The Gagliardo-Nirenberg-Sobolev Inequality for Non-Isotropic Riesz Potentials

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Abstract: In this study, the inequality of Gagliardo-Nirenberg-Sobolev type are established for non-isotropic Generalized Riesz Potential depending on λ -distance.

Keywords: Non-isotropic distance, potentials

1. Introduction

In this paper, we establish analogues well known Gagliardo-Nirenberg-Sobolev inequality [2] for Riesz potentials with non-isotropic kernel depended on λ - distance. Note that different problems for convolution type integrals with kernels depending on λ - distance were considered in [1], [3-5].

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be positive numbers with $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_n$ and

$\|x\|_\lambda = \left(|x_1|^{1/\lambda_1} + |x_2|^{1/\lambda_2} + \dots + |x_n|^{1/\lambda_n} \right)^n$, $x \in \mathbb{R}^n$. The

expression $\|x - y\|_\lambda$, $x, y \in \mathbb{R}^n$ is called the λ - distance between the points x and y .

Now we set

$$\Lambda_\alpha f(x) = \int_{\mathbb{R}^n} \|x - y\|_\lambda^{\alpha-n} f(y) dy, 0 < \alpha < n$$

this integral is called non-isotropic λ - Riesz potential.

Theorem 1. (Gagliardo-Nirenberg-Sobolev inequality). Assume $1 \leq p < q < n$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. There exists a constant C , depending only p , q and n , such that

$$\|\Lambda_\alpha f\|_{L_q(\mathbb{R}^n)} \leq C \|Df\|_{L_p(\mathbb{R}^n)}$$

for all $f \in C_c^1(\mathbb{R}^n)$.

Proof. It is obvious that,

$$\begin{aligned} \Lambda_\alpha f(x) &= \int_{\mathbb{R}^n} f(x-y) E_1(y) dy \\ &\quad + \int_{\mathbb{R}^n} f(x-y) E_\infty(y) dy \\ &= I_1(x) + I_2(x) \end{aligned}$$

where for some positive μ

$$E_1(y) = \begin{cases} \|y\|_\lambda^{\alpha-n}, & \text{if } \|y\|_\lambda \leq \mu \\ 0, & \text{if } \|y\|_\lambda > \mu \end{cases}$$

$$E_\infty(y) = \begin{cases} 0, & \text{if } \|y\|_\lambda \leq \mu \\ \|y\|_\lambda^{\alpha-n}, & \text{if } \|y\|_\lambda > \mu. \end{cases}$$

Applying the L_p norm we can see that

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} |I_1(x)|^p dx \right)^{1/p} \\ &= \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} E_1(y) f(x-y) dy \right|^p dx \right)^{1/p} \\ &\leq \int_{\mathbb{R}^n} |E_1(y)| \left(\int_{\mathbb{R}^n} |f(x-y)|^p dx \right)^{1/p} dy \\ &= \int_{\mathbb{R}^n} |E_1(y)| \left(\int_{\mathbb{R}^n} |f(t)|^p dt \right)^{1/p} dy \end{aligned}$$

$$\|I_1\|_{L_p} \leq C \|E_1(y)\|_{L_1} \quad (1)$$

Note that the norm $\|E_1\|_{L_1}$ can be easily calculated. Namely, passing to spherical coordinates by transformation $x_j = (\gamma \theta_j)^{\lambda_j}$, $j = 1, 2, \dots, n$, where θ_j are the coordinates of the point θ on unit sphere, we can see that Jacobian of this transformation is $\gamma^{2|\lambda|-1} \Omega(\theta)$ depend only on angles. Consequently, denoting $C_1 = \int_{S^{n-1}} \Omega(\theta) d\theta$ where S^{n-1} is the unit sphere in $\mathbb{R}^n$. We obtain

$$\|E_1(y)\|_{L_1} = \int_{\|y\|_{\lambda} \leq \mu} \|y\|_{\lambda}^{\alpha-n} dy \leq C_1 \mu^{2\frac{|\lambda|}{n}\alpha}.$$

Thus I_1 is finite. For $E_{\infty}(y)$ applying Hölder inequality

$$\begin{aligned} \|E_{\infty}(y)\|_q &= \left(\int_{\|y\|_{\lambda} > \mu} \|y\|_{\lambda}^{(\alpha-n)q} dy \right)^{1/q} \\ &= C \left(\int_{\mu}^{\infty} \rho^{2\frac{|\lambda|}{n}(\alpha-n)q} \rho^{2|\lambda|-1} dy \right)^{1/q} \\ &= C \left(\rho^{2\frac{|\lambda|}{n}(\alpha-n)q+2|\lambda|} \right)^{\frac{1}{q}} \Big|_{\mu}^{\infty}. \end{aligned}$$

By hypothesis

$$2\frac{|\lambda|}{n}(\alpha-n) + 2\frac{|\lambda|}{q} = 2|\lambda| \left(\frac{\alpha}{n} - \frac{1}{p} \right) < 0.$$

Thus

$$\|E_{\infty}\|_q < C. \quad (2)$$

By using (1) and (2) inequalities,

$$|\Lambda_{\alpha} f| \leq |f(t)|.$$

Now first, assume $p = 1$. f has compact support for each $i = 1, 2, \dots, n$, $x \in \mathbb{R}^n$, we have

$$f(t) = \int_{-\infty}^t U_{t_i}(t_1, \dots, t_{i-1}, y_i, t_{i+1}, \dots, t_n) dy_i$$

and so

$$\begin{aligned} |\Lambda_{\alpha} f| &\leq |f(t)| \leq \\ &\int_{-\infty}^{+\infty} |Df(t_1, \dots, t_{i-1}, y_i, t_{i+1}, \dots, t_n)| dy_i. \end{aligned}$$

Consequently,

$$\begin{aligned} |f(t)|^{\frac{n}{n-\alpha}} &\leq \\ \prod_{i=1}^n \left(\int_{-\infty}^{+\infty} |Df(t_1, \dots, t_{i-1}, y_i, t_{i+1}, \dots, t_n)| dy_i \right)^{\frac{1}{n-\alpha}} \end{aligned}$$

integrate this inequality with respect to t_1

$$\begin{aligned} \int_{-\infty}^{+\infty} |f|^{\frac{n}{n-\alpha}} dt_1 &\leq \left(\int_{-\infty}^{+\infty} |Df| dy_1 \right)^{\frac{1}{n-\alpha}} \\ &\cdot \left(\prod_{i=2}^n \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Df| dt_i dy_i \right)^{\frac{1}{n-\alpha}}. \end{aligned}$$

The last inequality resulting from general Hölder inequality.

We continue by integrating with respect to $t_2, t_3, \dots, t_n$ eventually to find.

$$\begin{aligned} &\int_{\mathbb{R}^n} |\Lambda_{\alpha} f|^{\frac{n}{n-\alpha}} dx \\ &\leq \prod_{i=1}^n \left(\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} |Df| dt_1 \dots dy_i \dots dt_n \right)^{\frac{1}{n-\alpha}} \\ &= \left(\int_{\mathbb{R}^n} |Df| dt \right)^{\frac{n}{n-\alpha}}. \end{aligned} \quad (3)$$

Consider now case that $1 < p < n$ we apply estimate (3) to $v := |\Lambda_{\alpha} f|^{\gamma}$ where $\gamma > 1$ is to be selected. Then

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} |\Lambda_{\alpha} f|^{\frac{\gamma n}{n-\alpha}} dx \right)^{\frac{n-\alpha}{n}} \\ &\leq C \left(\int_{\mathbb{R}^n} |\Lambda_{\alpha} f|^{(\alpha-1)\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Df|^p dx \right)^{\frac{1}{p}} \\ &= \left(\rho^{2\frac{|\lambda|}{n}(\alpha-n)\frac{p}{p-1}+2|\lambda|} \right)^{\frac{p-1}{p}(\gamma-1)} \left(\int_{\mathbb{R}^n} |Df|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

where

$$2\frac{|\lambda|}{n}(\alpha-n)\frac{p}{p-1} + 2|\lambda| = 2|\lambda| \left(\frac{\alpha}{n} - \frac{1}{p} \right) < 0$$

and $\gamma > 1$. Finally,

$$\left(\int_{\mathbb{R}^n} |\Lambda_{\alpha} f|^q dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |Df|^p dx \right)^{\frac{1}{p}}.$$

The proof is completed.

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