On Maximal, Discrete, and Area Operators

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Abstract: In this paper, we study the boundednesses of maximal operator g_* , the discrete operator g_d , and the area operator A on Bergman spaces.

Key words: Area operator, Bergman Space, Discrete Maximal operator, Hardy Space, maximal operator

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ denote the unit disk in the complex plane and $\mathbb{T} = \partial\mathbb{D} = \{z : |z| = 1\}$ its boundary, the unit circle. Littlewood-Paley operators have been well known, studied by a lot of people, and used to characterize function spaces, such as Hardy spaces, etc. In this paper, we mainly consider the characterizations of classical Bergman spaces by these operators. Here are two Littlewood-Paley operators we have studied:

$$g_*(f)(z) = \left(\int_0^1 (1-r) \sup_{\rho < r} |f'(\rho z)|^2 dr \right)^{1/2}$$

$$g_d(f)(z) = \left(\sum_{n=0}^{\infty} 2^{-2n} |f'(r_n z)|^2 dr \right)^{1/2}$$

where $r_n = 1 - 2^{-n}$ and $z \in \mathbb{D} \cup \mathbb{T}$.

A Luzin area function is defined to be

$$A(f)(z) = \int_{D_z} |f'(w)|^2 dm(w)$$

where

$D_z =$ the convex hull of $\{w : |w| < \frac{1}{2} |z|\} \cup \{z\}$.

For $0 < p < \infty$, $H^p(\mathbb{D})$ is the usual Hardy

space of functions analytic on the unit disk and the norm of f in H^p is as follows

$$\|f\|_{H^p} = \sup_{0 \leq r < 1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty.$$

It is standard that for $f \in H^p$, its nontangential limit exists almost everywhere on $\mathbb{T}$,

$$\lim_{r \rightarrow 1^-} f(re^{i\theta}) = f(e^{i\theta})$$

and

$$\|f\|_{H^p} = \|f(e^{i\theta})\|_{L^p(\mathbb{T})}.$$

we refer readers to [11] for Hardy space theory.

For $0 < p < \infty$, the Bergman space $A^p(\mathbb{D})$ is defined as the set of all functions f analytic in $\mathbb{D}$ with the following norm

$$\|f\|_{A^p(\mathbb{D})} = \left(\int_{\mathbb{D}} |f(z)|^p dm(z) \right)^{1/p} < \infty$$

where $z = x + iy$ and $dm(z) = \frac{dx dy}{\pi}$ in $\mathbb{D}$, the normalized area measure on $\mathbb{D}$. It is well known that $A^p(\mathbb{D})$ is a Banach space when $1 \leq p < \infty$ and when $0 < p < 1$, the space $A^p(\mathbb{D})$ is a quasi-Banach space with p -norm $\|f\|_{A^p(\mathbb{D})}^p$.

The main results of this note are to prove that for f is analytic, f in the Bergman space $\|f\|_{A^p(\mathbb{D})}$ if and only if $g_*(f)$ or $g_d(f)$ is in $L^p(\mathbb{D})$ and the similar results also obtained for the area operator A .

Throughout this paper, for two nonnegative expressions P and Q , $P \lesssim Q$ means that there exists a positive constant C , not depending on the main factors, such that $P \leq CQ$, and $P \approx Q$ means that $P \lesssim Q$ and $Q \lesssim P$.

2. Main Results and Their Proofs

For our convenience and reference, all main results and useful results are stated in this section and proofs are also given with some remarks.

When we prove Theorem 4 and 5 we need the following results on Hardy spaces due to Miroslav Pavlović, see Theorem 1 in [10].

Lemma 1. Let $0 < p < \infty$. For an analytic function, the following conditions are mutually equivalent:

- (a) $f \in H^p$;
- (b) $g_*(f) \in L^p(\mathbb{T})$;
- (c) $g_d(f) \in L^p(\mathbb{T})$.

Furthermore, there are constants C_1, C_2, C_3 independent of f such that

$$\begin{aligned} \|f\|_{H^p} &\leq C_1 \|g_*(f)\|_{L^p(\mathbb{T})} \leq \\ C_2 \|g_d(f)\|_{L^p(\mathbb{T})} &\leq C_3 \|f\|_{H^p}. \end{aligned}$$

In order to prove the second inequality of Theorem 4, we need a Theorem in [11] with $\alpha = 0$ (Theorem 2.30 on page 69). We state it as a lemma here.

Lemma 2. Suppose $0 < p \leq 1$. Then there exists a sequence $\{a_k\}$ in $\mathbb{D}$ such that $A^p(\mathbb{D})$ consists of functions of the form

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{1 - |a_k|^2}{(1 - \bar{a}_k z)^{\frac{2}{p}+1}}, \quad z \in \mathbb{D}$$

where $\{c_k\}$ belongs to the sequence space l^p and the series converges in the norm topology of $A^p(\mathbb{D})$.

Moreover,

$$\|f\|_{A^p(\mathbb{D})}^p \leq \sum_{k=1}^{\infty} |c_k|^p \|f_k\|_{A^p(\mathbb{D})}^p,$$

where

$$f_k(z) = \frac{1 - |a_k|^2}{(1 - \bar{a}_k z)^{\frac{2}{p}+1}}$$

and there is a constant C such that

$$C^{-1} \left(\sum_{k=1}^{\infty} c_k \right)^{1/p} \leq \|f\|_{A^p(\mathbb{D})} \leq C \left(\sum_{k=1}^{\infty} c_k \right)^{1/p}.$$

A relationship between Littlewood-Paley operators is listed here and we will need it in the proof of Theorem 5.

Lemma 3. For any $f \in A^p(\mathbb{D})$, $g_d(f) \lesssim g_*(f)$.

$$\begin{aligned} g_*(f)^2(z) &= \int_0^1 (1-s) \sup_{\rho < s} |f'(\rho z)|^2 ds \\ &= \sum_{n=0}^{\infty} \int_{r_n}^{r_{n+1}} (1-s) \sup_{\rho < s} |f'(\rho z)|^2 ds \\ &\geq \sum_{n=0}^{\infty} \sup_{\rho < r_n} |f'(\rho z)|^2 \int_{r_n}^{r_{n+1}} (1-s) ds \\ &\geq \frac{3}{8} g_d(f)^2(z). \end{aligned}$$

Theorem 4. For $0 < p < \infty$ and $f \in A^p(\mathbb{D})$ with $f(0) = 0$, the following are true

$$C_p' \|f\|_{A^p(\mathbb{D})} \leq \|g_*(f)\|_{L^p(\mathbb{D})} \leq C_p'' \|f\|_{A^p(\mathbb{D})}$$

where C_p' and C_p'' are constants depending only on p .

Moreover

$$\|f\|_{A^p(\mathbb{D})} \approx |f(0)| + \|g_*(f)\|_{L^p(\mathbb{D})}, \quad 1 \leq p < \infty$$

and

$$\|f\|_{A^p(\mathbb{D})}^p \approx |f(0)|^p + \|g_*(f)\|_{L^p(\mathbb{D})}^p, \quad 0 < p < 1$$

The basic idea of the proof of this theorem is due to Chen and Ouyang in [5] and based on some original results by Littlewood-Paley, Zygmund, and Lemma 1.

Proof of Theorem 4 For $0 < p < \infty$, let $f_r(z) = f(rz)$ for $0 < r < 1$ and $z \in \mathbb{D}$, $z = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, we have

$$\begin{aligned} \int_D |f(z)|^p dm(z) &= \int_0^{2\pi} \int_0^1 r |f(re^{i\theta})|^p \frac{dr d\theta}{\pi} \\ &= 2 \int_0^1 r \int_0^{2\pi} |f_r(e^{i\theta})|^p \frac{dr d\theta}{2\pi} \end{aligned}$$

By (1), it follows that

$$\begin{aligned} &\int_D |f(z)|^p dm(z) \\ &\approx 2 \int_0^1 r \int_0^{2\pi} |g_*(f_r)(e^{i\theta})|^p \frac{dr d\theta}{2\pi} \\ &\approx \int_0^1 \|g_*(f_r)\|^p r dr \\ &= \int_0^1 \int_0^{2\pi} \left[\int_0^1 (1-s) \sup_{\rho < s} |f_r(\rho e^{i\theta})|^2 ds \right]^{p/2} \frac{d\theta}{2\pi} r dr \\ &= \int_0^1 \int_0^{2\pi} \left[\int_0^1 (1-s) \sup_{\rho < s} |f'(r\rho e^{i\theta})|^2 r^2 ds \right]^{p/2} \frac{d\theta}{2\pi} r dr \\ &\lesssim \|g_*(f)\|_{L^p(\mathbb{D})}^p. \end{aligned}$$

The first inequality of Theorem 4 is proved.

For $0 < p \leq 1$, by Lemma 2, we get

$$f(z) = \sum_{k=1}^{\infty} c_k f_k(z).$$

Standard calculations give us

$$\begin{aligned} |f_k(\rho z)|^2 &= \\ (1 - |a_k|^2) \left(\frac{2}{p} + 1 \right)^2 |\bar{a}_k|^2 \frac{1}{|1 - \bar{a}_k \rho z|^{2(\frac{2}{p}+2)}} \end{aligned}$$

At this point we are in need of a very useful fact which is easy to check. For $0 < t \leq 1$, $z \in \mathbb{D}$, we have

$$|1 - tz| \leq (1 - t) + |1 - z| \leq 3|1 - tz|. \quad (1)$$

Now we have

$$\begin{aligned} &g_*(f_k)(z) \\ &\lesssim (1 - |a_k|^2) \left(\int_0^1 \sup_{\rho < r} \frac{1 - r}{[(1 - \rho) + |1 - \bar{a}_k z|]^{2(\frac{2}{p}+2)}} dr \right)^{1/2} \end{aligned}$$

$$\begin{aligned} &\lesssim (1 - |a_k|^2) \left(\int_0^1 \sup_{\rho < r} \frac{1}{[(1 - \rho) + |1 - \bar{a}_k z|]^{2(\frac{2}{p}+1)+1}} dr \right)^{1/2} \\ &= (1 - |a_k|^2) \left(\int_0^1 \frac{1}{[(1 - r) + |1 - \bar{a}_k z|]^{2(\frac{2}{p}+1)+1}} dr \right)^{1/2} \\ &\lesssim (1 - |a_k|^2) \frac{1}{|1 - \bar{a}_k z|^{\frac{2}{p}+1}}. \end{aligned}$$

Therefore, for $f = \sum_k c_k f_k$ with

$\sum_k |c_k|^p < \infty$, we have

$$\begin{aligned} &\int_D |g_*(f)(z)|^p dm(z) \\ &\leq \sum_{k=1}^{\infty} |c_k|^p \int_D |g_*(f_k)(z)|^p dm(z) \\ &\lesssim \sum_{k=1}^{\infty} |c_k|^p (1 - |a_k|^2)^p \int_D \frac{1}{|1 - \bar{a}_k z|^{2+p}} dm(z) \\ &\lesssim \sum_{k=1}^{\infty} |c_k|^p. \end{aligned}$$

Note that

$$\int_D \frac{1}{|1 - \bar{a}_k z|^{2+p}} dm(z) \approx \frac{1}{(1 - |a_k|^2)^p}.$$

Next for the case of $1 < p < \infty$, if $f \in A^{-p}(\mathbb{D})$, then f has the integral representation [11] (Theorem 2.2 on page 40),

$$f(z) = \int_D \frac{f(w)}{(1 - \bar{w}z)^2} dm(w), \quad \forall z \in \mathbb{D}.$$

So

$$|f'(\rho z)| \leq (1 - |\rho z|)^{-1/2} \int_D \frac{|f(w)|}{|1 - \rho \bar{w}z|^{5/2}} dm(w).$$

Hence we have

$$\begin{aligned} &g_*(f)^2(z) \lesssim \\ &\int_0^1 \sup_{\rho < r} \frac{1 - r}{(1 - |\rho z|)} \left[\int_D \frac{|f(w)|}{|1 - \rho \bar{w}z|^{5/2}} dm(w) \right]^2 dr. \end{aligned}$$

By (1), we have the following estimate

$$3|1 - \rho \bar{w}z| \geq \left(1 - \frac{\rho}{r}\right) + |1 - r\bar{w}z|.$$

Therefore, we obtain

$$g_*(f)^2(z) \lesssim \int_0^1 \left[\int_D \frac{|f(w)|}{|1 - r\bar{w}z|^{5/2}} dm(w) \right]^2 dr.$$

By Minkowski's integral inequality, we have

$$\begin{aligned} g_*(f)^2(z) &\lesssim \left[\int_D |f(w)| \left(\int_0^1 \frac{1}{|1 - r\bar{w}z|^5} dm(w) \right)^{1/2} dm(w) \right]^2 \\ &\lesssim \left[\int_D |f(w)| \left(\int_0^1 \frac{1}{[(1-r) + |1 - \bar{w}z|^2]^5} dm(w) \right)^{1/2} dm(w) \right]^2 \\ &\lesssim \left[\int_D \frac{|f(w)|}{|1 - \bar{w}z|^2} dm(w) \right]^2. \end{aligned}$$

But,

$$f \mapsto \int_D \frac{|f(w)|}{|1 - \bar{w}z|^2} dm(w)$$

is bounded on $L^p(\mathbb{D})$ for $1 < p < \infty$. The proof of Theorem 4 is completed.

The next theorem is regarding the characterization of analytic function f in Bergman space $A^p(\mathbb{D})$ by the discrete Littlewood-Paley operator $g_d(f)$.

Theorem 5. For $0 < p < \infty$ and $f \in A^p(\mathbb{D})$ with $f(0) = 0$, there exist constants C_p' and C_p'' depending only on p such that

$$C_p' \|f\|_{A^p(\mathbb{D})} \leq \|g_d(f)\|_{L^p(\mathbb{D})} \leq C_p'' \|f\|_{A^p(\mathbb{D})}.$$

Moreover

$$\|f\|_{A^p(\mathbb{D})} \approx |f(0)| + \|g_d(f)\|_{L^p(\mathbb{D})}, \quad 1 \leq p < \infty$$

and

$$\|f\|_{A^p(\mathbb{D})}^p \approx |f(0)|^p + \|g_d(f)\|_{L^p(\mathbb{D})}^p, \quad 0 < p < 1$$

The proof of the first half of the inequality of Theorem 5 is similar to that of Theorem 4 but we have to be careful with the change from continuous case to discrete case.

Proof of Theorem 5 For $0 < p < \infty$ and $f_r(z) = f(rz)$ $0 < r < 1$ and $z \in \mathbb{D}$, $z = re^{i\theta}$, by (1) and standard calculations, we have

$$\begin{aligned} \int_D |f(z)|^p dm(z) &= \int_0^{2\pi} \int_0^1 r |f(re^{i\theta})|^p \frac{dr d\theta}{\pi} \\ &= 2 \int_0^1 r \int_0^{2\pi} |f_r(e^{i\theta})|^p \frac{dr d\theta}{2\pi} \\ &\approx \int_0^1 \|g_d(f_r)\|_{L^p(\mathbb{D})}^p r dr \\ &\lesssim \|g_d(f)\|_{L^p(\mathbb{D})}^p. \end{aligned}$$

The last inequality holds because

$$g_d(f_r)(e^{i\theta}) \leq \left(\sum_{n=0}^{\infty} 2^{-2n} |f'(rr_n e^{i\theta})|^2 \right)^{1/2} = g_d(f)(z).$$

The second half of Theorem 5 follows immediately from Theorem 4 and Lemma 3. This completes the proof of Theorem 5.

In the next part, we are going to consider the classical area integral operator, also called the Luzin area operator on Bergman space. To state our results we need the following due to Calderón. In fact it is a special case of Calderón's theorem. See Theorem 1.3 in [10].

Lemma 6. Let $0 < p < \infty$. Then $f \in H^p(\mathbb{D})$ if and only if $A(f) \in L^p(\mathbb{D})$.

Theorem 7. For $0 < p < \infty$, $f \in A^p(\mathbb{D})$ with $f(0) = 0$, there exists a constant C such that

$$\|f\|_{A^p(\mathbb{D})} \leq C \|A(f)\|_{L^p(\mathbb{D})}.$$

Proof For $0 < p < \infty$, let $f_r(z) = f(rz)$, $0 < r < 1$. By Lemma 6, we have

$$\begin{aligned}
& \int_D |f(z)|^p dm(z) \\
&= 2 \int_0^1 \left[\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right] r dr \\
&\approx 2 \int_0^1 \left[\int_0^{2\pi} |A(f_r)(e^{i\theta})|^p \frac{d\theta}{2\pi} \right] r dr \\
&\leq C \|A(f)\|_{L^p(\mathbb{D})}.
\end{aligned}$$

Here in the last step we used the fact that

$$A(f_r)(z) \leq A(f)(z).$$

The converse of Theorem 7 is not as good as those of Theorem 4 and 5 when $1 \leq p < \infty$.

Theorem 8. For $0 < p < 1$, $f \in A^{-p}(\mathbb{D})$ with $f(0) = 0$, there exists a constant C such that

$$\|A(f)\|_{A^{-p}(\mathbb{D})} \leq C \|f\|_{L^p(\mathbb{D})}.$$

By Lemma 2, for $0 < p < 1$,

$$f(z) = \sum_{k=1}^{\infty} c_k f_k(z),$$

where

$$f_k(z) = \frac{1 - |a_k|^2}{(1 - \bar{a}_k z)^{\frac{2}{p}+1}}.$$

So we have

$$\begin{aligned}
& A(f_k)(z) \\
& \lesssim (1 - |a_k|^2) \left[\int_{D_z} \frac{1}{|1 - \bar{a}_k w|^{2(\frac{2}{p}+2)}} dm(w) \right]^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
& \int_D |A(f)(z)|^p dm(z) \leq \\
& \sum_{k=1}^{\infty} |c_k|^p (1 - |a_k|^2)^p \left[\int_{D_z} \frac{1}{|1 - \bar{a}_k w|^{2(\frac{2}{p}+2)}} dm(w) \right]^{p/2}.
\end{aligned}$$

To get through our proof, we need to estimate the inner integral in the last line by variable changes.

$$\begin{aligned}
& \int_{D_z} \frac{1}{|1 - \bar{a}_k w|^{2(\frac{2}{p}+2)}} dm(w) \\
& \leq \int_{|w| \leq |z|} \frac{1}{|1 - \bar{a}_k w|^{2(\frac{2}{p}+2)}} dm(w) \\
& = \int_D \frac{1}{|1 - \bar{a}_k z| |z| |w|^{2(\frac{2}{p}+2)}} \frac{1}{|z|^2} dm(w) \\
& \approx \frac{1}{|z|^2} \frac{1}{(1 - |a_k z|^2)^{2(\frac{2}{p}+1)}}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
& \int_D |A(f)(z)|^p dm(z) \\
& \leq \int_D \frac{1}{|z|^p} \frac{1}{(1 - |a_k z|^2)^{2+p}} dm(z) \\
& \leq \frac{1}{(1 - |a_k|^2)^2} \int_D \frac{1}{|z|^p} \frac{1}{(1 - |a_k z|^2)^p} dm(z) \\
& \leq \frac{1}{(1 - |a_k|^2)^2} \int_D \frac{1}{|z|^p} \frac{1}{(1 - |z|^2)^p} dm(z) \\
& \lesssim \frac{1}{(1 - |a_k|^2)^2} \int_0^{2\pi} \int_0^1 r^{1-p} (1-r)^{-p} dr d\theta \\
& \lesssim \frac{1}{(1 - |a_k|^2)^2} B(2-p, 1-p).
\end{aligned}$$

where $B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt$ is the Beta function. Hence

$$\int_D |A(f)(z)|^p dm(z) \lesssim \sum_k |c_k|^p.$$

We are done with the proof of Theorem 8.

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