 . The axial force due to residual strain from fabrication process is denoted by N_a and is positive for a tensile force. Upon the application of an applied voltage V which consists of a DC (direct

Fig. 1 A beam model for a micro-switch.

current) component V_0 and a small time-varying AC (alternating current) component $\Delta V(t)$, the micro-beam deflects towards the fixed electrode and vibrates under the combined action of distributed electrostatic force F_e , Casimir force F_c and temperature change. A rectangular coordinate system is applied here where the origin is located at the left end of the movable electrode with deflection denoted by w . It should be noted that the small AC voltage $\Delta V(t)$ will induce electromagnetic forces in both x- and z-directions. These forces, however, are neglected in the present study as they are much lower than both F_e and F_c .

Taking into account the first-order fringing field correction, the electric field force per unit length can be written as:

$$F_e = \frac{\epsilon_0 b V^2}{2(g_0 - w)^2} + \frac{0.65 \epsilon_0 V^2}{2(g_0 - w)} \quad (1)$$

where, $\epsilon_0 = 8.854 \times 10^{-12} \text{ C}^2 \text{N}^{-1} \text{m}^{-2}$ is the permittivity of vacuum. The Casimir force takes the form of:

$$F_c = \frac{\pi^2 \hbar c b}{240(g_0 - w)^4} \quad (2)$$

In this paper, the FGM micro-beam is a mixture of silicon nitride (Si_3N_4) as material phase 1 whose volume fraction is V_1 , and nickel (Ni) as material phase 2 whose volume fraction is V_2 . The mixing ratio changes smoothly and continuously along the thickness direction. The volume fraction of silicon nitride V_1 and that of nickel V_2 are given by Eq. (3) [10]:

$$V_1 + V_2 = 1, V_2(z) = \left(\frac{2z + h}{2h} \right)^n \quad (3)$$

where, the superscript “ n ” is a power law index that defines the volume fraction profile through beam thickness and can be varied to tailor for desired performance. It is noted that $n = 0$ corresponds to a pure N_i micro-beam. The volume fraction profile thus defined is Si_3N_4 -rich at the top beam surface ($z = -h/2$) whereas it is N_i -rich at the bottom surface ($z = h/2$).

According to the Voigt model, the effective material properties P_f of the FGM layer, like Young's modulus E_f , Poisson's ratio ν_f and thermal expansion coefficient α_f can then be expressed as:

$$P_f = \sum_{j=1} P_j V_j, \quad \sum_{j=1} V_j = 1 \quad (4)$$

where, P_j and V_j are the material properties and volume fraction of the constitute material j .

Since FGM structures are most commonly used in the high-temperature environment where significant changes in mechanical properties of the constituent materials are to be expected [11], it is essential to take this temperature-dependency for accurate prediction of the mechanical response into consideration. Thus, the effective material properties P_f can be expressed as a nonlinear function of temperature [11].

$$P_j = P_0 (P_{-1} T^{-1} + 1 + P_1 T + P_2 T^2 + P_3 T^3) \quad (5)$$

where, P_0 , P_{-1} , P_1 , P_2 and P_3 are the coefficients of temperature T (in K) and are unique to the constituent materials. Typical values for Si_3N_4 and N_i are listed in Tables 1 and 2. From Eq. (4), one has [12]:

$$E_f(z, T) = [E_2(T) - E_1(T)](0.5 + \bar{z})^n + E_1 \quad (6)$$

$$\alpha_f(z, T) = [\alpha_2(T) - \alpha_1(T)](0.5 + \bar{z})^n + \alpha_1 \quad (7)$$

$$\nu_f(z, T) = [\nu_2(T) - \nu_1(T)](0.5 + \bar{z})^n + \nu_1 \quad (8)$$

2.2 Geometrically Nonlinear Governing Equations

It is noted that the ratio h/L is usually small so that the shear deformation is negligible. The total strain at the x direction is the sum of the mechanical strain ε_m and thermal strain ε_T , i.e., $\varepsilon_x = \varepsilon_m + \varepsilon_T$. For micro-beams undergoing moderately large deformation, von Karman type nonlinear strain ε_x is:

$$\varepsilon_x = \frac{du}{dx} - z \frac{d^2 w}{dx^2} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \quad (9)$$

and $\varepsilon_T = \alpha_f(T - T_0) = \alpha_f \Delta T$, where ΔT is the temperature change, measured with respect to the initial temperature T_0 . It is noted that this paper takes into account the uniform temperature change. The principle of virtual work requires $\delta(W_q + W_N + V) = 0$, which yields the dimensionless governing equations.

$$\begin{aligned} \frac{\partial^2 \bar{w}}{\partial \tau^2} + \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + \frac{(J_1 \Delta T - N_a) L^2}{k} \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \\ - \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \int_0^1 \left[\zeta_1 \left(\frac{\partial \bar{w}}{\partial \bar{x}} \right)^2 - \zeta_2 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \right] d\bar{x} = \bar{q} \end{aligned} \quad (10)$$

in which,

$$\bar{q} = \frac{B(V_0 + \Delta V(\tau))^2}{(1 - \bar{w})^2} + \frac{fB(V_0 + \Delta V(\tau))^2}{(1 - \bar{w})} + \frac{R_c}{(1 - \bar{w})^4}$$

The associated dimensionless boundary conditions are $\bar{w} = 0$, $\partial \bar{w} / \partial \bar{x} = 0$. Where,

$$\bar{w} = w/g_0, \quad \bar{x} = x/L,$$

$$\zeta_1 = k_1 g_0^2 / 2k, \quad \zeta_2 = k_2 g_0 / k \tau = t \sqrt{k / \rho_l L^4},$$

$$k = k_3 - k_2^2 / k_1, \quad k_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \hat{E}_f b dz,$$

$$k_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \hat{E}_f b z dz, \quad k_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \hat{E}_f b z^2 dz \quad (11)$$

$$\rho_l = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho b dz, \quad J_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \hat{E}_f \alpha_f b dz, \quad B = \frac{\varepsilon_0 b L^4}{2 g_0^3 k},$$

$$f = 0.65 \frac{g_0}{b}, \quad R_c = \frac{\pi^2 \bar{h} c b L^4}{240 g_0^5 k}$$

Table 1 Material properties of silicon nitride (Si_3N_4).

Material property	P_0	P_{-1}	P_1	P_2	P_3
Young's modulus E (Pa)	348.43×10^9	0	-3.070×10^{-4}	2.160×10^{-7}	-8.946×10^{-11}
Poisson's ratio ν	0.24	0	0	0	0
Thermal expansion coefficient α (K^{-1})	5.8723×10^{-6}	0	9.095×10^{-4}	0	0

Table 2 Material properties of nickel (Ni).

Material property	P_0	P_{-1}	P_1	P_2	P_3
Young's Modulus E (Pa)	223.95×10^9	0	-2.794×10^{-4}	-3.998×10^{-9}	0
Poisson's Ratio ν	0.31	0	0	0	0
Thermal expansion coefficient α (K^{-1})	9.9209×10^{-6}	0	8.705×10^{-4}	0	0

The deformation of micro-beam caused by the combined electrostatic and intermolecular forces and axial residual stress is composed of a static component induced by the DC voltage denoted by $(\bar{w}_0, \bar{u}_0)$, and a small dynamic part due to the AC voltage, denoted by $(\bar{w}_f, \bar{u}_f)$, that is:

$$\bar{w} = \bar{w}_0 + \bar{w}_f, \bar{u} = \bar{u}_0 + \bar{u}_f \quad (12)$$

Setting the time derivatives and the AC forcing terms in Eq. (11) to zero yields the equations from which the static deformation $(\bar{w}_0, \bar{u}_0)$ can be obtained. The present paper considers a small AC voltage which induces a small additional dynamic deflection of the micro-beam. By substituting the static deformation equations into Eq. (10), dropping the coupling terms between the dynamic response and the AC voltage part as they are very small, and noting that for harmonic motion, the additional dynamic response takes the form of:

$$\bar{w}_f(\bar{x}) = W_f(\bar{x})e^{i\omega\tau}, \bar{u}_f(\bar{x}) = U_f(\bar{x})e^{i\omega\tau} \quad (13)$$

The equations of motion governing the free vibration of the micro-beam at the deflected state can be obtained as:

$$\begin{aligned} & \frac{d^4 W_f}{d\bar{x}^4} + \frac{(J_1 \Delta T - N_a) L^2}{k} \frac{d^2 W_f}{d\bar{x}^2} - \\ & \frac{d^2 \bar{w}_0}{d\bar{x}^2} \int_0^1 \left(2\zeta_1 \frac{d\bar{w}_0}{d\bar{x}} \frac{dW_f}{d\bar{x}} - \zeta_2 \frac{d^2 W_f}{d\bar{x}^2} \right) d\bar{x} - \\ & \frac{d^2 W_f}{d\bar{x}^2} \int_0^1 \left[\zeta_1 \left(\frac{d\bar{w}_0}{d\bar{x}} \right)^2 - \zeta_2 \frac{d^2 \bar{w}_0}{d\bar{x}^2} \right] d\bar{x} - \bar{q}_f = \omega^2 W_f \end{aligned} \quad (14)$$

where, $W_f(\bar{x})$ is dimensionless amplitudes and ω denotes the dimensionless natural frequency. The associated boundary conditions become $W_f = 0$, $dW_f/d\bar{x} = 0$. Where,

$$\bar{q}_f = \left[\frac{4R_c}{(1-\bar{w}_0)^5} + \frac{2\bar{B}V_0^2}{(1-\bar{w}_0)^3} + \frac{f\bar{B}V_0^2}{(1-\bar{w}_0)^2} \right] W_f \quad (15)$$

3. Solution Procedures

The dynamic governing Eq. (14) and the associated boundary condition form a nonlinear ordinary differential equation system. Because analytical closed-form solution is impossible, the DQM is employed in the present study. The derivatives of dimensionless deflection $\bar{w}$ at an arbitrary point $\bar{x}_i$ are approximated by DQM as [13]:

$$\frac{d^k}{d\bar{x}^k} \{W_f, \bar{w}_0\} \Big|_{\bar{x}=\bar{x}_i} = \sum_{j=1}^N C_{ij}^{(k)} \{W_f(x_j), \bar{w}_0(x_j)\} \quad (16)$$

where, N is the total number of sampling points $\bar{x}_i$ unevenly distributed over the domain.

$$\bar{x}_1 = 0.0, \bar{x}_2 = 0.001, \bar{x}_j = \frac{1}{2} \left[1 - \cos \frac{\pi(j-2)}{N-3} \right], \quad (17)$$

$$\bar{x}_{N-1} = 0.999, \bar{x}_N = 1.0$$

The weighting coefficients $C_{ij}^{(k)}$ are dependent on the distribution of sampling points only and can be calculated from the recursive formulae given in Ref. [13]. The static deformations $\bar{w}_0(\bar{x})$ can be determined according the solution process detailed in Ref. [7]. Application of DQM approximations to equations of motion (14) gives:

$$\begin{aligned} & \sum_{j=1}^N C_{ij}^{(4)} W_{fj} + \frac{(J_1 \Delta T - N_a) L^2}{k} \sum_{j=1}^N C_{ij}^{(2)} W_{fj} - \\ & \sum_{j=1}^N C_{ij}^{(2)} \bar{w}_{0j} \sum_{k=1}^N (\mathbf{w}_{Nk}' - \mathbf{w}_{1k}') \left[2\zeta_1 \sum_{j=1}^N C_{ij}^{(1)} \bar{w}_{0j} \sum_{j=1}^N C_{ij}^{(1)} W_{fj} - \zeta_2 \sum_{j=1}^N C_{ij}^{(2)} W_{fj} \right] - \\ & \sum_{j=1}^N C_{ij}^{(2)} W_{fj} \sum_{k=1}^N (\mathbf{w}_{Nk}' - \mathbf{w}_{1k}') \left[\zeta_1 \left(\sum_{j=1}^N C_{ij}^{(1)} \bar{w}_{0j} \right)^2 - \zeta_2 \sum_{j=1}^N C_{ij}^{(2)} \bar{w}_{0j} \right] - \\ & \left\{ \frac{2\bar{B}V_0^2}{[1-\bar{w}_0(\bar{x}_i)]^3} + \frac{f\bar{B}V_0^2}{[1-\bar{w}_0(\bar{x}_i)]^2} + \frac{4R_c}{[1-\bar{w}_0(\bar{x}_i)]^5} \right\} W_{fi} = \omega_i^2 W_{fi} \end{aligned} \quad (18)$$

The boundary conditions considered here can be approximated as:

$$W_{f1} = 0, \sum_{j=1}^N C_{2j}^{(1)} W_{fj} = 0, \sum_{j=1}^N C_{N-1j}^{(1)} W_{fj} = 0, W_{fN} = 0 \quad (19)$$

The geometrically linear and nonlinear natural frequencies ω_i can be determined through the same procedure in Ref. [8] and will not be repeated herein for brevity.

4. Results and Discussions

4.1 Validation Example

Unless stated otherwise, the geometric parameters of the micro-beams considered in this section are given in Table 3. To validate the present analysis, the first natural frequency of C-C micro-beams is compared with experimental results provided by Tilmans and Legtenberg [2]. The micro-beam is made of pure poly-silicon in the validation example, and the Young's modulus $E = 151$ GPa. The comparisons are shown in Table 4, which show that convergent results can be achieved when $N \geq 11$.

4.2 Effect of Volume Fraction Index and Temperature

Fig. 2a exhibits the fundamental frequency ω_1 versus DC voltage for functionally graded micro-beams with varying volume fraction index. In which, the solid lines and the dashed lines geometrically represent nonlinear and linear results. The effect of temperature T and temperature change ΔT on the nonlinear frequency ω_1 was shown in Figs. 2b and 2c. In Fig. 2, curved lines plus squares, circles, triangles and stars represent the results for micro-beams with $n = 0.0, 1.0, 10.0, 50.0$, respectively.

It is shown that for the same voltage and temperature, the micro-beam with a larger n has a higher ω_1 because

Table 3 Geometric parameters of the poly-silicon micro-beam.

L (μm)	b (μm)	g_0 (μm)	h (μm)
410	100	1.18	1.5

Table 4 Comparison of nonlinear fundamental frequencies (kHz) of C-C micro-beams.

V_0 (V)	L (μm)	$N=9$	$N=11$	$N=13$	Experiment [2]
6	210	323.257	323.271	323.271	322.05
3	310	162.545	162.541	162.541	163.22
3	410	101.905	101.893	101.893	102.17
2	510	73.676	73.659	73.659	73.79

(a) Effect of volume fraction index n

(b) Effect of temperature T

(c) Effect of temperature change ΔT

Fig. 2 Fundamental frequency for a functionally graded micro-beam.

it contains more Si_3N_4 which has a higher Young's modulus than N_i . The same functionally graded micro-beam has a lower fundamental frequency ω_1 with a higher temperature T and larger temperature

change ΔT . It is noted that the temperature change has more noticeable effect. As can be observed from Fig. 2a, the effect of geometric nonlinearity on ω_1 tends to be more important as n decreases.

Figs. 3 and 4 display the effects of varying beam length L and beam thickness h on the nonlinear frequency for free vibration, respectively. Where the dash lines denote the results for micro-beams with $n = 0.0$, while the solid lines represent that with $n = 10.0$.

It is shown that the nonlinear frequency increases with an increase in h due to the fact that a larger h leads to the stiffen effect. As expected, the longer the beam is, the smaller the nonlinear frequency that the beam is able to get. It is worth noting that the effects of L and h are much more obvious for micro-beams with larger temperature change.

(a) $T = 300 \text{ K}, \Delta T = 0 \text{ K}$

(b) $T = 300 \text{ K}, \Delta T = 1 \text{ K}$

Fig. 3 The effect of beam length on the nonlinear frequency with different temperature change.

(a) $T = 300 \text{ K}, \Delta T = 0 \text{ K}$

(b) $T = 300 \text{ K}, \Delta T = 1 \text{ K}$

Fig. 4 The effect of beam thickness on the nonlinear frequency with different temperature change.

5. Conclusions

The free vibration of micro-switches under electrostatic force and temperature change is studied in this paper based on the Euler-Bernoulli beam theory and von Karman type geometric nonlinearity. The governing equations and boundary conditions are derived based on the principle of virtual work and they are subsequently solved numerically for the natural frequencies using the DQM approach. Comparison with published experimental results shows excellent agreement of results. It is noted that geometric nonlinearity causes stiffening effect to loading for a micro-beam. The nonlinear frequency increases with growing volume fraction index for a functionally graded micro-beam. It is also concluded that the nonlinear frequency decreases with growing

temperature and larger temperature change. Besides, the nonlinear frequency increases with an increase in beam thickness h and a decrease in beam length L . The results reported herein provide important information which serves as a reference for the design of micro-devices in MEMS applications with improved or desired performance.

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